Configuration spaces of n lines in Affine (n+k)-space

Margareta Boege and Luis Montejano
luis@matem.unam.mx
margaret@matcuer.unam.mx

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Abstract

In this paper we study the space of configurations of n lines in affine space of dimension n+k. We give a topological description of these spaces, in terms of some fibre spaces, and describe more specifically some interesting cases.

[1]

1 Introduction

The origin of the problem we are interested in can be traced back to a paper by Gelfand, Goresky, Mac Pherson and Serganova [4]. They study the complex Grassmann manifold of $(n-k)$-dimensional subspaces of $\mathbb{C}^n$. More specifically, they study a descomposition of the Grassmannian into strata, governed by matroids, that comes from projective configurations of points in a complex projective space. The same descomposition can be obtained from three different points of view: the first one arising from the geometry of Schubert cells in the Grassmann manifold, a second one from the theory of of convex polyhedra and a third one from the theory of combinatorial geometries.

Something similar happens with real affine configurations of points, i.e. k -tuples of points that affinely span modulo the affine group. This space is also a Grassmann manifold, but with a different descomposition into strata, corresponding to oriented matroids [2]. In [3], Bracho, Montejano and Oliveros studied the space of m-transversals to a family $\mathcal{F}$ of convex sets, i.e. the space of all m-planes transversal to all elements of $\mathcal{F}$, and proved that the homotopy of this space is governed by the strata of the space of affine configurations of points.

In [1] Arocha, Bracho and Montejano raise the following question: what happens when one considers flats other than 0-dimensional? what are the spaces

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so obtained and what are the combinatorics that govern their natural stratifications? This is the context of this paper. We answer the question of what are the spaces obtained by taking configurations of n lines in the affine space of dimension $n+k$, modulo the affine group.

The quotient space of an arbitrary space modulo the action of a non compact group, as is the affine group, is not necessarily a manifold. There might be, for example, a dense orbit. Or the space obtained might not be Haussdorff, as in the case of 4 lines in the affine plane $\mathbb{A}^2$ studied in [1]. In our case this is what happens, we do not obtain a "good" quotient space unless we introduce certain "rules", that is, we have to restrict ourselves to an open dense subset, which we define in section II of this paper. Section III contains some more or less technical considerations, and in section IV we define for each configuration of lines a set of invariants, which contains all the information about the configuration. At first we define our invariants only as points in some topological spaces (Grassmannians and products of Grassmannians), and then, in section VI we describe the topology of our configuration spaces by means of a fibre bundle (defined in section V), obtaining a topological description of the space of configurations of n lines in the affine space of dimension $n+k$. Finally we focus on some particular cases.

2 Definitions.

Given two non-negative integers $n \neq 0$ and k, we first will consider the set of all n-tuples of lines in $\mathbb{R}^{n+k}$:

$$F_1 = \{(L_1, L_2, ..., L_n) : \text{ each } L_i \text{ is a line in } \mathbb{R}^{n+k} \}.$$

The Affine group $Af(n+k)$ acts on $\mathbb{R}^{n+k}$ and induces an action on F_1: given $T \in Af(n+k)$ and $(L_1, ..., L_n) \in F_1$,

$$T((L_1 ..., L_n)) = (T(L_1), ..., T(L_n))$$

where $T(L_i)$ is the line $\{T(x) : x \in L_i\}$.

Definition. $(L_1, ..., L_n)$ *fixes* if $T(L_1, ..., L_n) = (L_1, ..., L_n)$ for $T \in Af(n+k)$ implies that T is the identity in $Af(n+k)$.

Let F_2 be the following subset of F_1:

$$F_2 = \{(L_1, ..., L_n) \in F_1 : (L_1, ..., L_n) \text{ fixes}\}.$$

Then F_2 is the subset of F_1 where the action of $Af(n+k)$ is free. Now let F_3 be the open subset of F_2 consisting of the n-tuples of lines whose directions are linearly independent in $\mathbb{R}^{n+k}$.

Definition. The configuration space of n lines in Affine $(n+k)$-space, ${}^{1}_{n}\mathbb{A}^{n+k}$, is the quotient space

$${}^{1}_{n}\mathbb{A}^{n+k} = F_3 / Af(n+k).$$

By a configuration of n lines in Affine $(n+k)$-space we mean $[L_1, ..., L_n] \in F_3/Af(n+k)$.

In all that follows, $G(m,r)$ is the Grassmannian space of m-dimensional subspaces of $\mathbb{R}^{m+r}$.

Since F_3 is a open subset of the n-fold product of $G(2, n+k-1)$, the dimension of ${}_n^1\mathbb{A}^{n+k}$ is $2n(n+k-1)-(n+k)(n+k+1)$. Hence, $2n(n+k-1)$ can not be less than $(n+k)(n+k+1)$, and this will happen only if k is less or equal to $n-3$.

3 Preliminaries

Let $(L_1, ..., L_n)$ be an element of F_3. If we complete the directions of $L_1, ..., L_n$ to a base of $\mathbb{R}^{n+k}$, there is always a linear transformation that takes this base to the canonical base $e_1, ..., e_{n+k}$. This means that in each equivalence class $[L_1, ..., L_n] \in_n^1 \mathbb{A}^{n+k}$ there is $(L_1, ..., L_n)$ such that, for $i = 1, ..., n$, L_i can be written as

$$L_i = \{te_i + \sum_{j=1}^{n} a_j^i e_j + \sum_{j=n+1}^{n+k} b_j^i e_j : t \in \mathbb{R}\}.$$

where $a_i^i = 0$. Let E be the space of n-tuples of lines that fix and are in the form described above.

Let G be the subgroup of $Af(n+k)$ consisting of the affine transformations whose linear part is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & y_{n+1}^1 & ... & y_{n+k}^1 \\ 0 & \lambda_2 & 0 & 0 & y_{n+1}^2 & ... & y_{n+k}^2 \\ 0 & 0 & ... & 0 & ... & ... & ... \\ 0 & 0 & 0 & \lambda_n & y_{n+1}^n & ... & y_{n+k}^n \\ 0 & 0 & 0 & 0 & \lambda_{n+1} & ... & y_{n+k}^{n+1} \\ 0 & 0 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & 0 & y_{n+1}^{n+k} & ... & \lambda_{n+k} \end{pmatrix}$$

with none of $\lambda_1, ..., \lambda_{n+k}$ equal to zero. It is not hard to prove that G is a subgroup of $Af(n+k)$, that the action of G on E is free and that E/G is homeomorphic to ${}_n^1\mathbb{A}^{n+k} = F_3/Af(n+k)$.

Every element $(L_1, ..., L_n)$ of E is completely determined by the coefficients a_j^i y b_j^i, $i = 1, ..., n$ and $j = 1, ..., n+k$. Therefore, $(L_1, ..., L_n)$ can be represented as a pair (A, B), where

(i) $A = \begin{pmatrix} 0 & a_2^1 & ... & ... & a_n^1 \\ a_1^2 & 0 & a_3^2 & a_4^2 & a_n^2 \\ ... & ... & 0 & ... & ... \\ a_1^{n-1} & a_2^{n-1} & ... & 0 & a_n^{n-1} \\ a_1^n & a_2^n & ... & a_{n-1}^n & 0 \end{pmatrix}$ is an $n \times n$ matrix with all diagonal elements equal to zero.

(ii) $B = \begin{pmatrix} b_{n+1}^1 & ... & b_{n+k}^1 \\ ... & ... & ... \\ b_{n+1}^n & ... & b_{n+k}^n \end{pmatrix}$ is a $k \times n$ matrix.

If $(A_1, ..., A_n)$ are the columns of A, and $(B_{n+1}, ..., B_{n+k})$ are the columns of B, then $B_{n+1}, ..., B_{n+k}$ and each A_i, $i = 1, ..., n$, are vectors in $\mathbb{R}^n$.

We will use the following

Notation.

We will denote by π_i the projection of $\mathbb{R}^n$ onto the subspace whose elements have i-th coordinate equal to zero.

We denote by $\mathbb{R}_i^n$ the subspace $\pi_i(\mathbb{R}^n)$

The vector $(1, 1, 1, ..., 1) \in \mathbb{R}^n$ will be written as $\overline{1}$.

And $< Y_1, ...Y_r >$ is the subspace spanned by $Y_1, ...Y_r$, where $Y_1, ...Y_r$ can be vectors or other subspaces.

Now we write the condition of fixing in terms of the pair (A, B) representing $(L_1, ..., L_n)$.

Lemma 1. Let $(L_1, ..., L_n)$ be such that for each $i = 1, ..., n$, L_i can be written as

$$L_i = \{te_i + \sum_{j=1}^{n} a_j^i e_j + \sum_{j=n+1}^{n+k} b_j^i e_j : t \in \mathbb{R}\}.$$

Then $(L_1, ..., L_n) \in E$ if and only if

(i) The vectors $B_{n+1}, ..., B_{n+k}, \overline{1}$ are linearly independent.

(ii) For every $i \in \{1, ..., n\}$, the vectors $A_i, \pi_i(B_{n+1}), ..., \pi_i(B_{n+k}), \pi_i(\overline{1})$ are linearly independent.

Proof.

By definition, $(L_1, ..., L_n)$ fix if and only if given $g \in Af(n+k)$ with $g(L_i) = L_i$ for all $i = 1, ..., n$, g has to be the identity. Since g has to preserve the directions of the lines, g has to be in G, that is g has to be the composition of a linear part in the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & y_{n+1}^1 & ... & y_{n+k}^1 \\ 0 & \lambda_2 & 0 & 0 & y_{n+1}^2 & ... & y_{n+k}^2 \\ 0 & 0 & ... & 0 & ... & ... & ... \\ 0 & 0 & 0 & \lambda_n & y_{n+1}^n & ... & y_{n+k}^n \\ 0 & 0 & 0 & 0 & y_{n+1}^{n+1} & ... & y_{n+k}^{n+1} \\ 0 & 0 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & 0 & y_{n+1}^{n+k} & ... & y_{n+k}^{n+k} \end{pmatrix}$$

and a translation $T(x) = x + \overline{r}$, where

$$\overline{r} = \begin{pmatrix} r_1 \\ r_2 \\ ... \\ r_{n+k} \end{pmatrix}.$$

The condition $g(L_i) = L_i$ for $i = 1, ..., n$, yields for each $i = 1, ..., n$ a system of equations (I)

$$\begin{array}{rcl}
a_i^1\lambda_i + b_{n+1}^1 y_{n+1}^i + b_{n+2}^1 y_{n+2}^i + ... + b_{n+k}^1 y_{n+k}^i + r_i & = & a_i^1 \\
a_i^2\lambda_i + b_{n+1}^2 y_{n+1}^i + b_{n+2}^2 y_{n+2}^i + ... + b_{n+k}^2 y_{n+k}^i + r_i & = & a_i^2 \\
 & & ... \\
a_i^{i-1}\lambda_i + b_{n+1}^{i-1} y_{n+1}^i + b_{n+2}^{i-1} y_{n+2}^i + ... + b_{n+k}^{i-1} y_{n+k}^i + r_i & = & a_i^{i-1} \\
a_i^{i+1}\lambda_i + b_{n+1}^{i+1} y_{n+1}^i + b_{n+2}^{i+1} y_{n+2}^i + ... + b_{n+k}^{i+1} y_{n+k}^i + r_i & = & a_i^{i+1} \\
 & & ... \\
a_i^n\lambda_i + b_{n+1}^n y_{n+1}^i + b_{n+2}^n y_{n+2}^i + ... + b_{n+k}^n y_{n+k}^i + r_i & = & a_i^n
\end{array}$$

and for each $j = 1, ..., k$ a system of equations (II)

$$\begin{array}{rcl}
b_{n+1}^1 y_{n+1}^{n+j} + b_{n+2}^1 y_{n+2}^{n+j} + ... + b_{n+k}^1 y_{n+k}^{n+j} + r_{n+j} & = & b_{n+j}^1 \\
b_{n+1}^2 y_{n+1}^{n+j} + b_{n+2}^2 y_{n+2}^{n+j} + ... + b_{n+k}^2 y_{n+k}^{n+j} + r_{n+j} & = & b_{n+j}^2 \\
 & & ... \\
b_{n+1}^n y_{n+1}^{n+j} + b_{n+2}^n y_{n+2}^{n+j} + ... + b_{n+k}^n y_{n+k}^{n+j} + r_{n+j} & = & b_{n+j}^n
\end{array}$$

The configuration fixes if and only if all this systems have a unique solution. The identity is a solution. It is unique for the systems of type (I) if and only if $A_i, \pi_i(B_{n+1}), ..., \pi_i(B_{n+k}), \pi_i(\overline{1})$ are linearly independent; And for the systems of type (II) if a

$$\begin{array}{rcl}
b_{n+1}^1 y_{n+1}^{n+j} + b_{n+2}^1 y_{n+2}^{n+j} + ... + b_{n+k}^1 y_{n+k}^{n+j} + r_{n+j} & = & b_{n+j}^1 \\
b_{n+1}^2 y_{n+1}^{n+j} + b_{n+2}^2 y_{n+2}^{n+j} + ... + b_{n+k}^2 y_{n+k}^{n+j} + r_{n+j} & = & b_{n+j}^2 \\
 & & ... \\
b_{n+1}^n y_{n+1}^{n+j} + b_{n+2}^n y_{n+2}^{n+j} + ... + b_{n+k}^n y_{n+k}^{n+j} + r_{n+j} & = & b_{n+j}^n
\end{array}$$

The configuration fixes if and only if all this systems have a unique solution. The identity is a solution. It is unique for the systems of type (I) if and only if $A_i, \pi_i(B_{n+1}), ..., \pi_i(B_{n+k}), \pi_i(\overline{1})$ are linearly independent; And for the systems of type (II) if and only if $B_{n+1}, ..., B_{n+k}, \overline{1}$ are linearly independent. □

4 Invariants

We will assign to each n-tuple of lines in F_3 a set of invariants, consisting of a point P in $G(k+1, n-k-1)$ and an n-tuple of pairs, $((P_1, H_1), ..., (P_n, H_n))$, each pair (P_i, H_i) in $G(k+1, n-k-2) \times G(k+2, n-k-3)$ and with the property that $P_i \subset H_i$.

We start by defining the invariants for lines in E, using the canonical basis $e_1, ..., e_{n+k}$ of $\mathbb{R}^{n+k}$. This definition can be extended to all elements of F_3, using another basis and then proving that the definition is independent of the choice of basis.

We will prove that $(L'_1, ..., L'_n)$ is affine image of $(L_1, ..., L_n)$ if and only if their invariants coincide. Thus, every configuration in $F_3/Af(n+k) =^1_n \mathbb{A}^{n+k}$ is uniquely determined by its invariants.

The main invariant.

Let $\phi : E \to G(k+1, n-k-1)$ be defined as follows: given $(A, B) \in E$, we know that each of the columns $B_{n+1}, ..., B_{n+k}$ belongs to $\mathbb{R}^n$. Let $\phi(A, B)$ be

$$\phi(A, B) =< B_{n+1}, ..., B_{n+k}, \overline{1} > .$$

Then ϕ is well defined because one of the conditions on the elements of E is that $B_{n+1}, ..., B_{n+k}, \overline{1}$ are linearly independent.

Now we prove that ϕ is constant on the orbits of the action of G in E. Let (A', B') be in $G(A, B)$. Then there is $g \in G$ with $g(A, B) = (A', B')$. By the definition of the action of G, we have that for all $j = n+1, ..., n+k$ there are $y^j_{n+1}, ..., y^j_{n+k}, r_j \in \mathbb{R}$ such that

$$B'_j = y^j_{n+1} B_{n+1} + ... + y^j_{n+k} B_{n+k} + r_j \overline{1}.$$

Hence, $< B'_{n+1}, ..., B'_{n+k}, \overline{1} > \subseteq < B_{n+1}, ..., B_{n+k}, \overline{1} >$. Both subspaces being of the same dimension, they have to be equal.

Secondary invariants.

For each $i \in \{1, ..., n\}$, we will define functions φ_i that assign to each $(A, B) \in E$ a pair $(P_i(A, B), H_i(A, B))$, where $P_i(A, B)$ is a $(k+1)$-dimensional subspace of $\mathbb{R}^n_i$, $H_i(A, B)$ is a $(k+2)$-dimensional subspace of $\mathbb{R}^n_i$, and $P_i(A, B) \subset H_i(A, B)$.

$$\text{Define } P_i(A, B) = \pi_i(\phi(A, B)) =< \pi_i(B_{n+1}), ..., \pi_i(B_{n+k}), \pi_i(\overline{1}) > .$$

This is a $(k+1)$-dimensional subspace because of the conditions on the elements of E.

We know that $A_i \in \mathbb{R}^n_i$.

$$\text{Define } H_i(A, B) =< P_i(A, B), A_i > .$$

We have hereby defined a function φ_i from E to the set

$$\{(P, H) : P \in G(k+1, n-k-2), H \in G(k+2, n-k-3), P \subset H\}.$$

Now we prove that φ_i is constant on the orbits of the action of G in E. Take (A', B') in $G(A, B)$. Then there is $T \in G$ with $T(A, B) = (A', B')$, which means that for each $i \in \{1, .., n\}$ there are $\lambda_i, y^i_{n+1}, y^i_{n+2}, ..., y^i_{n+k}, r_i \in \mathbb{R}$, where $\lambda_i \neq 0$, such that

$$A'_i = \lambda_i A_i + y^i_{n+1}\pi_i(B_{n+1}) + y^i_{n+2}\pi_i(B_{n+2}) + ... + y^i_{n+k}\pi_i(B_{n+k}) + r_i\pi_i(\overline{1}).$$

We already proved that $P_i(A', B') = P_i(A, B)$. We have then that $H_i(A', B') \subseteq H_i(A, B)$. Both subspaces being of the same dimension, they have to be equal.

Suppose now that (A, B) and (A', B') have the same invariants. Since $\phi(A, B) = \phi(A', B')$ we have that $< B_{n+1}, ..., B_{n+k}, \overline{1} > = < B'_{n+1}, ..., B'_{n+k}, \overline{1} >$. In other words, for all $j = n+1, ..., n+k$ there are $y^j_{n+1}, ..., y^j_{n+k}, r_j \in \mathbb{R}$ such that

$$B'_j = y^j_{n+1}B_{n+1} + ... + y^j_{n+k}B_{n+k} + r_j\overline{1}.$$

Since $(P_i(A, B), H_i(A, B)) = (P_i(A', B'), H_i(A', B'))$ for all $i \in \{1, ..., n\}$, there are $\lambda_i, y^i_{n+1}, y^i_{n+2}, ..., y^i_{n+k}, r_i \in \mathbb{R}$ with $\lambda_i \neq 0$ such that

$$A'_i = \lambda_i A_i + y^i_{n+1}\pi_i(B_{n+1}) + y^i_{n+2}\pi_i(B_{n+2}) + ... + y^i_{n+k}\pi_i(B_{n+k}) + r_i\pi_i(\overline{1}).$$

Let $g \in G$ be such that the linear part of g is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & y^1_{n+1} & ... & y^1_{n+k} \\ 0 & \lambda_2 & 0 & 0 & y^2_{n+1} & ... & y^2_{n+k} \\ 0 & 0 & ... & 0 & ... & ... & ... \\ 0 & 0 & 0 & \lambda_n & y^n_{n+1} & ... & y^n_{n+k} \\ 0 & 0 & 0 & 0 & y^{n+1}_{n+1} & ... & y^{n+1}_{n+k} \\ 0 & 0 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & 0 & y^{n+k}_{n+1} & ... & y^{n+k}_{n+k} \end{pmatrix}$$

followed by the traslation $T(x) = x + \overline{r}$ with

$$r = \begin{pmatrix} r_1 \\ r_2 \\ ... \\ r_{n+k} \end{pmatrix}.$$

Then $g(A, B) = g(A', B')$. Hence, $(L_1, ..., L_n)$ and $(L'_1, ..., L'_n)$ represent the same configuration if and only if their invariants coincide.

Invariants in F_3.

Let $(L_1, ..., L_n)$ be any element in F_3. Let $d_1, ..., d_n$ be the directions of the lines $L_1, ..., L_n$. Let H be the subspace of $\mathbb{R}^{n+k}$ orthogonal to $< d_1, ..., d_n >$ and $d_{n+1}, ..., d_{n+k}$ a basis for H. Each of the lines $L_1, ..., L_n$ can be written as a linear combination of $d_1, ..., d_{n+k}$. As before, we can use this expresion for $L_1, ..., L_n$ to associate to $(L_1, ..., L_n)$ a pair of matrices (A, B), where A consists of the coefficients of $d_1, ..., d_n$ and B of the coefficients of $d_{n+1}, ..., d_{n+k}$. Again, the columns of B are in $\mathbb{R}^n$ and each column A_i of A is in $\mathbb{R}^{n-1}_i$. Using this matrices (A, B), we define the set of invariants as we did before, for lines in E.

Theorem 1. The invariants are well defined in F_3. Moreover, $(L_1, ..., L_n)$ and $(L'_1, ..., L'_n)$ represent the same configuration in ${}^1_n\mathbb{A}^{n+k} = F_3/Af(n+k)$ if and only if their invariants coincide.

Proof.
Given $(L'_1, ..., L'_n) \in F_3$ with directions $d_1, ..., d_n$, the linear transformation that takes $d_1, ..., d_{n+k}$ to $e_1, ..., e_{n+k}$ takes $(L'_1, ..., L'_n)$ to an n-tuple of lines $(L_1, ..., L_n)$ in E, such that the matrices A and B associated to $(L_1, ..., L_n)$ are identical to those associated to $(L'_1, ..., L'_n)$. This, together with the other results in this section, implies the theorem.□

Up to this point our invariants are only points in certain sets. We have not taken into account the topological properties of our configuration spaces. We will do this by using a fibre bundle, which we describe in what follows.

5 The fibre bundle ξ.

Definition of $\vartheta_{k,n-2}$.

Let $\vartheta_{k,n-2}$ be the fibre bundle

$$\begin{array}{ccc} P^{n-k-3} & \hookrightarrow & E_{k,n-k} \\ & & \downarrow \\ & & G(k, n-k-1) \end{array}$$

where the base space is the Grassmannian space $G(k, n-k-2)$; the total space is

$$E_{k,n-2} = \{(P, H) : P \in G(k, n-k-2),\, H \in G(k+1, n-k-3),\, P \subset H\}$$

together with the projection map $\pi : E_{k,n-2} \to G(k, n-k-2)$ given by $\pi(P, H) = P$. The fibre $\pi^{-1}(x)$ is the Projective space P^{n-k-3}.

Definition of a map $f : (G(k, n-k-1) - \Delta) \to \prod^n G(k, n-k-2)$.
Consider now the Grassmannian space $G(k, n-k-1)$. Let $\mathcal{H} \cong \mathbb{R}^{n-1}$ be the subspace of $\mathbb{R}^n$ orthogonal to $\overline{1}$. Using the embedding $j : \mathbb{R}^{n-1} \to \mathcal{H} \subset \mathbb{R}^n$, each $P \in G(k, n-k-1)$ can be regarded as a k-dimensional subspace of $\mathcal{H} \subset \mathbb{R}^n$.

Given a k-dimensional subspace P of $\mathcal{H} \subset \mathbb{R}^n$ and $i \in \{1, ..., n\}$, the image of P under the projection π_i is again a k-dimensional subspace, now of $\mathbb{R}^{n-1}_i$, which may or may not contain $\pi_i(\overline{1})$. Define Δ as

$$\Delta = \{P \in G(k, n-k-1) : \text{for some } i = 1, ..., n,\, \pi_i(\overline{1}) \in (\pi_i \circ j)(P)\}.$$

Let q_i be the projection of $\mathbb{R}^{n-1}_i$ onto the subspace of $\mathbb{R}^{n-1}_i$ orthogonal to $\pi_i(\overline{1})$. Then the composition $q_i \circ \pi_i \circ j$ assignes to each $P \in G(k, n-k-1) - \Delta$, a k-dimensional subspace in $q_i(\mathbb{R}_i^{n-1}) \cong \mathbb{R}^{n-2}$. We have hereby defined for each $i \in \{1, ..., n\}$, a map

$$f_i = q_i \circ \pi_i \circ j : (G(k, n-k-1) - \Delta) \to G(k, n-k-2)$$

Define $f : (G(k, n-k-1) - \Delta) \to \prod^n G(k, n-k-2)$ as $f(P) = (f_1(P), ..., f_n(P))$.

Definition of ξ.

The space $\prod^n G(k, n-k-2)$ is the base space of the product bundle $\vartheta_{k,n-2} \times ... \times \vartheta_{k,n-2}$. Let ξ be the pullback $f^*(\vartheta_{k,n-2} \times ... \times \vartheta_{k,n-2})$. This is equivalent to the Whitney sum $f_1^*(\vartheta_{k,n-2}) \oplus ... \oplus f_n^*(\vartheta_{k,n-2})$.

The codimension of the set Δ.

Recall that

$$\Delta = \{P \in G(k, n-k-1) : \text{for some } i = 1, ..., n,\ \pi_i(\overline{1}) \in (\pi_i \circ j)(P)\}.$$

Since j is an embedding, Δ is homeomorphic to the set

$$\{P \subset \mathcal{H} : \text{for some } i = 1, ..., n,\ \pi_i(\overline{1}) \in \pi_i(P)\}.$$

Suppose that P belongs to this set. Then $P \cap \pi_i^{-1}(\pi_i(\overline{1})) \neq \varnothing$ for some $i \in \{1, ..., n\}$. But observe that $\mathcal{H} \cap \pi_i^{-1}(\pi_i(\overline{1}))$ is a one point set, say $\{p\}$, with $p \neq 0$. Then $< p > \subset P$. That is, P belongs to the set of all k-dimensional subspaces of $\mathcal{H} \cong \mathbb{R}^{n-1}$ that contain $< p >$. This set is homeomorphic to $G(k-1, n-k-1)$. Since Δ is contained in $G(k-1, n-k-1)$, and this set is of codimension at least 2, the codimension of Δ is greater or equal to 2.

6 The space ${}^{1}_{n}\mathbb{A}^{n+k}$.

Let ξ be the bundle defined in the above section, $\mathcal{E}$ its total space and $\pi^* : \mathcal{E} \to G(k, n-k-1) - \Delta$ its projection map.

Theorem 2. The space ${}^{1}_{n}\mathbb{A}^{n+k}$ is homeomorphic to $\mathcal{E}$. Moreover, the space of principal invariants is homeomorphic to $G(k, n-k-1) - \Delta$ and each fiber $(\pi^*)^{-1}(P)$ corresponds to configurations of lines having the same main invariant.

Proof.

First we prove that the space of main invariants is homeomorphic to $G(k, n-k-1) - \Delta$. Recall that the main invariant function $\phi : E \to G(k+1, n-k-1)$ is constant on the orbits of the action of G in E, and so we have:

$$\overline{\phi} : E/G \to G(k+1, n-k-1).$$

The space of all main invariants is the image of $\overline{\phi}$. We shall describe this image. Any $(k+1)$-dimensional subspace of $\mathbb{R}^n$ in the image of $\overline{\phi}$ contains, by definition of $\overline{\phi}$, the vector $\overline{1}$. Let P be any $(k+1)$-dimensional subspace of $\mathbb{R}^n$ containing $\overline{1}$. Let $B_{n+1}, ..., B_{n+k}, \overline{1}$ be a set of vectors spanning P. If for some $i \in \{1, ..., n\}$

we have that $\pi_i(B_{n+1}),...,\pi_i(B_{n+k}),\pi_i(\overline{1})$ are not linearly independent, then P is not in the image of $\overline{\phi}$ (see lemma 1).

Suppose then that for all $i = 1,...,n$ the set $\pi_i(B_{n+1}),...,\pi_i(B_{n+k}),\pi_i(\overline{1})$ is linearly independent. Then they span a $(k+1)$-dimensional subspace in $\mathbb{R}_i^{n-1} \cong \mathbb{R}^{n-1}$. Since $k \le n-3$, we can choose A_i in $\mathbb{R}_i^{n-1}$ linearly independent of $\pi_i(B_{n+1}),...,\pi_i(B_{n+k}),\pi_i(\overline{1})$. And then, by lemma 1, the line configuration corresponding to $(A,B) = ((A_1,...,A_n),(B_{n+1},...,B_{n+k}))$ is in E, and its image under $\overline{\phi}$ is P.

Therefore, the image of $\overline{\phi}$ is the subset of $G(k+1,n-k-1)$ consisting of those $(k+1)$-dimensional subspaces of $\mathbb{R}^n$ that contain $\overline{1}$ and such that for all $i = 1,...,n$ their projection in $\mathbb{R}_i^{n-1}$ is again of dimension $k+1$. Projection onto the subspace $\mathcal{H}$ orthogonal to $\overline{1}$ gives a homeomorphism of the image of $\overline{\phi}$ and $G(k,n-k-1) - \Delta$.

For each $i = 1,...,n$ we have the secondary invariant function φ_i and, as above, this gives us a function

$$\overline{\varphi_i} : E/G \to \{(P,H) : P \in G(k+1,n-k-2), H \in G(k+2,n-k-3), P \subset H\}.$$

Now we prove that the image of $\overline{\varphi_i}$ is homeomorphic to $E_{k,n-2}$. Take $(P_i(A,B),H_i(A,B))$ in the image of $\overline{\varphi_i}$. Then, by definition of $\overline{\varphi_i}$, $\pi_i(\overline{1}) \in P_i(A,B) \subset H_i(A,B)$. Composition with q_i (the projection onto the subspace orthogonal to $\pi_i(\overline{1})$) gives a homeomorphism between the image of $\overline{\varphi_i}$ and $E_{k,n-2}$.

Now we define a homeomorphism $\varphi : E/G \to \mathcal{E}$. By definition, the space $\mathcal{E}$ is

$$\begin{aligned}\mathcal{E} &= \{(P,((P_1,H_1),...,(P_n,H_n))) \in (G(k,n-k-1)-\Delta) \times (E_{k,n-2} \times ... \times E_{k,n-2}) \\ &: f(P) = (P_1,...,P_n)\}.\end{aligned}$$

Given $(A,B) \in E/G$, define

$$\varphi(A,B) = ((p \circ \overline{\phi})(B),((q_1 \circ \overline{\varphi_1})(A,B),...,(q_n \circ \overline{\varphi_n})(A,B)).$$

Recall that $\varphi_i(A,B) = (P_i(A,B),H_i(A,B))$. For φ to be well defined we only need

$$f((p \circ \overline{\phi})(B)) = (q_1(P_1(A,B)),...,q_n(P_n(A,B))).$$

But

$$\begin{aligned} f((p \circ \overline{\phi})(B)) &= (f_1(p(\overline{\phi}(A,B))),...,f_1(p(\overline{\phi}(A,B)))) \\ &= (q_1(\pi_1(\overline{\phi}(A,B))),...,q_n(\pi_n(\overline{\phi}(A,B))) \\ &= ((q_1 \circ \overline{\varphi_1})(A,B),...,(q_n \circ \overline{\varphi_n})(A,B)). \end{aligned}$$

This suffices to prove the theorem since we have already proved the bijectivity of φ when we proved that the set of invariants is unique for each configuration, and we also know that the maps assigning to a set of vectors the subspace spanned by them and the projections are continuous and open, and E/G has the quotient topology.

Finally, take $P \in G(k, n-k-1) - \Delta$. Then the fibre $(\pi^*)^{-1}(P)$ is

$$(\pi^*)^{-1}(P) = \{(P, ((P_1, H_1), ..., (P_n, H_n))) \in \mathcal{E} : f(P) = (P_1, ..., P_n)\}$$

Taking $\varphi^{-1}((\pi^*)^{-1}(P))$, the fibre over P corresponds to

$$\{(A, B) : \varphi(A, B) \in (\pi^*)^{-1}(P)\} = \{(A, B) : (p \circ \overline{\phi})(A, B) = P\}.$$

Since, in the image of $\overline{\phi}$, p is a homeomorphism, two configurations are in the same fibre if and only if they have the same main invariant.□

7 Some examples.

The space ${}^1_n\mathbb{A}^n$ of configurations of n lines in $\mathbb{A}^n$ is homeomorphic to the product of n Projective spaces P^{n-3}.

When $k = 0$, a configuration $(L_1, ..., L_n)$ is described by the matrix A of section III, and the condition for a configuration to fix reduces to: for all $i = 1, ..., n$, A_i and $\pi_1(\overline{1})$ are linearly independent.

The main invariant is given by the one dimensional subspace $< \overline{1} >$ in $\mathbb{R}^n$, and is the same for every configuration. The secondary invariants are, for each $i \in \{1, ..., n\}$ a pair (P_i, H_i), where $P_i =< \pi_i(\overline{1}) >$ for every configuration and H_i is a 2-dimensional subspace of $\mathbb{R}^{n-1}_i$ that contains $< \pi_i(\overline{1}) >$. By theorem 2, ${}^1_n\mathbb{A}^n$ is homeomorphic to a fibre bundle with base space a single point and fibre the n-fold product of projective spaces P^{n-3}, each arising from projecting all planes in $\mathbb{R}^{n-1}_i$ containing $< \pi_i(\overline{1}) >$ onto the space orthogonal to $< \pi_i(\overline{1}) >$. Thus, the space of configurations of 4 lines in $\mathbb{R}^4$ is homeomorphic to $T^4 = P^1 \times P^1 \times P^1 \times P^1$; the space of configurations of 5 lines in $\mathbb{R}^5$ is homeomorphic to the product of five Projective spaces P^2, and so on.

The space ${}^1_n\mathbb{A}^{2n-3}$ of configurations of n lines in $\mathbb{A}^{2n-3}$ is homeomorphic to the Grassmannian $G(n-3, 2) - \Delta$, where Δ is a simplicial complex of dimension $2(\mathbf{n-4})$.

In this case, $k = n-3$. Since $0 \leq k \leq n-3$, in this case k takes its maximum value. Here, the main invariant is given by a $(n-2)$-dimensional subspace of $\mathbb{R}^n$ containing $\overline{1}$; and for each $i \in \{1, ..., n\}$, the secondary invariant is given by a pair (P_i, H_i) where P_i is a $(n-2)$-dimensional subspace of $\mathbb{R}^{n-1}$, and H_i must be a $(n-1)$-dimensional subspace of $\mathbb{R}^{n-1}_i$, and therefore, for all configurations, H_i must be $\mathbb{R}^{n-1}_i$. Since P_i can be obtained from the main invariant by means of the projection π_i, all the information about a configuration must be contained in the main invariant.

Theorem 2 reflects this fact: according to it, ${}^1_n\mathbb{A}^{2n-3}$ is homeomorphic to a fibre bundle with base space $G(n-3, 2) - \Delta$ (the space of the main invariants), and fibre the n-fold product of $P^{n-(n-3)-3} = P^0$, which is a point.

In section V we showed that Δ is contained in the union of n spaces $G(n-4, 2)$ (one for each $i = 1, ..., n$). In fact, from the same argument it can be seen that Δ is equal to the union of this spaces. Then we have that, ${}^1_4\mathbb{A}^5$ is homeomorphic to the projective space P^2 with 4 points removed; $\frac{1}{5}\mathbb{A}^7$ is homeomorphic to $G(2,2)$ with 5 projective spaces P^2 removed, and so on.

The space $\frac{1}{5}\mathbb{A}^6$of configurations of five lines in $\mathbb{A}^6$.

In this case, neither the space of main invariants nor the space of secondary invariants are trivial. For each configuration, the main invariant is a 2-dimensional plane in $\mathbb{R}^5$ and for each $i \in \{1, ..., 5\}$, the secondary invariant is a pair (P_i, H_i) where P_i is a 2-dimensional plane in $\mathbb{R}_i^4$ containing $\overline{1}$ and H is a 3-dimensional subspace of $\mathbb{R}_i^4$ containing P_i.

Let us look more closely at the construction of the bundle ξ for this case. Recall that $\vartheta_{1,3}$ is the bundle with base space the projective space P^2; total space the set of all pairs (P, H) where P is a line in $\mathbb{R}^3$, and H a plane containing P; and the projection map takes every pair (P, H) to P. This is the projectivization of the tangent bundle of P^2 (see [5]). Then we take the product of five of these bundles, and the map $f : (P^3 - \Delta) \to (P^2 \times P^2 \times P^2 \times P^2 \times P^2)$. Here Δ consists of five points and in each coordinate the map f retracts P^3 without these points to a P^2 not containing one of the points.

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