HELLY TYPE THEOREMS ON THE HOMOLOGY OF THE SPACE OF TRANSVERSALS

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ABSTRACT. In this paper we "measure" the size of the set of n-transversals of a family F of convex sets in R^{n+k} according to its homological complexity inside the corresponding Grassmannian manifold. Our main result states that the "measure" μ of the set of n-transversals of F is greater or equal than k if and only if every k+1 members of F have a common point and also if and only if for some integer $m, \ 1 \le m \le n$, and every subfamily F' of F with k+2 members, the "measure" μ of the set of m-transversals of F' is greater or equal than k.

1. Introduction.

For a family $F = \{A^1, ..., A^d\}$ of d convex sets in R^{n+k} , let $T_n(F)$ be the set of n-transversals to F, that is, the set of all n-planes in R^{n+k} which intersect every member of F.

If X is a set of n-planes in R^{n+k} , we say that $\mu(X) \geq r$ if X has "homologically" as many n-planes as the set of n-planes through the origin in R^{n+r} . Thus, μ "measures" the homological complexity of X inside the corresponding Grassmannian manifold. We will use this "measure" to prove that if subfamilies of F with few members have enough transversals of small dimension, then the whole family F has many transversals of a fixed dimension. That is, after a formal definition of μ , in Section 2, we shall prove in Section 3 the equivalence of the following three properties.

- * Every k+1 members of F have a point in common;
- * $\mu(T_n(F)) > k$;
- * For some integer m where $1 \le m \le n$ and every subfamily F' of F with k+2 members $\mu(T_m(F')) \ge k$.

The first equivalence can be thought of as a homological version of Horn and Klee's classical results [5,6]. See also [4]. They proved that the following assertions are equivalent.

- a) Every k + 1 members of F have a point in common;
- b) Every linear n-subspace of R^{n+k} admits a translate which is a member of $T_n(F)$;
 - c) Every (n-1)-plane Λ lies in a member of $T_n(F)$.

First note that b) is just assertion c), when Λ lies at infinity. In fact, the set of all n-planes that contain Λ is a manifold embedded in the corresponding Grassmannian manifold, which represents an element of its cohomology. So, by using the product structure of the cohomology we shall prove that

$$\mu(T_n(F)) \ge k$$
 \Rightarrow b) and c).

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If X is a set of n-planes in R^{n+k} and for every linear n-subspace of R^{n+k} we can choose a translate which is a member of X, then $\mu(T_n(F))$ is not necessarily greater or equal that k, unless, of course, according with our definition of μ , the choice can be done continuously. If $X = T_n(F)$, the existence of a member of $T_n(F)$ parallel to every linear n-subspace of R^{n+k} implies that we can choose continuously this member and hence that:

$$\mu(T_n(F)) \ge k \qquad \Leftrightarrow \qquad \text{b) and c)}.$$

The spirit of the complete equivalences follows the topological study of the space of transversals initiated in [1] and [2].

We shall consider Euclidean n-space R^n and complete it to the n-projective space P^n by adding the hyperplane at infinity. Let G(n+k,n) be the Grassmannian nk-manifold of all n-planes through the origin in euclidean space R^{n+k} . Although we summarize what we need in Section 2, good references for the homology and cohomology of Grassmannian manifolds are Milnor and Stasheff [7], Pontryagin [9] and Chern [3]; see also [8]. In this paper, we will use reduced Cech-homology and cohomology with Z_2 -coefficients.

2. The topology of Grassmannian manifolds.

Let $\lambda_1,...,\lambda_n$ be a sequence of integers such that $0 \le \lambda_1 \le ... \le \lambda_n \le k$. Let us denote by:

- $(2.1) \ \{\lambda_1,...,\lambda_n\} = \{H \in G(n+k,n) \mid \dim(H \cap R^{\lambda_j+j}) \geq j, \ j=1,...,n\}. \text{ For example, } \{0,\lambda,...,\lambda\} = \{H \in G(n+k,n) \mid R^1 \subset H \subset R^{n+\lambda}\} \text{ and } \{k-\lambda,...,k-\lambda,k\} = \{H \in G(n+k,n) \mid \dim(H \cap R^{n-1+k-\lambda}) \geq n-1\}.$
- (2.2) It is known that $\{\lambda_1,...,\lambda_n\} \subset G(n+k,n)$ is a closed connected λ -manifold, where $\lambda = \sum_1^n \lambda_i$, except possibly for a closed connected subset of codimension three. Thus, $H^{\lambda}(\{\lambda_1,...,\lambda_n\};Z_2) = Z_2 = H_{\lambda}(\{\lambda_1,...,\lambda_n\};Z_2)$. Let $(\lambda_1,...,\lambda_n) \in H_{\lambda}(G(n+k,n);Z_2)$ be the λ -cycle which is induced by the inclusion $\{\lambda_1,...,\lambda_n\} \subset G(n+k,n)$. These cycles are called Schubert-cycles. A canonical basis for $H_{\lambda}(G(n+k,n);Z_2)$ consists of all Schubert-cycles $(\xi_1,...,\xi_n)$ such that $0 \le \xi_1 \le ... \le \xi_n \le k$ and $\sum_1^n \xi_i = \lambda$.
- (2.3) Let us denote by $[\lambda_1,...,\lambda_n] \in H^{\lambda}(G(n+k,n);Z_2)$ the λ -cocycle whose value is one for $(\lambda_1,...,\lambda_n)$ and zero for any other Schubert-cycle of dimension λ . Thus a canonical basis for $H^{\lambda}(G(n+k,n);Z_2)$ consists of all Schubert-cocycles $[\xi_1,...,\xi_n]$ such that $0 \leq \xi_1 \leq ... \leq \xi_n \leq k$ and $\sum_1^n \xi_i = \lambda$. The isomorphism $D: H_{\lambda}(G(n+k,n);Z_2) \to H^{nk-\lambda}(G(n+k,n);Z_2)$ given by:

The isomorphism $D: H_{\lambda}(G(n+k,n); Z_2) \to H^{nk-\lambda}(G(n+k,n); Z_2)$ given by: $D((\lambda_1,...,\lambda_n)) = [k-\lambda_n,...,k-\lambda_1]$ is the classical *Poincaré Duality Isomorphism*.

(2.4) By the above, if $X \subset G(n+k,n)$ is such that $X \cap \{\lambda_1,...,\lambda_n\} = \phi$ and $i_X: X \to G(n+k,n)$ is the inclusion, then

$$i_X^*(D((\lambda_1,...,\lambda_n))) = i_X^*([k - \lambda_n,...,k - \lambda_1]) = 0$$

(2.5) Let M(n+k,n) be the set of all n-planes in R^{n+k} . Thus, $G(n+k,n) \subset M(n+k,n)$. We shall regard M(n+k,n) as an open subset of G(n+k+1,n+1), making the following identifications:

Let $z_0 \in R^{n+k+1} - R^{n+k}$ be a fixed point and, without loss of generality, let G(n+k+1,n+1) be the space of all (n+1)-planes in R^{n+k+1} through z_0 . Let us

identify $H \in M(n+k,n)$ with the unique (n+1)-plane $H' \in G(n+k+1,n+1)$ which contains H and passes through z_0 . Thus

$$G(n+k,n) \subset M(n+k,n) \subset G(n+k+1,n+1),$$

where M(n+k,n) is an open subset of G(n+k+1,n+1) and $G(n+k,n) \subset G(n+k+1,n+1)$ may be regarded as $\{0,k,...,k\}$, the set of all (n+1)-planes in R^{n+k+1} which contains R^1 . In other words, if $j:G(n+k,n)\to G(n+k+1,n+1)$ is the inclusion, then $j(\{\lambda_1,...,\lambda_n\})=\{0,\lambda_1,...,\lambda_n\}$. So, if $0\leq \lambda\leq k,\,\{0,\lambda,...,\lambda\}$ as a subset of M(n+k,n) is the set of all n-planes H through the origin in R^{n+k} with the property that $H\subset R^{n+\lambda}$.

If $X \subset M(n+k,n)$, then $i_X: X \to G(n+k+1,n+1)$ will denote the inclusion.

(2.6) Let A be a subset of X, $i: A \to X$ the inclusion and let $\gamma \in H^*(X; Z_2)$. We say that γ is zero or not zero in A, provided $i^*(\gamma)$ is zero or not zero respectively, in $H^*(A; Z_2)$.

Now we are ready to state our main definition which captures the basic idea of having as many n-planes as the set of all n-planes through the origin in R^{n+r} .

Definition. Let $X \subset M(n+k,n) \subset G(n+k+1,n+1)$. For $0 \le r \le k$, we say that the "measure" of X is at least r,

$$\mu(X) \ge r$$
,

if [0, r, ..., r] is not zero in X.

It is easy to verify that if $\mu(X) \geq r$, then, for any integer $0 \leq r_o \leq r$, $\mu(X) \geq r_o$. Furthermore, observe that for m > 0, then X is also naturally contained in M(n+m+k,n) and the definition of the "measure" μ is independent of m.

Example 2.1. Let $F = \{A^0, ..., A^d\}$ be a family of convex sets. We say that F has a cycle of transversal lines if there is a transversal line that moves continuously until it comes back to itself with the opposite orientation. Observe that, F has a cycle of transversal lines if and only if $\mu(T_1(F)) \geq 1$

The following lemma will be very useful for our purposes

Lemma 2.1. Let $X \subset M(n+k,n)$ be a collection of n-planes and let H be a r-plane of R^{n+k} , $1 \le r \le k$. If $\mu(X) \ge r$, then there is $\Gamma \in X$ such that $\pi_H(\Gamma)$ is a single point, where $\pi_H : R^{n+k} \to H$ is the orthogonal projection.

Proof. Let $Y \subset M(n+k,n)$ be the set of all n-planes Γ in R^{n+k} such that $\pi_H(\Gamma)$ is a single point. As in (2.5), we regard $Y \subset M(n+k,n)$ as a subset of G(n+k+1,n+1). Let Δ be the (n+k-r)-plane in R^{n+k+1} through z_0 orthogonal to the (r+1)-plane that contains H and passes through z_0 . Note that $\Gamma \in Y$ if and only if the (n+1)-plane Γ' that contains Γ and passes through z_0 is such that $\dim(\Gamma' \cap \Delta) \geq n$. Consequently, if we regard Y as a subset of G(n+k+1,n+1), by (2.1) and (2.5), $Y = \{k-r, ..., k-r, k\}$.

Let us regard X as a subset of G(n+k+1,n+1) and suppose that $X \cap Y = \phi$. Then, by (2.4), $i_X^*([0,r,...,r]) = 0$, which means that [0,r,...,r] is zero in X, but this is a contradiction because $\mu(X) \geq r$. Then, $X \cap Y \neq \phi$. This completes the proof of Lemma 2.1. \blacksquare

Remark 2.1. If in the above proof, k = r, and $Y \subset M(n + k, n)$ is the set of all n-planes Γ in R^{n+k} such that $\Gamma \subset \Lambda$, where Λ is a (n-1)-plane in P^{n+k} , then we obtain the following result. Let $X \subset M(n+k,n)$ be a collection of n-planes with the

property that $\mu(X) \geq k$, then: every linear n-subspace of R^{n+k} admits a translate which is a member of X; and every (n-1)-plane Λ lies in a member of X.

3. The Space of Transversals

Let $F = \{A^0, ..., A^d\}$ be a family of convex sets in R^{n+k} and let $T_n(F)$, the space of n-transversals of F, be the subset of the Grassmannian manifold M(n + k, n) of n-planes that intersect all members of F.

Before stating our first result we need the following technical lemma.

Lemma 3.1. Let A^0, A^1, A^k be k+1 convex sets in R^{n+k} , $n \geq 0$, such that $\bigcap_0^k A^i = \phi$. Then there is a k-dimensional linear subspace H of R^{n+k} with the property that $\bigcap_0^k \pi_H(A^i) = \phi$, where $\pi_H : R^{n+k} \to H$ is the orthogonal projection.

Proof. The proof is by induction on k. If k = 1, the proof follows by the separation theorem for disjoint convex sets. Suppose the theorem is true for k, we will prove it for k + 1.

Let A^0, A^1, \dots, A^{k+1} be k+2 convex sets in R^{n+k} , such that $\bigcap_0^{k+1} A^i = \phi$. Since $(\bigcap_0^k A^i) \cap A^{k+1} = \phi$, then there is a hyperplane Λ that separates $\bigcap_0^k A^i$ from A^{k+1} . Suppose $\bigcap_0^k A^i \subset \Lambda^-$ and $A^{k+1} \subset \Lambda^+$, where Λ^+ and Λ^- are the closed half-spaces determined by Λ . Note that $\bigcap_0^k (A^i \cap \Lambda^+) = \phi$.

By induction hypothesis, there is a k-dimensional linear subspace H_0 such that $\bigcap_{0}^{k} \pi_{H_0}(A^i \cap \Lambda^+) = \phi$. Let H be a (k+1)-dimensional linear subspace containing H_0 and the 1-dimensional linear subspace orthogonal to Λ . We shall prove that

$$\bigcap_{0}^{k+1} \pi_H(A^i) = \phi.$$

Assume the opposite and take $x \in \bigcap_0^{k+1} \pi_H(A^i)$. Since $x \in \pi_H(A^{k+1}) \subset \pi_H(\Lambda^+)$, then $x \in \pi_H(A^i \cap \Lambda^+)$, for i = 0, ..., k, which is a contradiction because $\bigcap_0^k \pi_H(A^i \cap \Lambda^+) \neq \phi$ implies $\bigcap_0^k \pi_{H_0}(\pi_H(A^i \cap \Lambda^+)) = \bigcap_0^k \pi_{H_0}(A^i \cap \Lambda^+) \neq \phi$.

Our first result characterizes families of convex sets with the (k+1)-intersection property.

Theorem 3.2. Let $F = \{A^1, ..., A^d\}$ be a family of d convex sets in R^{n+k} , $d \ge k+1$. Every subfamily of F with k+1 members has a common point if and only if

$$\mu(T_n(F)) \geq k$$
.

Proof. Suppose every subfamily of F with k+1 members has a common point. We start by constructing a continuous map $\psi: G(n+k,n) \to T_n(F)$ as follows: for every n-plane H through the origin, let $\pi_H: R^{n+k} \to H^\perp$ be the orthogonal projection, where H^\perp is the k-plane through the origin orthogonal to H. Let us consider the family $\pi_H(F) = \{\pi_H(A^1), ..., \pi_H(A^d)\}$ of d convex sets in H^\perp . Note that every subfamily of $\pi_H(F)$ with k+1 members has a common point. Therefore, by Helly's Theorem, the convex set $F(H) = \bigcap_{1}^{d} \pi_H(A^i)$ is not empty. Note also that $F(H) \subset H^\perp$ depends continuously on $H \in G(n+k,n)$. Let $\psi(H)$ be the n-plane through the center of mass of F(H) and orthogonal to H^\perp . By construction, $\psi(H) \in T_n(F)$.

Let $i: T_n(F) \to G(n+k+1,n+1)$ and note that $i\psi: G(n+k,n) \to G(n+k+1,n+1)$ is homotopic to the inclusion. Therefore, by (2.1) and (2.3), [0,k,...,k] is not zero in $T_n(F)$ and hence $\mu(T_n(F)) \geq k$.

Suppose now $\mu(T_n(F)) \geq k$ and suppose that $\bigcap_{1}^{k+1} A^i = \phi$. By Lemma 3.1, there is a k-dimensional linear subspace H of R^{n+k} with the property that $\bigcap_{1}^{k+1} \pi_H(A^i) = \phi$, where $\pi_H : R^{n+k} \to H$ is the orthogonal projection. This is a contradiction because, by Lemma 2.1, there is $\Gamma \in T_n(F)$ such that $\pi_H(\Gamma)$ is a single point which lies in $\bigcap_{1}^{d} \pi_H(A^i)$. This completes the proof of Theorem 3.2.

Example 3.1. For k = 1 and n = 2, Theorem 3.2 states that every two members of F have a common point if and only if for every direction there is a transversal plane to F orthogonal to it.

Our next result characterizes families of k + 2 convex sets with the (k + 1)-intersection property. Note that this time our transversals need not to be of dimension k.

Theorem 3.3. Let $F = \{A^1, ..., A^{k+2}\}$ be a family of k+2 convex sets in R^{n+k} and let us consider an integer $1 \le m \le n$. Every subfamily of F with k+1 members has a common point if and only if

$$\mu(T_m(F)) \geq k$$
.

Proof. Suppose every subfamily of F with k+1 members has a common point. For i=1,...,k+2, let $a_i\in\bigcap_{j\neq i}\{A^j\in F\}\neq \phi$ and let Γ be a (m+k)-plane containing $\Theta=\{a_1,...,a_{k+2}\}$. Furthermore, for i=1,...,k+2, let $B^i\subset \Gamma$ be the convex hull of the set $\{a_j\in\Theta\mid i\neq j\}$. Therefore, $F'=\{B^1,...,B^{k+2}\}$ is a family of convex sets in the (m+k)-plane Γ with the property that $T_m(F')\subset T_m(F)$ because for i=1,...,k+2, $B^i\subset A^i$. By Theorem 3.2, for $n=m,\,\mu(T_m(F'))\geq k$, which immediately implies that $\mu(T_m(F))\geq k$.

Suppose now $\mu(T_m(F)) \geq k$ and suppose $\bigcap_1^{k+1} A^i = \phi$. By Lemma 3.1, there is a k-dimensional linear subspace H of R^{n+k} with the property that $\bigcap_1^{k+1} \pi_H(A^i) = \phi$, where $\pi_H: R^{n+k} \to H$ is the orthogonal projection. Note now that $T_m(F) \subset M(m+(n-m+k),m)$ is a collection of m-planes in $R^{m+(n-m+k)}$ with the property that $\mu(T_m(F)) \geq k$, and H is a k-plane, $1 \leq k \leq n-m+k$. By Lemma 2.1, there is $\Gamma \in T_m(F)$ such that $\pi_H(\Gamma)$ is a single point which lies in $\bigcap_1^{k+1} \pi_H(A^i)$. This is a contradiction. \blacksquare

Example 3.2. For k = 1 and m = 1, Theorem 3.3 states that three convex sets have the property that every two of them have a common point if and only if there is a cycle of transversal lines to them.

We conclude with our main result, whose proof follows immediately from Theorems 3.2 and 3.3.

Theorem 3.4. Let $F = \{A^1, ..., A^d\}$ be a family of d convex sets in R^{n+k} , $d \ge k+2$, and let us consider an integer $1 \le m \le n$. Every subfamily F' of F with k+2 members has the property that $\mu(T_m(F')) \ge k$ if and only if $\mu(T_n(F)) \ge k$.

Example 3.3. Following Horn and Klee's spirit, for k = 1, n = 2, and m = 1, Theorem 3.4 states that every 3 convex sets of F have a cycle of transversal lines if and only if F has transversal planes orthogonal to every direction.

Example 3.4. For m = n, Theorem 3.4 states that if for every subfamily F' of F with k+2 members and for every linear n-subspace of R^{n+k} there is a translate which is a n-transversal to F', then every linear n-subspace of R^{n+k} admits a translate which is a n-transversal to F.

Example 3.5. Let $F = \{A^1, ..., A^d\}$ be a family of convex sets in \mathbb{R}^{n+k} . According to [1], F has a virtual n-point, if there are (homologically) as many n-transversals to F as if F had a common point, that is, as many n-transversals as there are n-planes through the origin in \mathbb{R}^{n+k} . More precisely, F has a virtual n-point and only if $\mu(T_n(F)) \geq k$. For m = n, Theorem 3.4 states that every subfamily F' of F with k+2 members has a virtual n-point if and only if F has a virtual n-point

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