Luna-Vust theory of spherical varieties

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This talked is based on the paper "The Luna-Vust Theory of Spherical Embeddings" by Friedrich Knop that can be found at `www.math.rutgers.edu/~knop/papers/LV.html`, that is an expository paper about [**?**].

Thoughout this talk, G will be a connected reductive group over the complex numbers, and B a Borel subgroup. Let's remind some definitions from the previous talk.

Definition 0.1. *An homogenous space G/H is called spherical, if B acts on it with an open orbit.*

Definition 0.2. *An equivariant open embedding of a spherical variety $G/H \hookrightarrow X$ with X normal is called* ***spherical embedding****.*

We also found out in the previous talk that

$$\mathbb{C}[G/H] \cong \bigoplus_{\Lambda(G/H)} V(\lambda)$$

for $\Lambda(G/H)$ the intersection of a lattice and a cone in Λ the set of weights for G.

The aim of this talk is to describe how to classify all spherical embeddings of a given spherical variety G/H.

1 Divisors and G-invariant valuations

Let $G/H \hookrightarrow X$ be a spherical embedding, and let $D \subseteq X$ be an irreducible G-invariant Cartier divisor; note that a divisor is G-invariant if and only if it is contained in the boundary $X \setminus G/H$. Let us now take a function $f \in \mathbb{C}[G/H]$; this can be extended uniquely to a rational function on X, having a fixed order of zero or pole along D; this gives rise to a function

$$\nu_D : \mathbb{C}[G/H] \setminus 0 \to \mathbb{Z}$$

that is a *G-invariant valuation* on G/H.

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Definition 1.1. *A* G***-invariant valuation*** *on* G/H *is a* G*-invariant function* $\nu : \mathbb{C}[G/H] \setminus 0 \to \mathbb{Z}$ *such that*

i) $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$*;*

ii) $\nu(fg) = \nu(f) + \nu(g)$*.*

Let us now study properties of such objects. Given a G-invariant valuation ν, we can consider its restriction to the subring of B-semiinvariant

$$\mathbb{C}[G/H]^B = \bigoplus_{\Lambda(G/H)} V(\lambda)^{(B)} = \bigoplus_{\Lambda(G/H)} \mathbb{C} f_\lambda.$$

Considering the values $\nu(f_\lambda)$, this gives rise to a group homomorphism $\rho_\nu : \Lambda \to \mathbb{Z}$.

Proposition 1.2. ρ_ν *determines uniquely the* G*-invariant valuation* ν*.*

So, we can talk about G-invariant valuation only using the combinatorial data of the homomorphism ρ_ν. This gives us the first ingredient of the combinatorial data that will classify spherical embeddings; given an embedding $G/H \hookrightarrow X$, we consider the boundary

$$X \setminus G/H = D_1 \cup \ldots \cup D_k$$

and we decompose it into irriducible components, and each such divisor D_i will give a G-invariant valuation ν_i, that is, an homomorphism $\rho_{\nu_i}\Lambda(G/H) \to \mathbb{Z}$.

2 B-invariant divisors and orbits

Given a spherical embedding X, we can consider now the irreducible B-invariant divisors $\mathcal{D}(X)$; every such divisor that is not in contained in the boundary (i.e. it is not G-invariant), and so it is determined by its intersection with G/H; in fact, there is only a finite set of such divisors, and they depend only on G/H; let's call the set of such divisors $\mathcal{D} = \mathcal{D}(G/H)$; note that a B-invariant divisor D gives a valuation

$$\nu_D : \mathbb{C}[G/H]^B \setminus 0 \to \mathbb{Z},$$

that descends to an homomorphism $\rho_{\nu_D} : \Lambda(G/H) \to \mathbb{Z}$ as well.

We have the following proposition, relating G-orbits of X and invariant divisors.

Theorem 2.1. *Every* G*-orbit* $Y \subseteq X$ *is uniquely determined by the set of* B*-invariant divisors in* $\mathcal{D}(X)$ *that contains it.*

Given the orbit Y, we will call $\mathcal{B}_Y(X)$ the set of G-stable divisors containing it, and $\mathcal{F}_Y(X)$ the set of B-stable (not G-stable) divisors containing it.

Example 2.2. Let us explain the last theorem by a simple example. Let G be PGL_3, H be PO_3 so that G/H is the space of smooth plane conics, and let us consider the embedding into $\mathbb{P}^5$ the space of degree two polynomials. In this situation, we have only one irreducible G-stable divisor, that is the entire boundary, parametrizing not smooth plane conics, that we will call E. Let us now find B-stable divisors in G/H; remember that B is the subgroup fixing a complete flag composed of a point p and a line L containing p; this means that the divisor of conics through the point p, that we will call D_1 is fixed, and also the divisor of conics tangent to L, that we will call D_2; so, we have $\mathcal{D} = \{D_1, D_2\}$. So, by the previous theorem, every orbit in X can be read by the divisors containing it; now in X we have three orbits: the orbit G/H, the orbit $\mathring{E}$ of the conics that split in two lines, and the orbit of double lines Z. We have

Orbit Y	$\mathcal{B}_Y(X)$	$\mathcal{F}_Y(X)$
G/H	$\emptyset$	$\emptyset$
$\mathring{E}$	E	$\emptyset$
Z	E	D_2

This theorem, by the way, also proves the finiteness of G-orbits in a spherical embedding.

3 Simple embeddings

Definition 3.1. *An embedding $G/H \hookrightarrow X$ is called **simple** if it has a unique closed G-orbit.*

Simple embeddings are simpler to classify; in fact, we have the following, that follows from the theorem in the previous section.

Theorem 3.2. *Let X be a simple spherical embedding, with Z the unique closed G-orbit; then X is uniquely determined by the couple*

$$(\mathcal{B}_Z(X), \mathcal{F}_Z(X)) \in Hom_{\mathbb{Z}}(\Lambda(G/H), \mathbb{Z}) \times \mathcal{F}.$$

Example 3.3. In the previous example of conics, the closed orbit is the locus of double lines F; the divisors containing it are what we called E (that is G-invariant) and D_2, so this embedding is determined by the couple

$$(\{E\}, \{D_2\}) \in Hom_{\mathbb{Z}}(\Lambda(G/H), \mathbb{Z}) \times \{D_1, D_2\}.$$

This theorem still doesn't quite complete the classification of simple embeddings, because it doesn't say what couples in $Hom_{\mathbb{Z}}(\Lambda(G/H),\mathbb{Z})\times\mathcal{F}$ can occur; let's now fill this gap, using the notion of colored cone.

All our combinatorics will live inside the lattice $Hom(\Lambda(G/H),\mathbb{Z})$; inside it we have the cone $\mathcal{V}$ of objects coming from G-equivariant valuations (that is, related to boundary divisors), and we have the finite set $\mathcal{D}$.

Definition 3.4. *A **colored cone** is a pair $(\mathcal{C},\mathcal{F})$ where $\mathcal{C}\subseteq Hom(\Lambda(G/H),\mathbb{Z})$ and $\mathcal{F}\subseteq\mathcal{D}$ (called the **colors**) such that*

CC1: *$\mathcal{C}$ is a cone generated by $\mathcal{F}$ and finitely many elements of $\mathcal{V}$;*

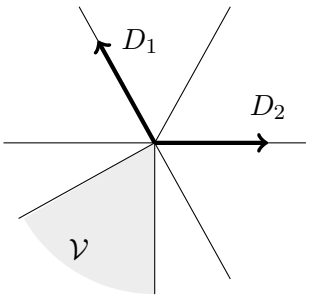
CC2: *the interior $\mathring{C}$ of C intersects $\mathcal{V}$;*

SCC: *the cone $\mathcal{C}$ is strictly convex, that means, it does not contain any line.*

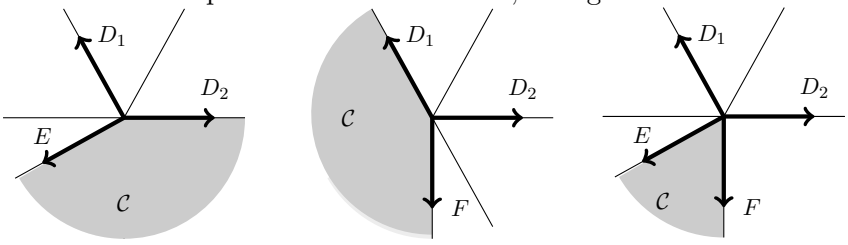
Given the data $(\mathcal{B}_Z(X),\mathcal{F}_Z(X))$ for a simple spherical embedding X with closed orbit Z, the associated colored cone will be given by the cone generated by all elements in $\mathcal{B}_Z(X)$ and $\mathcal{F}_Z(X)$, and the set of colors $\mathcal{F}_Z(X)$.

Theorem 3.5 (Luna-Vust classification of simple embeddings)**.** *Isomoprhism classes of simple spherical embeddings of G/H are in one-to-one correspondence with colored cones.*

Example 3.6. Let's go back again to the case PGL_3/PO_3; the lattice $Hom(\Lambda(G/H),\mathbb{Z})$ lies in a two dimensional vector space; the picture is the following.

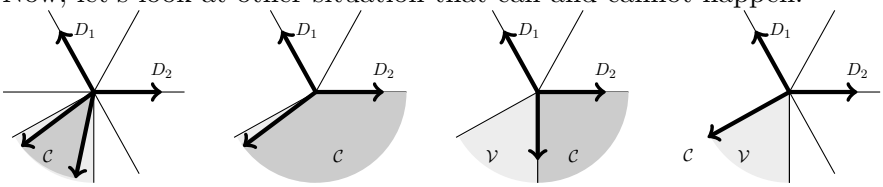


Let us look as some pictures of colored cones, that give different embeddings.



$\mathbb{P}^5$ $\mathbb{P}^{5*}$ complete conics

Now, let's look at other situation that can and cannot happen.



YES YES NO YES

Remark 3.7. Note that in the symmetric case $\mathcal{V}$ is a convex cone, so that there is a canonical choice of colored cone, obtained choosing $\mathcal{V}$ itself and no colors. In case G is adjoint, this gives the wonderful compactification of G/H described in [**?**].

4 The Luna-Vust classification

Now, we want to extend the classification to any embedding, possibly having more then one orbit. As for toric varieties, the combinatorial step we need to take is the one from cones to *fans*.

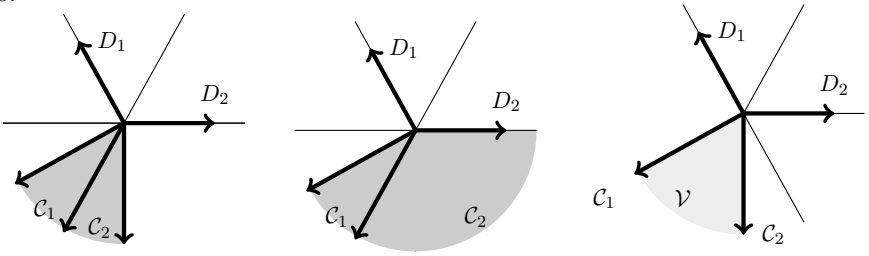
Definition 4.1. *A* ***colored fan*** *is a collection $\mathfrak{F}$ of colored cones such that*

CF1: *every colored face $\mathcal{C}'$ of a colored cone $\mathcal{C} \in \mathfrak{F}$ belongs to $\mathfrak{F}$ as well;*

CF2: *the interiors of all cones in $\mathfrak{F}$ are disjoint;*

Theorem 4.2 (Luna-Vust classification)**.** *Isomorphism classes of spherical embeddings of G/H are in one-to-one correspondence with colored fans.*

Example 4.3. In the same example as before, let us show some colored fans.



5 Further properties

Let us now describe briefly how combinatorial properties of colored fans correspond to geometric properties of the embedding X.

Lemma 5.1. *An embedding X is complete if and only if its colored fan covers the whole $\mathcal{V}$.*

Lemma 5.2. *G-orbits of X correspond to colored cones in the colored fan; moreover, when colored cones have the same number of colors, the codimension is given by the codimension of the cones.*

Lemma 5.3. *Let X, X' be two spherical embeddings, and $\mathfrak{F}$, $\mathfrak{F}'$ their colored fans; then we say $\mathfrak{F}$ **maps to** $\mathfrak{F}'$ if every cone of $\mathfrak{F}$ is contained in a cone of $\mathfrak{F}'$. We have a morphism between two spherical embeddings $X \to X'$ if and only if $\mathfrak{F}$ maps into $\mathfrak{F}'$.*

We can check all this properties in the examples given.

Many other properties can be read from the geometry of colored cones, such as Picard groups and sections of line bundles, intersection theory, and much more.

References

[1] C. De Concini, C. Procesi, *Complete symmetric varieties*, Invariant Theory, Lecture Notes in Math., 996, Springer-Verlag (1983), pp. 1–44

[2] D. Luna, T. Vust, *Plongements d'espaces homogènes*, Commentarii mathematici Helvetici 58 (1983), 186-245.