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Two Applications of Topology to Convex Geometry

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Abstract— The purpose of this paper is to prove two theorems of convex geometry using the techniques of topology. The first theorem states that if, for a strictly convex body K, one may choose continuously a centrally symmetric section, then K must be centrally symmetric. The second theorem states that if every section of a three-dimensional convex body K through the origin has an axis of symmetry, then there is a section of K through the origin which is a disk.

1. A SHAKEN ROGERS THEOREM

Let K be a convex body and let p_0 be a point. Suppose that every section of K through p_0 is centrally symmetric. Then, Rogers proved in [5] that K is centrally symmetric, although p_0 may not be the center of K. If this is the case, Aitchison, Petty, and Rogers [1] and Larman [2] proved that K must be an ellipsoid. The purpose of this section is to prove a shaken version of the Rogers theorem; that is, we essentially prove that if, for every direction, one can choose continuously a section of K that is centrally symmetric, then K is centrally symmetric.

Let $\delta\colon S^2 \to \mathbb{R}$ be a continuous function such that $\delta(-x) = -\delta(x)$. Let us denote by $[\delta]$ the following set of hyperplanes in $\mathbb{R}^3$:

$$[\delta] = \left\{H_y^\delta = \{x \in \mathbb{R}^3 \mid x \cdot y = \delta(y)\}\right\}_{y \in S^2}.$$

If this is the case, we say that $[\delta]$ is a 2-cycle of planes in E^3. This 2-cycle should be considered as a subset of the Grassmannian manifold $G(3,4) = P^3$ (identifying E^3 with a hyperplane of E^4 that does not contain the origin and every plane of E^3 with the hyperplane of E^4 that passes through the origin and contains the plane). The cohomology ring $H^*(G(3,4), Z_2) = \{Z_2[x]; \varkappa^4 = 0\}$, where the generator $\varkappa \in H^1(G(3,4), Z_2) = Z_2$, by duality, can be realized through every 2-cycle of planes.

Let now $K \subset E^3$ be a convex body and, for every $y \in S^2$, let $K_y^\delta = K \cap H_y^\delta$. We say that $\{K_y^\delta\}_{y \in S^2}$ is a 2-cycle of sections of K if

$$K_x^\delta \cap K_y^\delta \cap \operatorname{int}(K) \neq \varnothing$$

for every $x, y \in S^2$.

Lemma 1.1. *Let $[\delta_1]$ and $[\delta_2]$ be two 2-cycles of planes in E^3 and let $p \in E^3$. Then, there is $x_0 \in S^2$ such that $p \in H_{x_0}^{\delta_1} = H_{x_0}^{\delta_2}$.*

Proof. Let $[\delta_3]$ be the 2-cycle of planes that pass through p. It is enough to prove that $[\delta_1] \cap [\delta_2] \cap [\delta_3] \neq \varnothing$, that is, that there exists $x_0 \in S^2$ such that $H_{x_0}^{\delta_1} = H_{x_0}^{\delta_2} = H_{x_0}^{\delta_3}$; but this is true because any 2-cycle of planes realizes the generator $\varkappa \in H^1(G(3,4), Z_2) = Z_2$ and we know that $\varkappa^3$, the generator of $H^3(G(3,4), Z_2) = Z_2$, is not zero.

Lemma 1.2. *Let $[\delta]$ be a 2-cycle of planes in E^3 and let L be a line in E^3. Then, there is $x_0 \in S^2$ such that $L \subset H_{x_0}^\delta$.*

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Proof. Let p_1 and p_2 be two different points of L and let $[\delta_i]$ be the 2-cycle of planes that pass through p_i, $i = 1, 2$. As in the above lemma, we have $[\delta_1] \cap [\delta_2] \cap [\delta] \neq \varnothing$; that is, there exists $x_0 \in S^2$ such that $L \subset H^{\delta}_{x_0}$.

Theorem 1.1. *Let K and L be strictly convex bodies and let δ_i be such that $\{K^{\delta_1}_y\}_{y \in S^2}$ is a 2-cycle of sections of K and $\{L^{\delta_2}_y\}_{y \in S^2}$ is a 2-cycle of sections of L. Suppose that, for every $x \in S^2$,*

$$K^{\delta_1}_{x_0} \text{ is a translation of } L^{\delta_2}_{x_0}.$$

Then, K is a translation of L.

Proof. Let $\aleph$ be the set of diametral lines of K. Since K is a strictly convex body, there is exactly one diametral line in every direction. Note first that, by Lemma 1.1, if ℓ is a diametral line of K, then there is $x \in S^2$ such that $\ell \subset H^{\delta_1}_x$.

We start proving that there are $x_0 \neq y_0 \in S^2$ such that $H^{\delta_1}_{x_0} \cap H^{\delta_1}_{y_0}$ is a diametral line of K. Suppose not. Then, for every diametral line ℓ of K, there is a unique plane $H \in [\delta_1]$ such that $\ell \subset H$. Define $\varphi\colon \aleph \to [\delta_1]$ in such a way that $\varphi(\ell) \in [\delta_1]$ is the unique plane of $[\delta_1]$ that contains the diametral line ℓ. Note that φ is continuous. To see this, suppose that $\ell_i \to \ell$ but $\varphi(\ell_i) \nrightarrow \varphi(L)$. Since $[\delta_1]$ is compact, there is a subsequence $\varphi(\ell_{i_j})$ such that $\varphi(\ell_{i_j}) \to H$ in $[\delta_1]$ but $\varphi(\ell) \neq H$. By hypothesis, $\ell_{i_j} \subset \varphi(\ell_{i_j})$, which implies that $\ell \subset H$; hence, by the definition of φ, $\varphi(\ell) = H$, which is a contradiction. This proves that φ is continuous. Next, to every plane Γ through the origin, we associate continuously a line $\Phi(\Gamma) \subset \Gamma$ as follows: let Γ be a plane through the origin and let $\ell \in \aleph$ be orthogonal to Γ; define $\Phi(\Gamma)$ as the unique line through the origin contained in Γ that is parallel to $\Gamma \cap \varphi(\ell)$. Clearly, the map $\Gamma \to \Phi(\Gamma)$ is continuous, but this is impossible. Then, there exist $x_0 \neq y_0 \in S^2$ such that $\mathcal{L} = H^{\delta_1}_{x_0} \cap H^{\delta_1}_{y_0}$ is a diametral line of K.

Let $\mathcal{L}'$ be the diametral line of L parallel to $\mathcal{L}$. Then, we have $\operatorname{length}(\mathcal{L} \cap K) = \operatorname{length}(\mathcal{L}' \cap L)$. If not, suppose, without loss of generality, that $\operatorname{length}(\mathcal{L} \cap K) < \operatorname{length}(\mathcal{L}' \cap L)$. Hence, by Lemma 1.2, there is $z_0 \in S^2$ such that $\mathcal{L}' \subset H^{\delta_2}_{z_0}$. If this is so, $\mathcal{L}'$ is also a diametral line of $L^{\delta_2}_{z_0}$, and by hypothesis, since $L^{\delta_2}_{z_0}$ is a translation of $K^{\delta_1}_{z_0}$, there is a chord of K whose length is the length of $\mathcal{L}' \cap L$, which contradicts the fact that $\mathcal{L}$ is the diametral line of K. By the same arguments and since L is strictly convex, we have $\mathcal{L}' = H^{\delta_2}_{x_0} \cap H^{\delta_2}_{y_0}$.

Assume, without loss of generality, perhaps after a translation of L, that $K^{\delta_1}_{x_0} = L^{\delta_2}_{x_0}$ and $K^{\delta_1}_{y_0} = L^{\delta_2}_{y_0}$. Therefore, $H^{\delta_1}_{x_0} = H^{\delta_2}_{x_0}$, $H^{\delta_1}_{y_0} = H^{\delta_2}_{y_0}$, and $\mathcal{L} = \mathcal{L}'$. We will prove that $K = L$. For this purpose, it will be enough to prove that $\operatorname{bd} K \subset \operatorname{bd} L$. Let $p \in \operatorname{bd} K$ and, by Lemma 1.1, let $w_0 \in S^2$ be such that $p \in H^{\delta_1}_{w_0} = H^{\delta_2}_{w_0}$. Note that $K^{\delta_1}_{w_0}$ is a translation of $L^{\delta_2}_{w_0}$ and both convex figures lie in the same plane. Furthermore, by definition, $H^{\delta_1=\delta_2}_{w_0} \cap H^{\delta_1=\delta_2}_{x_0} \cap \operatorname{int} K \neq \varnothing$, $H^{\delta_1=\delta_2}_{w_0} \cap H^{\delta_1=\delta_2}_{x_0} \cap \operatorname{int} L \neq \varnothing$, $H^{\delta_1=\delta_2}_{w_0} \cap H^{\delta_1=\delta_2}_{y_0} \cap \operatorname{int} K \neq \varnothing$, and $H^{\delta_1=\delta_2}_{w_0} \cap H^{\delta_1=\delta_2}_{y_0} \cap \operatorname{int} L \neq \varnothing$. Consequently, either $\operatorname{bd} K^{\delta_1}_{w_0} \cap \operatorname{bd} L^{\delta_2}_{w_0}$ has more than three points or $\operatorname{bd} K^{\delta_1}_{w_0} \cap \operatorname{bd} L^{\delta_2}_{w_0}$ has exactly two points that are the extreme points of a common diametral chord. In any case, since $K^{\delta_1}_{w_0}$ is a translation of $L^{\delta_2}_{w_0}$, we have $K^{\delta_1}_{w_0} = L^{\delta_2}_{w_0}$; therefore, $p \in \operatorname{bd} L$. This completes the proof of the theorem.

Theorem 1.2. *Let K be a strictly convex body and let δ be such that $\{K^{\delta}_y\}_{y \in S^2}$ is a 2-cycle of sections of K. Suppose that, for every $x \in S^2$, $K^{\delta_1}_x$ is centrally symmetric. Then, K is centrally symmetric.*

Proof. Let $\delta_1 = \delta$, $L = -K$, and $\delta_2 = -\delta$. Therefore, by hypothesis, for every $x \in S^2$, we have that $L^{\delta_2}_x = -K^{\delta_1}_x$ is a translation of $K^{\delta_1}_x$. Consequently, by the above theorem, $L = -K$ is a translation of K, which implies that K is centrally symmetric.

Corollary 1.1. *Suppose that K is a strictly convex body with the property that the hyperplane that divides the area (surface) in two is centrally symmetric. Then, K is centrally symmetric.*

Corollary 1.2. *Let K be a strictly convex body and let δ be such that $\{K_y^\delta\}_{y\in S^2}$ is a 2-cycle of sections of K. Suppose that, for every $x \in S^2$, $K_x^{\delta_1}$ is affinely equivalent to a fixed convex body L. Then, K is centrally symmetric.*

Proof. We follow the ideas and notation of [4]. As in the proof of Theorem 1 of [4], there is a complete turning of L in E^3. So, by Lemma 2 of [4], L is centrally symmetric; hence, by our Theorem 1.2, K is centrally symmetric.

2. ON THE BEZDEK CONJECTURE

Let $K \subset E^3$ be a convex body. Suppose that every section of K is axially symmetric. Then, K. Bezdek conjectured that K must be a body of revolution or an ellipsoid. Consider two different and concentric circles in E^3 of the same radius with center at the origin, and let K be the convex hull of these two circles. Then, it is not difficult to see that every section of K through the origin is axially symmetric. The purpose of this section is to prove the following theorem.

Theorem 2.1. *Let $K \subset \mathbb{R}^3$ be a convex body and let $p_0 \in \operatorname{int} K$. Suppose that every plane through p_0 has an axis of symmetry. Then, there is a section of K through p_0 that is a disk.*

First, we need a definition.

Definition. A collection of lines $\{L_1, \dots, L_n\}$ is called an *n-starline with vertex x_0* if the lines L_i lie in a plane and are concurrent at x_0 and the angle between two consecutive lines is $\frac{2\pi}{n}$. If H is a plane and $L \subset H$ is a line through a point x_0, we denote by $\operatorname{st}_n(H, L, x_0)$ the unique n-starline with vertex x_0 contained in H that has L as a member.

The following theorem is our main topology ingredient and follows immediately from the results of Mani in [3] (see also [4]).

Theorem 2.2. *Let $1 \le n < \infty$ be a positive integer. It is impossible to choose continuously, for every plane H through the origin in $\mathbb{R}^3$, an n-starline contained in H with vertex at the origin.*

Proof. A continuous selection of an n-starline for every plane through the origin in $\mathbb{R}^3$ gives rise to a field of regular n-gons tangent to S^2, contradicting the main theorem of [3].

The proofs of the following results are straightforward and are left to the reader.

I. Let Φ be a plane convex figure and let $\{L_1, \dots, L_n\}$ be the collection of all its orthogonal lines of symmetry. Then, $\{L_1, \dots, L_n\}$ is an *n-starline.*

II. Let $K \subset \mathbb{R}^3$ be a convex body and let $\{H_i\}$ be a sequence of planes that intersect $\operatorname{int} K$. If $H_i \to H$, then $H_i \cap K \to H \cap K$ in the Hausdorff metric.

III. Let $K \subset \mathbb{R}^3$ be a convex body and let $\{H_i\}$ be a sequence of planes that intersect $\operatorname{int} K$. Suppose that $H_i \to H$, $L_i \subset H_i$ is an axial line of symmetry of $H_i \cap K$, and $L_i \to L$; then, by II, L is an axial line of symmetry of $H \cap K$.

Definition. Let K be a convex body and let H be a plane that intersects $\operatorname{int} K$. We say that $\mu(H) = n$ if $H \cap K$ has exactly n axial lines of symmetry. Note that $\mu(H) = \infty$ implies that $H \cap K$ is a disk.

IV. Let $K \subset E^3$ be a convex body and let $\{H_i\}$ be a sequence of planes that intersect K. If $H_i \to H$ and $\mu(H_i) \to \infty$, then, by II, $H \cap K$ is a disk.

Proof of Theorem 2.1. By IV, we may assume that $1 \le \mu(H) \le n$ for every plane H through p_0. Let $m = n!$. For every plane Γ through the origin, let $\operatorname{st}_m \Gamma = \operatorname{st}_m(\Gamma, L, 0)$, where L is parallel to an axial line of symmetry of $p_0 + \Gamma$. Note that $\operatorname{st}_m \Gamma$ does not depend on the choice of the axial line of symmetry. We will prove that the function $\Gamma \to \operatorname{st}_m \Gamma$ is a continuous function. For this purpose, let $\Gamma_i \to \Gamma$ and suppose that $\operatorname{st}_m \Gamma_i$ does not converge to $\operatorname{st}_m \Gamma$. Then, there is a subsequence Γ_{i_j} such that $\operatorname{st}_m \Gamma_{i_j}$ converges to an m-starline Ω contained in Γ with vertex 0, different from $\mathrm{st}_m\,\Gamma$. Let $L_{i_j} \in \mathrm{st}_m\,\Gamma_{i_j}$ be such that there is an axis of symmetry of $(p_0 + \Gamma_{i_j}) \cap K$ parallel to L_{i_j}. We may assume, without loss of generality, perhaps by taking a subsequence, that $L_{i_j} \to L$. So, $L \in \Omega$, and therefore $L \notin \mathrm{st}_m\,\Gamma$. But, by III, L is parallel to an axis of symmetry of $(p_0+\Gamma)\cap K$, which is a contradiction. Therefore, $\Gamma \to \mathrm{st}_m\,\Gamma$ is continuous, contradicting Theorem 2.1.

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