Transversal lines to lines and intervals.

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Abstract

We prove three theorems. A set of lines in $\mathbb{RP}^n$ has a transversal line if and only if any six of them have a transversal line. The same holds when any five of them have a transversal line, provided that the set of lines is in general position and there are at least seven of them. A finite set of intervals in $\mathbb{R}^n$ has a transversal line if and only if any six of them have a transversal line compatible with a given linear order.

1 Introduction

Helly's Theorem reads: *let $\mathcal{C}$ be a family of compact convex sets in $\mathbb{R}^n$; if every $n+1$ of the sets in $\mathcal{C}$ have a common point, then all the family has a common point.* Hadwiger showed that an extra hypothesis is needed to prove an analogous theorem for "lines that cross" convex sets in the plane. Hadwiger's Theorem, [7], can be stated as follows. *Let $\{C_1, C_2, ..., C_n\}$ be a finite collection of convex sets in the plane such that for any three, C_i, C_j, C_k, $i < j < k$, there is a line crossing them precisely in that order; then there exists a line crossing all the sets in the collection.*

A fruitful direction in which Hadwiger's Theorem has been generalized is for hyperplane transversals. It was opened by Goodman and Pollack in [5], where they gave necessary and sufficient conditions for the existence of a hyperplane transversal to convex sets in any dimension. That work has been pushed further by them and other authors (see [1],[2] and the references there).

However, until now, and as far as we know, no Hadwiger-type theorem is known for transversal line in $\mathbb{R}^n$ for $n > 2$. Some work has been done to obtain criteria for the existence of transversal lines to special classes of convex sets. For example, there is an open conjecture due to Katchalski [9]: if every m members of a collection of pairwise disjoint unit balls in $\mathbb{R}^3$ have a transversal line, then the entire collection has a transversal line. For more information we refer the reader to the excellent surveys [3],[4],[6] and [10].

The goal of this paper is to study transversal lines to families of lines and intervals in n^{th} dimensional space ($n > 2$); we prove a Helly-type theorem for lines and a Hadwiger-type theorem for intervals. Consider lines first. Note that when talking about two lines in $\mathbb{R}^n$ that intersect, there is always a limiting case of "intersection at infinity" when they become parallel (and remain coplanar). To avoid awkward argumentations, or changing the concept of transversality for coplanarity, it is better to complete $\mathbb{R}^n$ to the real projective n-dimensional space $\mathbb{P}^n$, and simply define that two lines there are *transversal* if they intersect, and that a line is *transversal* to a set of lines if it is transversal to each of them. Then all our results will have obvious translations to the affine case. Our opening Theorem is the following.

Theorem 1 *Let $\mathcal{L}$ be a collection of lines in $\mathbb{P}^n$. If every 6 of them have a transversal line, then $\mathcal{L}$ has a transversal line.*

It seems strange that no Hadwiger-type assumption is needed, so that by adding the extra condition of partial transversal lines consistent with a given linear or cyclic ordering, one might expect that the "magic number", 6, can be lowered to 5 in the theorem. This is not the case. There are examples of 6 lines with 5 to 5 transversal lines that meet them in a given linear or cyclic ordering, but with no complete transversal. The study of such examples leads to a refinement in a different direction. Namely, the "magic number" can be lowered to 5 if $\mathcal{L}$ contains enough lines (at least 7) and they are in general position (Theorem 2 proved in Section 5).

The proof of Theorem 1 is remarkably easy for the general case (lines in general position). It relies on simple properties of *hyperboloids*, that is, quadratic surfaces with two line rulings. Most of these facts are well known or straightforward, however, we feel the need to write them down in Section 2 for the benefit of the unaware reader and to establish notation.

In Section 3 we prove the degenerate case of Theorem 1. In Section 4 we describe the examples of 6 lines with 5 to 5 transversals. Finally, in Section

6 we extend the results to the general case of intervals, rays or lines in $\mathbb{R}^n$ ($n \geq 3$), proving a Hadwiger-type theorem (Theorem 3).

2 Hyperboloids and their rulings

Given a family $\mathcal{L}$ of lines in $\mathbb{P}^n$, we will denote by $h(\mathcal{L})$ the set of all the lines transversal to $\mathcal{L}$, and by $|h(\mathcal{L})| \subset \mathbb{P}^n$, called its *support*, the union of all those lines. We will be working with these sets constantly, so that, even if its obvious, it is important to bear in mind that

$$\mathcal{L}' \subset \mathcal{L} \Rightarrow h(\mathcal{L}) \subset h(\mathcal{L}').$$

Our main tool will be the understanding of $h(\mathcal{L})$ for small sets of lines, $\mathcal{L} = \{\ell_1, \ell_2, \cdots, \ell_k\}$ with $k \leq 6$. For the remainder of this section, we will consider such a set of lines $\mathcal{L}$, for growing k, with the extra assumption that they are in *general position*, that is, that no two of them intersect. Until the next section we will address the degenerate case when some lines in $\mathcal{L}$ may meet.

Let us begin with $k = 2$. For any point $p_2 \in \ell_2$, we have a plane passing through ℓ_1 and p_2, which we denote $\ell_1 \vee p_2$. Observe that any point $p \in \ell_1 \vee p_2$ different from p_2 is in a unique line through p_2 and transversal to ℓ_1 (namely, $p \vee p_2$). As $p_2 \in \ell_2$ varies, the planes $\ell_1 \vee p_2$ span a 3-dimensional flat. Therefore $|h(\ell_1, \ell_2)| \cong \mathbb{P}^3$, and every point there not in ℓ_1 or ℓ_2 is in a **unique** transversal to them (in a unique line in $h(\ell_1, \ell_2)$). One can also think of $h(\ell_1, \ell_2)$ naturally parametrized by the torus $\ell_1 \times \ell_2$ (the projective line is a circle); namely, for each $p_1 \in \ell_1$, $p_2 \in \ell_2$, we have the transversal $p_1 \vee p_2 \in h(\ell_1, \ell_2)$.

Now, consider a new line. If ℓ_1 and ℓ_2 have a common transversal with ℓ_3 ($h(\ell_1, \ell_2, \ell_3) \neq \emptyset$), then ℓ_3 intersects the 3-flat $|h(\ell_1, \ell_2)|$. But if, moreover, ℓ_1, ℓ_2 and ℓ_3 have more than one transversal line ($\sharp h(\ell_1, \ell_2, \ell_3) > 1$), then ℓ_3 is contained in the 3-flat $|h(\ell_1, \ell_2)|$ (because it has two points there) and $h(\ell_1, \ell_2, \ell_3)$ grows to be a projective line: it can be naturally parametrized by the points in ℓ_3 where the transversals intersect, because of the uniqueness remark in the preceding paragraph. In this case, three lines in general position in $\mathbb{P}^3$, $h(\ell_1, \ell_2, \ell_3)$ is one *ruling* (naturally parametrized by –intersection with– either of the three lines) of the *hyperboloid* $|h(\ell_1, \ell_2, \ell_3)|$; the unique one that contains the three lines as subsets. For a beautiful exposition of this idea see the opening paragraphs of [8]. The main fact we need about hyperboloids is that $|h(\ell_1, \ell_2, \ell_3)|$ has another ruling which we call the *orthogonal ruling*, and denote it $h(\ell_1, \ell_2, \ell_3)^{\perp}$, in which the ℓ_i lie. Namely, consider any three (different) lines $\ell_1^{\perp}, \ell_2^{\perp}, \ell_3^{\perp} \in h(\ell_1, \ell_2, \ell_3)$, since they have at least the transversal lines ℓ_1, ℓ_2 and ℓ_3, then $\left|h\left(\ell_1^{\perp}, \ell_2^{\perp}, \ell_3^{\perp}\right)\right|$ is also a hyperboloid, and it happens that $|h(\ell_1, \ell_2, \ell_3)| = \left|h\left(\ell_1^{\perp}, \ell_2^{\perp}, \ell_3^{\perp}\right)\right|$. Then we define $h(\ell_1, \ell_2, \ell_3)^{\perp} := h\left(\ell_1^{\perp}, \ell_2^{\perp}, \ell_3^{\perp}\right) \supset \{\ell_1, \ell_2, \ell_3\}$. Summarizing, $h(\ell_1, \ell_2, \ell_3)$ is either empty, a unique line or the ruling (parametrized by the projective line) of a hyperboloid, which has another ruling containing the 3 lines.

The main property we will repeatedly use about hyperboloids is that if a line meets one in more than 2 points, then it belongs to one of its two rulings. This follows, of course, because they are given by quadratic equations, but also from the simple facts we have gadered.

For $k = 4$, suppose that $\sharp h(\ell_1, \ell_2, \ell_3, \ell_4) > 2$, we want to prove that in this case, ℓ_1, ℓ_2, ℓ_3 and ℓ_4 belong to a ruling of a hyperboloid. By hypothesis we can take three lines $\ell_1^{\perp}, \ell_2^{\perp}, \ell_3^{\perp} \in h(\ell_1, \ell_2, \ell_3, \ell_4)$. Since $h(\ell_1, \ell_2, \ell_3, \ell_4) \subset h(\ell_1, \ell_2, \ell_3)$, we then have that $h(\ell_1, \ell_2, \ell_3)$ is the ruling of a hyperboloid. Then $\ell_1, \ell_2, \ell_3, \ell_4 \in h\left(\ell_1^{\perp}, \ell_2^{\perp}, \ell_3^{\perp}\right) = h(\ell_1, \ell_2, \ell_3)^{\perp}$.

This is enough to prove Theorem 1 for the general case of lines in general position.

Proposition 1 *Let $\mathcal{L}$ be any family of lines in $\mathbb{P}^n$ in general position such that every* 6 *of them have a transversal line, then $\mathcal{L}$ has a transversal line.*

Proof. The hypothesis is that for every $\mathcal{L}' \subset \mathcal{L}$ with $\sharp\mathcal{L}' \leq 6$, $h(\mathcal{L}') \neq \emptyset$. Consider three lines in $\mathcal{L}$, ℓ_1, ℓ_2 and ℓ_3 say. They do have transversals, and if they have only one, t say, then t is transversal to every other $\ell \in \mathcal{L}$. So, we must assume they generate a hyperboloid. If $\mathcal{L} \subset h(\ell_1, \ell_2, \ell_3)^{\perp}$, we are done. So, assume there is a line $\ell_4 \in \mathcal{L}$ not in the ruling $h(\ell_1, \ell_2, \ell_3)^{\perp}$. Then $\sharp h(\ell_1, \ell_2, \ell_3, \ell_4) \leq 2$ (since it will appear repeatedly, let us address the following as the "4-2 argument"). Let $\{t_1, t_2\} = h(\ell_1, \ell_2, \ell_3, \ell_4)$ and assume there exists some $\ell_5 \in \mathcal{L}$ that meets only one of them, t_1 say; so that $\{t_1\} = h(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$. But then we still have a vacancy to fill, and every other line in $\mathcal{L}$ must meet t_1. ∎

3 The degenerate case

The aim of this section is to extend the preceding proposition to the case where two of the lines in our family $\mathcal{L}$ meet, and thus, to complete the proof of Theorem 1.

Proposition 2 *Let $\mathcal{L}$ be a family of lines in $\mathbb{P}^n$ such that two of them intersect. If every 6 of them have a transversal line, then $\mathcal{L}$ has a transversal line.*

Proof. Let $\ell_1, \ell_2 \in \mathcal{L}$ be transversal. Denote by p their intersection point and by P the plane they span (we shall write with its obvious generalizations, $p = \ell_1 \wedge \ell_2$ and $P = \ell_1 \vee \ell_2$). Observe that $h(\ell_1, \ell_2)$ consists of all the lines in P and the lines through p in $\mathbb{P}^n$. The proof brakes into three cases.

Case 1. There exists $\ell_3 \in \mathcal{L}$ such that $\ell_3 \subset P$ and $p \notin \ell_3$. Then $h(\ell_1, \ell_2, \ell_3)$ is exactly the set of lines in P. We may clearly disregard all the lines in $\mathcal{L}$ that are contained in P (any line in $h(\ell_1, \ell_2, \ell_3)$ is also transversal to them). The remaining lines in $\mathcal{L}$ intersect P at a point (because they have transversals with ℓ_1, ℓ_2 and ℓ_3). The hypothesis then gives us a set of points in a plane, each three of which are colinear. Hence they are all colinear, and the given line is transversal to $\mathcal{L}$.

Case 2. There exists $\ell_3 \in \mathcal{L}$ such that $\ell_3 \cap P = \emptyset$. Then $h(\ell_1, \ell_2, \ell_3)$ is the set of lines through p in the plane $p \vee \ell_3$ (this is, $h(\ell_1, \ell_2, \ell_3) = \{p \vee p_3 : p_3 \in \ell_3\}$). Now every line in $\mathcal{L}$ must meet the plane $p \vee \ell_3$. Suppose there exists $\ell_4 \in \mathcal{L}$ such that $p \notin \ell_4$ and $\ell_4 \not\subseteq p \vee \ell_3$ (if not, we are clearly done), and let $p_4 = \ell_4 \wedge (p \vee \ell_3)$. Then, $h(\ell_1, \ell_2, \ell_3, \ell_4) = \{p \vee p_4\}$ and every other line in $\mathcal{L}$ meets $p \vee p_4$. (Note that the magic number 6 could be lowered to 5 in this case).

Case 3. Otherwise. In this case, for each line $\ell_i \in \mathcal{L}$, we have a well defined point $p_i \in \ell_i \cap P$; namely $p_i := p$ if $p \in \ell_i$ and $p_i := \ell_i \wedge P$ otherwise. We may assume that this points are not colinear. Then we can find $p_3 \neq p_4$, and such that $p \notin p_3 \vee p_4$. Then, $h(\ell_1, \ell_2, \ell_3, \ell_4)$ contains the line $p_3 \vee p_4$ and at most one other line passing through p (either $p \vee (\ell_3 \wedge \ell_4)$ if ℓ_3 and ℓ_4 intersect, or a unique one if $p \in |h(\ell_3, \ell_4)|$). From here ($\sharp h(\ell_1, \ell_2, \ell_3, \ell_4) \leq 2$), the 4-2 argument completes the proof. ■

4 Examples

We will describe examples of 6 lines without a transversal line, but having transversals 5 to 5. They will prove that Theorem 1 is best possible for such a simple statement. However, they will also point out that it has a refinement in an unexpected direction.

All our examples consist of 6 lines $\ell_1, \ell_2, \cdots, \ell_6$ in $\mathbb{P}^3$ (though described in $\mathbb{R}^3 \subset \mathbb{P}^3$), and 6 "transversals" which we may denote $\ell_1^\perp, \ell_2^\perp, \cdots, \ell_6^\perp$ according to the rule:

$$\ell_i \cap \ell_j^\perp = \emptyset \Leftrightarrow i = j.$$

Example 1. Let ℓ_1, ℓ_2, ℓ_3 be three lines in a plane P meeting at different points $p_{ij} = \ell_i \wedge \ell_j$. Consider three non colinear points in P, called p_4, p_5, p_6, and any point p not in the plane P. Define $\ell_i = p \vee p_i$ for $i = 4, 5, 6$. The 5 to 5 transversals are then uniquely determined by $\ell_i^\perp = p \vee p_{jk}$ for $\{i, j, k\} = \{1, 2, 3\}$, and $\ell_i^\perp = p_j \vee p_k$ for $\{i, j, k\} = \{4, 5, 6\}$. This example may clearly grow for any $k > 6$ by taking new points $p_i \in P$, $7 \leq i \leq k$, and defining $\ell_i = p \vee p_i$; they maintain 5 to 5 transversals but not all 6 to 6.

Example 2. We now avoid the concurrence of many lines in the example above, by taking ℓ_4, ℓ_5, ℓ_6 in general position. Let $C = |h(\ell_4, \ell_5, \ell_6)| \cap P$ for some transversal plane P, and as before, let $p_i = \ell_i \wedge P$. Now, consider three new points in C, and let ℓ_1, ℓ_2, ℓ_3 be the three lines through them (denote again $p_{ij} = \ell_i \wedge \ell_j$). For a subset of 5 containing $\{1, 2, 3\}$ the transversal is as before (generated by the corresponding $p_j$and p_k), and if it intersects in only two elements, say 1 and 2, it is the rule in $h(\ell_4, \ell_5, \ell_6)$ passing through p_{12}. Again, this example may grow by taking more lines in the ruling $h(\ell_4, \ell_5, \ell_6)^\perp$. We are now using a little more about hyperboloids: that planes that do not contain any of their rules (what we called transversal), intersect them in conic curves which parametrize via intersection both of the rulings, and that conics do not have three colinear points.

Our aim now is to give examples in general position.

Example 3. Consider two parallel copies of a circle in $\mathbb{R}^3$. To be precise, let $C^0 = \mathbb{S}^1 \times \{0\}$ and $C^1 = \mathbb{S}^1 \times \{1\}$ ($\subset \mathbb{R}^3$) where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle. Let $C = C^0 \cup C^1$, and let

$$R_\alpha : C \rightarrow C$$

be the map that interchanges the two components (maintaining the first two coordinates) and then rotates both by an angle α. For $i = 0, 1$, let

$$h_\alpha^i = \{p \vee R_\alpha(p) : p \in C^i\}.$$

It is not hard to see that, for $0 < \alpha < \pi$, h_α^0 and h_α^1 are the two rulings of a hyperboloid Q_α.

Now, consider $\beta \neq \alpha$ such that $3(\alpha+\beta) = 2\pi$ and let Q_β be the analogous hyperboloid. Note that $Q_\alpha \cap Q_\beta = C$. Our example will have three rules in h^0_α and three in h^0_β. Let us construct the first three lines together with their corresponding transversals.

Consider any point $p^0_1 \in C^0$. Let $p^1_1 := R_\alpha(p^0_1)$ and define $\ell_1 := p^0_1 \vee p^1_1 \in h^0_\alpha$; observe that $p^1_1 \in C^1$. Now, let $p^0_2 := R_\beta(p^1_1)$, and $\ell^\perp_3 := p^1_1 \vee p^0_2 \in h^1_\beta$; we have gone up through a rule in h^0_α and come down again through one in h^1_β, so that p^0_2 is obtained by rotating p^0_1 an angle $\alpha+\beta$ in C^0. Do this two more times: $p^1_2 := R_\alpha(p^0_2)$, with $\ell_2 := p^0_2 \vee p^1_2 \in h^0_\alpha$; $p^0_3 := R_\beta(p^1_2)$, with $\ell^\perp_1 := p^1_2 \vee p^0_3 \in h^1_\beta$; and then $p^1_3 := R_\alpha(p^0_3)$, with $\ell_3 := p^0_3 \vee p^1_3 \in h^0_\alpha$, and finally, $p^0_1 = R_\beta(p^1_3)$ (because of our choice of angles), with $\ell^\perp_2 := p^1_3 \vee p^0_1$. Observe that, by definition, $\ell^\perp_i \in h(\ell_j, \ell_k)$ for $\{i,j,k\} = \{1,2,3\}$; and moreover, $\ell^\perp_i \cap \ell_i = \emptyset$ because $\ell^\perp_i$, being a rule of $Q_\beta \neq Q_\alpha$, touches Q_α at most (and in fact) twice, namely $\ell^\perp_i \cap Q_\alpha = \{p^0_j, p^1_k\}$ for $(i\ j\ k) = (1\ 2\ 3)$, where they are considered as cyclic orders.

To define ℓ_4, ℓ_5, ℓ_6 and their corresponding transversals, we do the same procedure starting at a new point but going up by R_β and coming down by R_α. Namely, let $p^0_4 \in C^0$ be any point different from p^0_1, p^0_2 and p^0_3. Then define $p^1_4 := R_\beta(p^0_4)$, $p^0_5 := R_\alpha(p^1_4)$, $p^1_5 := R_\beta(p^0_5)$, $p^0_6 := R_\alpha(p^1_5)$, $p^1_6 := R_\beta(p^0_6)$; and $\ell_i := p^0_i \vee p^1_i$, $\ell^\perp_i := p^1_j \vee p^0_k$ for $(i\ j\ k) = (4\ 5\ 6)$. We must finally observe that for $i = 1,2,3$, $\ell^\perp_i \in h(\ell_4, \ell_5, \ell_6) = h^1_\beta$ (and likewise $\ell^\perp_j \in h(\ell_1, \ell_2, \ell_3) = h^1_\alpha$ for $j = 4,5,6$), because they belong to orthogonal rulings of the same hyperboloid. So that our 12 lines satisfy the required conditions. It is also clear that there is no transversal to the six lines, it would have to be a rule of both Q_α and Q_β and there are no such lines $(\alpha \neq \beta)$.

It is not hard to see that the 5 to 5 transversals $\ell^\perp_1, \ell^\perp_2, \cdots, \ell^\perp_6$ in this example, do not intersect the lines $\ell_1, \ell_2, \cdots, \ell_6$ in a consistent linear or cyclic order (which is the natural thing to ask in $\mathbb{P}^3$) –we avoid the technicalities involved in the proof, because they will not be relevant for the results. Thus, we were lead to believe that a Hadwiger-type theorem was at hand, lowering the magic number to 5 but imposing a compatible order, to rule out this example. But this turned out not to be the case as the following, and last, example shows.

Example 4. Consider two hyperboloids Q_a and Q_b, and let C be their curve of intersection, that is, $C = Q_a \cap Q_b$. Then C is a curve of (algebraic) degree 4. It may have many different topological types. In our previous example it has two components each of which is (homologically) an essential cycle in both hyperboloids –and, believe us, this has to do with the impossibility of giving them an order. But it may also happen that C has two components C^0 and C^1 which are topological circles, and each of which bounds a disk in both hyperboloids –with the obvious notation, we can call these disks D_a^0, D_b^0, D_a^1 and D_b^1, so that $\partial D_*^i = C^i$ and $D_*^i \subset Q_*$. For an example, consider two ellipses centered at the origin in $\mathbb{R}^2$ having 4 intersection points; let them be sections of hyperboloids Q_a and Q_b with central axis on the z-axis but expanding, as hyperbolas, at different rates (see Figure 2). This is the case we want to study.

We have two spheres $S^0 = D_a^0 \cup D_b^0$ and $S^1 = D_a^1 \cup D_b^1$ with "equators" C^0 and C^1, respectively. Consider one of them, $S = D_a \cup D_b$ dropping the superindices for a moment. We visualize it as a baseball. The stitching is the equator and the two patches are the hemispheres. So let us call this topological type of intersection of two hyperboloids a *double baseball intersection*. Each hemisphere of S, say D_a (but the same holds for D_b), is ruled by intervals that start and end at the equator –the intersections of the rules of Q_a with D_a. If, with these rules, we can form a cycle of length 6 that goes alternatively from one hemisphere to the other, we can play the game of the example above, obtaining three lines (on one hyperboloid) and the corresponding transversals (on the other). The remaining three lines and transversals will come from the other sphere using the appropriate rulings of the hyperboloids to impose the needed intersections. The main point being that a rule of Q_a, say, is transversal at most to two rules in a given ruling of Q_b. Lets be more precise.

Denote by h_a^+ and h_a^- the two rulings of Q_a, and likewise h_b^+ and h_b^- for Q_b –our change in notation from Example 3 is because 0 and 1 have now a different meaning. For any Q_a and Q_b (with no common rules) we have a map

$$R_a^+ : C \to C$$

defined as follows. Given $p \in C = Q_a \cap Q_b$, let $h_a^+(p)$ be the rule in h_a^+ through p; since $h_a^+(p) \cap Q_b \subset C$ has at most one other point than p, define it to be $R_a^+(p)$ if there is such, or $R_a^+(p) = p$ if $h_a^+(p) \cap Q_b = \{p\}$. Similarly, we have three other maps $R_a^-, R_b^+, R_b^- : C \to C$, which will be addressed as the *ruling involutions* of C. Clearly, they are involutions, that is, e.g., $(R_a^+)^2 = R_a^+ \circ R_a^+ = \text{id}$. In Example 3, R_α and R_β are not globally the ruling involutions, but they are so in each component, so that the example can be constructed using them.

Now, suppose that Q_a and Q_b have double baseball intersection. Then, each of the ruling involutions keeps the components of C fixed (as opposed to Example 3 where they transpose them) because now they are inessential. Therefore, they act as (topological) reflections on C^0 and C^1. Consider one of the components, C^0 say. The composition of two ruling involutions (corresponding to the two hyperboloids, say $R_b^- \circ R_a^+$) is then an orientation preserving homeomorphism in $C^0 \cong \mathbb{S}^1$, with a rotation angle α. Examples show, and the algebraic nature of the context makes plausible, that all the orbits of this "rotation" have the same behavior depending on α. Our examples arise when $\alpha = \pi/3$.

More precisely, suppose Q_a and Q_b are such that for some $p_1^+ \in C^0$ we have that

$$\left(R_b^- \circ R_a^+\right)^3 \left(p_1^+\right) = p_1^+. \tag{1}$$

Then, as in Example 3, define $p_1^- := R_a^+\left(p_1^+\right)$, $p_2^+ := R_b^-\left(p_1^-\right)$, $p_2^- := R_a^+\left(p_2^+\right)$, $p_3^+ := R_b^-\left(p_2^-\right)$, $p_3^- := R_a^+\left(p_2^+\right)$ (our condition then gives $R_b^-\left(p_3^-\right) = p_1^+$); and $\ell_i := h_a^+(p_i^+) = h_a^+(p_i^-)$, $\ell_i^\perp := h_b^-(p_j^-) = h_b^-(p_k^+)$ for $(i\ j\ k) = (1\ 2\ 3)$. If we also have $p_4^+ \in C^1$ such that $\left(R_a^- \circ R_b^+\right)^3 \left(p_4^+\right) = p_4^+$, then, using R_b^+ and then R_a^-, the lines $\ell_4, \ell_5, \ell_6 \in h_b^+$ and $\ell_4^\perp, \ell_5^\perp, \ell_6^\perp \in h_a^-$ can analogously be defined giving an example of 6 lines without transversals, but with 5 to 5 transversals.

By sending an appropriate plane to infinity, and probably relabeling the lines among the subsets $\{1, 2, 3\}$ and $\{4, 5, 6\}$, it is clear that they look like Figure 1. Thus the transversals are compatible with the natural ordering.

It remains to prove the existence of such Q_a and Q_b. For this, let them be defined, for $a, b > 0$, by the following equations in $\mathbb{R}^3$.

$$\begin{aligned} Q_a &: \quad a^2x^2 + \frac{y^2}{a^2} - z^2 = 1, \\ Q_b &: \quad x^2 + y^2 - b^2z^2 = 1. \end{aligned}$$

As we will shortly see, they do have double baseball intersection when $0 < a < 1$, and $b > 1/a$. So it will be convenient to define $\dot{c} := 1/b$, and use parameters a and c subject to the condition $0 < c < a < 1$. Indeed, if we contract the z-axis by a factor c, Q_b becomes the canonical hyperboloid $(x^2 + y^2 - z^2 = 1)$ and Q_a is given by the equation $a^2x^2 + a^{-2}y^2 - c^2z^2 = 1$.

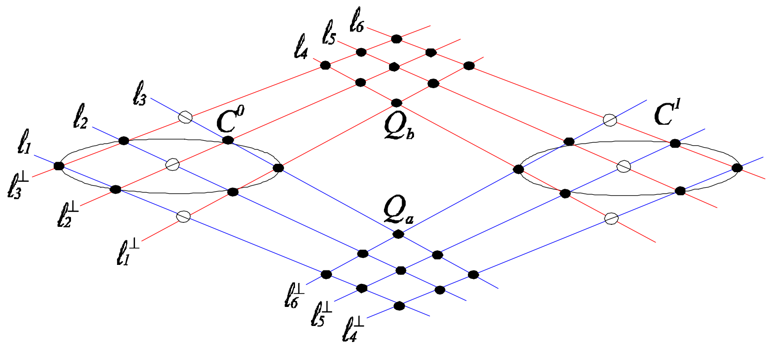

Figure 1:

These are the hyperboloids we will study, referring to them still as Q_a and Q_b. The projection of their intersection $C = Q_a \cap Q_b$ to the yz-plane may be derived by solving for x^2 in the equation of Q_b and then substituting in that of Q_a, giving

$$E : \left(\frac{1+a^2}{a^2}\right) y^2 + \left(\frac{a^2-c^2}{1-a^2}\right) z^2 = 1,$$

which is an ellipse, E, for $0 < c < a$. The two components of C are then obtained by lifting each point in this ellipse E (on the yz-plane) to the appropriate x-coordinate (positive for C^0 say, and its negative in C^1). So that all the analysis can now be done in the yz-plane, whose hyperbolas of intersection with Q_a and Q_b are, lets say, H_a and H_b respectively, (see Figure 2). The rulings of Q_a (respectively, Q_b), project to the pencil of tangent lines to H_a (resp., H_b), so that the ruling involutions can be seen on E as the transposition of the feet of the chords in E belonging to the appropriate part of those pencils. Let us call the ruling involutions $R_a^+, R_a^-, R_b^+, R_b^- : E \to E$ as before. We are seeking for the appropriate a and c such that (1) holds. Observe, that by the symmetry of our example, it suffices to do this on E and then lift it to the actual components C^0 and C^1.

Though hard to write down explicitly, observe that $R_b \circ R_a$ (where we will drop the superindices by making an specific choice) is, by its algebraic nature, an analytic function, which can be thought of as going from the circle to the circle. It then has a topological rotation angle α. Which, by a well known theorem in dynamical systems, makes $R_b \circ R_a$ topologically equivalent

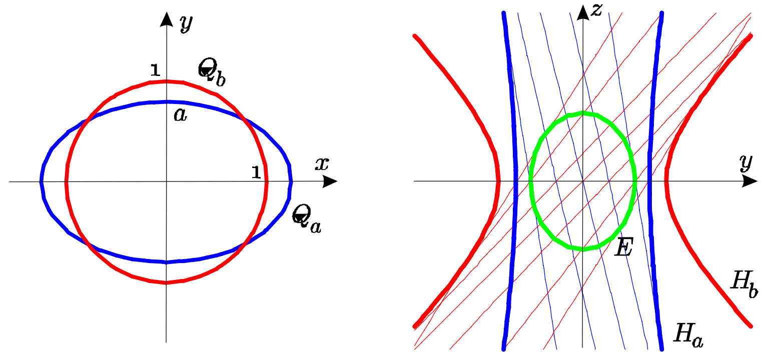

Figure 2:

to the rotation of angle α. This implies that if the angle is rational, then all the orbits are finite and of the same length. But moreover, if it has a finite orbit then all the others have that same length. We are looking for one orbit of length 3. Our example tells us exactly where to look for it.

To fix ideas, let h_a be the pencil of Q_a with negative slope and h_b the one with positive slope. Observe that R_a has exactly two fixed points, namely the two points of E where lines of the pencil h_a are tangent to it. Let p_a be one of them (with $z > 0$, say), and let ℓ_b be the line of h_b that passes through it. So that $(R_b \circ R_a)(p_a)$ is the other foot of ℓ_b, say p. Observe that the only possible way that $(R_b \circ R_a)^3(p_a) = p_a$, is then that the other foot of the chord $h_a(p)$ is precisely a fixed point of R_b (see Figure 3).

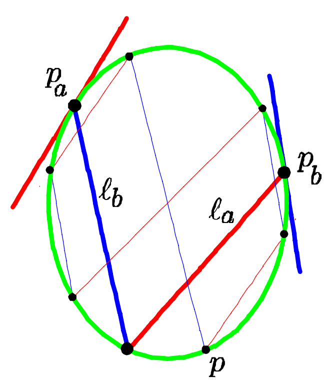

Figure 3:

Therefore, we have a simple way of finding out if $(R_b \circ R_a)$ has an orbit, and hence all the orbits, of length three. Namely, calculate the fixed points p_a and p_b for R_a and R_b respectively (chosen among the respective pairs with obvious geometric arguments), and find the lines $\ell_b := h_b(p_a)$ and $\ell_a := h_a(p_b)$. Then, $(R_b \circ R_a)^3 = \mathrm{id}_E$ if and only if $\ell_a \cap \ell_b \in E$. In principle, this procedure can be carried on. An algebraic "tour de force", involving the "right" parametrizations and a miracoulous solution of the equations that appear (with no further important contribution to the present paper), yields that $\ell_a \cap \ell_b \in E$ if and only if

$$c = \mu(a) := \frac{1 + a^4 + (a^2 - 1)\sqrt{a^4 + 6a^2 + 1}}{2\,(a + a^2 - a^3)}, \tag{2}$$

where, of course, we are still assuming that $0 < c < a < 1$.

Observe that the denominator has roots at 0, the golden ratio $\Phi := (1 + \sqrt{5})/2$ and $-\Phi^{-1}$. Surprisingly, the numerator has roots at the inverse golden ratio Φ^{-1}, and also at $-\Phi^{-1}$ and a double one at 0. So that μ has a single pole at Φ and zeros at 0 and Φ^{-1} (Figure 4). Curiously $\mu(a)\mu(1/a) = 1$. In particular, $\mu(\Phi^{-1}) = 0$; also important to us are the facts that $\mu(1) = 1$, and that $0 < \mu(a) < a$ for $\Phi^{-1} < a < 1$. Because then we have proved the existence of our examples: for every a between Φ^{-1} and 1, take $c = \mu(a)$. Then any point in E (or C^0 and C^1) different from the four points of tangency to the rulings h_a an h_b serves to form the needed cycle of length 6 with lines alternating on the rulings of Q_a and Q_b.

Figure 4:

Summarizing, there exist examples of 6 lines in $\mathbb{R}^3$ with 5 to 5 transversals compatible with a linear order, but without transversals. For the last Section, it will be interesting to note now that the lines in these examples can be reduced to closed intervals.

5 The improvement

Examples 3 and 4 outlined above, and their obvious generalization to other kinds of topological intersection of hyperboloids, turn out to be the only ones that make it impossible to lower the magic number 6 of Proposition 1 to 5. However, they cannot grow as Examples 1 and 2 did.

Theorem 2 *Let $\mathcal{L}$ be a family of lines in general position in $\mathbb{P}^n$. If $\sharp\mathcal{L} \geq 7$ and every 5 lines in $\mathcal{L}$ have a transversal line, then $\mathcal{L}$ has a transversal line.*

Proof. Suppose that there are 6 lines in $\mathcal{L}$, say $\ell_1, \ell_2, \cdots, \ell_6$ with no common transversal. By hypothesis they have 5 to 5 transversals, which we denote $\ell_1^\perp, \ell_2^\perp, \cdots, \ell_6^\perp$, labelled so that

$$\ell_i \cap \ell_j^\perp = \emptyset \Leftrightarrow i = j.$$

Observe that then $\ell_i^\perp \neq \ell_j^\perp$ for $i \neq j$. Our first goal is to prove that there are no more transversals than the obvious ones for all subsets of 4 and 5 elements.

We claim that for all i, j (indices understood between 1 and 6), we have that

$$\begin{aligned} h\left(\{\ell_k : i \neq k \neq j\}\right) &= \{\ell_i^\perp, \ell_j^\perp\}, \\ h\left(\{\ell_k : k \neq i\}\right) &= \{\ell_i^\perp\}. \end{aligned} \tag{3}$$

We will refer to these equalities as “uniqueness” of transversals because the sets on the right hand side are, by definition, contained in the corresponding left hand sides. To ease notation, we may take $i = 6$ and $j = 5$. So consider $\ell_1, \ell_2, \ell_3, \ell_4$. By the general position hypothesis, and the fact that $\sharp h(\ell_1, \ell_2, \ell_3) > 1$, we have that $h(\ell_1, \ell_2, \ell_3)$ is the ruling of a hyperboloid. If ℓ_4 is an element of its orthogonal ruling $h(\ell_1, \ell_2, \ell_3)^\perp$, then every transversal to ℓ_1, ℓ_2, ℓ_3 is also transversal to ℓ_4, but then $\ell_4^\perp$ is transversal to ℓ_4, which does not happen. Therefore $\sharp h(\ell_1, \ell_2, \ell_3, \ell_4) \leq 2$, and the first equation is proved. Since $h(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \subset h(\ell_1, \ell_2, \ell_3, \ell_4) = \{\ell_5^\perp, \ell_6^\perp\}$, but $\ell_5^\perp \notin h(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)$ by definition, the second uniqueness equation follows.

Now, consider any other line $\ell \in \mathcal{L}$. Since

$$\emptyset \neq h(\ell_1, \ell_2, \ell_3, \ell_4, \ell) \subset h(\ell_1, \ell_2, \ell_3, \ell_4) = \{\ell_5^\perp, \ell_6^\perp\},$$

then ℓ is transversal to $\ell_5^\perp$ or to $\ell_6^\perp$. This clearly holds for any subset of two. Namely, for each i and j, we have that ℓ is transversal to one of $\ell_i^\perp$ or $\ell_j^\perp$.

If we assume that ℓ is not transversal to one of the "transversals" $\ell_i^\perp$, say to $\ell_6^\perp$, then, using all the sets of 2 elements containing 6, we conclude that

$$\ell \in h(\ell_1^\perp, \ell_2^\perp, \ell_3^\perp, \ell_4^\perp, \ell_5^\perp). \tag{4}$$

Observe that this is also the case if ℓ meets all of the $\ell_i^\perp$. To conclude from here that $\ell = \ell_6$, we need the equivalent version of (3) with the "transversals" playing the role of the "original" lines and viceversa. Observe that all of the conditions on them are symmetric except for the general position of the "original" lines. But the general position of the "transversals" follows because every pair of them are in a ruling, e.g., $\ell_5^\perp, \ell_6^\perp \in h(\ell_1, \ell_2, \ell_3)$. Thus, we have the corresponding uniqueness equations (3) for the "transversals", which imply, from (4), that $\ell = \ell_6$.

We have proved that any line $\ell \in \mathcal{L}$ is equal to one of $\ell_1, \ell_2, \cdots, \ell_6$. Therefore $\sharp\mathcal{L} = 6$. The condition $\sharp\mathcal{L} \geq 7$ then implies that every 6 lines have a transversal line and the Theorem follows from Proposition 1. ■

Remark 1. The general lines considered thoroughly in the preceding proof belong to the family outlined in Examples 3 and 4. Indeed, if we let $Q_a = |h(\ell_1, \ell_2, \ell_3)|$ and $Q_b = |h(\ell_4, \ell_5, \ell_6)|$, then $\ell_1^\perp, \ell_2^\perp, \ell_3^\perp \in h(\ell_4, \ell_5, \ell_6)$ and $\ell_4^\perp, \ell_5^\perp, \ell_6^\perp \in h(\ell_1, \ell_2, \ell_3)$. The intersection points $\ell_i \wedge \ell_j^\perp$ for $\{i, j\} \subset \{1, 2, 3\}$ or $\{i, j\} \subset \{4, 5, 6\}$ are then in $C = Q_a \cap Q_b$. They form two cycles of length 6 (joined by the corresponding lines) and are related by the corresponding ruling involutions just as in Example 4. They also have the property of behaving like this for any other subset of 3 to start with.

Remark 2. The "grown up" Examples 1 and 2 show that some hypothesis like "general position" is needed. It could possibly be weakened, but hardly stated more simply.

6 Intervals

The purpose of this last section is to obtain a criterion for the existence of transversals to a set of intervals in $\mathbb{R}^n$. By an *interval* we mean any connected subset of a line, so that they may also be open, rays or complete lines. A line being *transversal* to a set of intervals means that it does intersect all of them. We denote, as before, by $h(I_1, I_2, \cdots, I_k)$ the set of lines transversal to the intervals $I_1, I_2, \cdots, I_k$.

The extension of Theorem 1 to intervals, requires the Hadwiger hypothesis of partial transversals compatible with a given linear order. Because we have been working in projective space, where the general problem really lies, one other natural Hadwiger-type hypothesis to consider is compatibility with a given cyclic order. So, before going into the main theorem, let us give an example which will also come handy for the proof.

Example 5. Consider the hyperboloid Q_α of Example 3 for any angle α, $0 < \alpha < \pi$. But now with the circle C^0 at height $z = -1$ ($C^0 = \mathbb{S}^1 \times \{-1\}$), so that Q_α becomes symmetric with respect to the xy-plane, where it intersects at its smallest horizontal circle, which we call S. Consider k points $p_1, p_2, \cdots, p_k$ that form an equilateral k-gon in S, observe they have a natural cyclic order. Choose one of the rulings of Q_α, say h^0_α. And then for any $t \geq 0$, let $I_i(t)$ be the interval of the corresponding rule $h^0_\alpha(p_i)$ that satisfies $-t \leq z \leq t$. Line transversals to three or more of our intervals must lie in the orthogonal ruling h^1_α. The best way to detect them is by *projecting to S by the ruling h^1_α*. Formally, this projection is defined by

$$\begin{aligned} \Pi &: Q_\alpha \to S \\ \Pi(q) &= h^1_\alpha(q) \cap S. \end{aligned}$$

So that, for any subset $A \subset \{1, \cdots, k\}$, with $\sharp A \geq 3$, we have that

$$h(\{I_i(t) : i \in A\}) \neq \emptyset \ \Leftrightarrow \ \bigcap_{i \in A} \Pi(I_i(t)) \neq \emptyset.$$

Moreover, the transversals correspond one-to-one to the intersection points; the former are precisely the rules of h^1_α passing through the latter.

By symmetry, the projections $\Pi(I_i(t))$ are arcs in S centered at p_i. Their angle grows monotonously with t, so that it is better to change the meaning of t for half of that angle. Now, t goes from 0 to π (never reaching it). For small values of t there are no transversals to the intervals $I_i(t)$ in the ruling h^1_α, but they will gradually appear as t grows. For $t = \pi/k$ we have the first (rule) transversals for consecutive intervals. For $t = 2(\pi/k)$, the first three-way transversals appear, but only for three consecutive intervals. This goes on, so that for $t = (k-2)(\pi/k)$, every subset of $k-1$ intervals (which is necessarily consecutive) has a unique common transversal; but there is no transversal to them all.

This example $(\{I_i(t)\}, t = (k-2)(\pi/k))$ shows that for any k, there are intervals such that every $k-1$ of them have transversal lines compatible with a given cyclic order, but with no total transversals; thus, that an extension of Theorem 1 to intervals, requires an extra hypothesis which cannot be compatibility with a cyclic order. It also shows, by the way, the impossibility of a simple Helly-type theorem for connected subsets of the circle.

The existence of transversals to all the intervals, $I_i(t)$, first happens when $t = (k-1)(\pi/k)$; where exactly k transversals to the k intervals appear. Each transversal intersects the intervals in one of the k compatible linear orders of their natural cyclic order.

The example just developed is, in many ways, not as particular as it seems; because all hyperboloids are projectively equivalent, so that Q_α serves as a good model to think about any of them. Let us remark some of these general facts. First, for any set $\mathcal{L}$ of k intervals along one ruling, h^0, of a hyperboloid Q, they naturally acquire a cyclic order such that all the orthogonal rules hit them (or their supporting lines) in that order. Second, once an orientation is given to a rule, it spreads uniquely to all the lines in its ruling (in Example 5 this is implicitly given according to the z-axis); cyclic orders implicitly use this fact. Third, thinking of the hyperboloid in affine space, as a rule of h^1 moves, the intersection points with the distinguished lines (in $\mathcal{L}$) move, in the positive direction say; the "largest" goes to infinity and reappears on the other side so that all compatible linear orders are shown, each of them in a connected interval of rules. Finally, observe that the projection by a ruling, can also be made to any orthogonal rule, and they are topologically equivalent. We have now enough information to prove our last result.

Theorem 3 *Suppose $\mathcal{L} = \{I_1, I_2, \cdots, I_k\}$ is a family of intervals in $\mathbb{R}^n$. If for every 6 of them there is a transversal line that intersects them compatibly with their linear order, then they have a transversal line.*

Proof. First of all, note that if for some subset of 4 intervals, say I_1, I_2, I_3, I_4, we have that $\sharp h(I_1, I_2, I_3, I_4) \leq 2$, the 4-2 argument goes through, taking intervals instead of lines, without any use of the order assumption. So we will assume henceforth that every four of our intervals have at least 3 transversals, addressing it as the "4-3 assumption".

Let ℓ_i be the line in which I_i lies, considered as a line in $\mathbb{P}^n$. The proof falls into cases, corresponding to Propositions 1 and 2.

Case 1. Suppose $\ell_1, \ell_2, \cdots, \ell_k$ are in general position. By the 4-3 assumption, each four of these lines, and hence all, lie in the ruling h^0 of a hyperboloid Q. Furthermore, all of their (at least 3-way) transversals lie in the other ruling h^1 of Q, which may then be oriented to keep the cyclic ordering $(12 \cdots k)$ –there are enough partial transversals to assure this is the cyclic order. We must find a rule in h^1 that intersects all the intervals.

To get the idea of the general argument and to establish notation on the way, suppose there exists a rule $t \in h^1$ that misses all the intervals $I_1, I_2, \cdots, I_k$. Then, the projection, Π, of Q by the ruling h^1 unto any rule of h^0, say to ℓ_0, gives us k intervals in the real line $(\ell_0 \backslash \{t \wedge \ell_0\})$, with 6 to 6 intersections. More than enough ("4 too generous") to apply Helly's Theorem on the line and obtain a common point of the projections, and thus a transversal to the intervals.

For the general argument we use the linear order. Consider the two rules $\ell_1, \ell_k \in h^0$. By the remarks above, we have that ℓ_0 breaks into two intervals such that the rules in h^1 that pass through one of them, say I_0, hit ℓ_1 and ℓ_k in the order $1 < k$, and the rules through the complement hit them in the order $k < 1$. Consider now the set of intervals $\hat{I}_i := \Pi(I_i) \cap I_0$, $i = 1, \ldots, k$, as convex sets in the real line (sending to infinity any point outside I_0). Our hypothesis of transversals compatible with the order $1, 2, \cdots, k$, gives us that each 4 of the $\hat{I}_i$ intersect. Because if we add I_1 and I_k, to the four corresponding I_i, by hypothesis we get a transversal line with 1 and k as extremes, and such a rule intersects I_0. Classical Helly in the line (now with "generosity 2") gives us the total transversal.

The remaining cases correspond precisely to those of Proposition 2. They follow them step by step making the convenient, or necessary adjustments for intervals. So suppose ℓ_1 and ℓ_2 meet at the point p and span the plane P. Observe that now the prescribed linear order of the Theorem need not correspond to the indices, but it will be much simpler to follow the notation of Proposition 2, keeping this in mind.

Case 2. We assume ℓ_1, ℓ_2 and ℓ_3 lie in the plane P but now we can generalize to I_1, I_2, I_3 not concurrent. In this case, $h(I_1, I_2, I_3)$ need not be all the set of lines in P, but it is certainly contained there, and that's enough. We must now consider, for every other interval, its intersection with P. They are all intervals or points. So that the case follows from the classical Hadwiger's Theorem (with "generosity 0"). Indeed, for each three of the intersections, $I_i \cap P$, adding I_1, I_2, I_3 to them we get a compatibly ordered transversal on the plane. Observe that the argument extends to the case $p \notin I_1 \cap I_2$, because then $h(I_1, I_2)$ is contained in the lines of the plane P and we can apply Hadwiger (with "generosity 1").

Case 3. We can assume that $p = I_1 \cap I_2$, and the case extends to the existence of I_3 such that $I_3 \cap P = \emptyset$. Since $h(I_1, I_2, I_3) = \{p \vee p_3 : p_3 \in I_3\}$ which is contained in the lines of the plane $p \vee \ell_3$. This case follows from Hadwiger's Theorem as the previous one, now arguing in the plane $p \vee \ell_3$.

Case 4. Now, we have that every interval contained in P passes through p, and the remaining ones intersect it at a point. Assigning points as in the corresponding Case, we obtain that they are either colinear or a contradiction to the 4-3 assumption. ■

A final remark about the notion of "compatibility" is in order because of the degenerate cases. We understand that a transversal line is *compatible* with the given order if the map of the indices to the intersection points is monotonous (in the transversal), and not necessarily strictly monotonous. Hadwiger's Theorem clearly holds in this case.

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