CHARACTERIZATION OF ELLIPSOIDS AND POLARITY IN CONVEX SETS

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Abstract. By introducing the concept of polarity in convex sets, we are able, in a natural way, to generalize several classic characterizations of ellipsoids, showing that all of them depend upon and are related to the concept of projective center of symmetry. Using these ideas, we are also able to develop new characterizations of ellipsoids and to propose new problems.

1. Introduction

The classical and geometrically elementary characterizations of ellipsoids, studied by Buseman in [5], are fundamental tools in the proofs of a number of powerful and interesting geometrical results, such as the False Centre Theorem [1], [14]; Gruber's Theorem about caustics and ellipsoids [12], [3]; Hobinger-Burton-Larman's Theorem [8], [2], and many other important geometric results (see for example [6], [7], [9], [10], [11], [15] and [16]).

As we shall see later, from different points of view, these classical characterizations are formulated in a restricted manner. So, the purpose of this paper is to place them in the general setting of "polarity in convex sets". This will allow us to extend naturally these classical characterizations and to show that all of them are related and have as a basic notion the concept of projective centre of symmetry. Under this new point of view, natural generalizations and variations of interesting results, new proofs and conjectures will arise naturally.

2. Poles and Polars in Convex Sets

From now on, K will be a smooth, compact, strictly convex body in euclidean n-space E^n, $n \geq 2$. We complete E^n to the n-dimensional projective space P^n by adding the hyperplane at infinity. If $\Omega \in P^n - K$, then the union of all lines through Ω that intersect K will be denoted by $C_\Omega(K)$. If Δ is a hyperplane of P^n and $\Omega \in P^n - (K \cup \Delta)$, then *the projection of K from Ω into Δ*, is defined by: $P_\Omega(\Delta, K) = C_\Omega(K) \cap \Delta$. Finally, if $\Gamma \subset P^n - K$ is a k-plane, $0 \leq k \leq n-2$, *the shadow boundary of K with respect Γ*, $S\partial(K,\Gamma)$, is the topological $(n-k-2)$-sphere contained in ∂K, defined by:

$$S\partial(K,\Gamma) = \{H \cap K \ / \ H \text{ is a supporting hyperplane of } K \text{ through } \Gamma\} = \{\Lambda \cap K \ / \ \Lambda \text{ is a supporting } (k+1)\text{-plane of } K \text{ through } \Gamma\}.$$

Let $O \in P^n - \partial K$. We say that O is a *pole of K* if there is a hyperplane H of P^n with the property that for every line L through O such that $\partial K \cap L = \{A, B\}$, we have that the cross section of A, B, O and the intersection of L and H is minus one. That is:

$$[A, B; O, L \cap H] = -1.$$

If this is so, we say that H is a *polar hyperplane* of K and also that H is the *polar* of the pole O.

If $O \in \text{int}K$ is a pole of K, then we say that O is a *projective center of symmetry* of K because, in this case, the polar of K is a hyperplane H that does not intersect K and if π is a projective isomorphism that sends H to the hyperplane at infinity, then $\pi(K)$ is a compact centrally symmetric convex body with center $\pi(O)$. In fact, $O \in \text{int}K$ is a pole of K, or a projective center of symmetry of K if and only if there is a projective isomorphism $\pi : P^n \to P^n$ with the property that $\pi(K)$ is a compact, centrally symmetric, convex body with center $\pi(O)$.

If $O \in P^n - K$ is a pole of K, then its polar H will be called a *projective hyperplane of symmetry* of K. In this case, $H \cap \partial K \neq \phi$, in fact, it is not difficult to see that

$$H \cap \partial K = S\partial(K, O). \tag{1}$$

If π is a projective isomorphism that sends $\pi(K)$ into a compact convex body and $\pi(O)$ into a point at infinity, then $\pi(H)$ is a hyperplane of symmetry of K. In fact, a hyperplane H is a projective hyperplane of symmetry of K if and only if there is a projective isomorphism $\pi : P^n \to P^n$ with the property that $\pi(H)$ is a hyperplane of symmetric of the compact convex body $\pi(K)$.

Let $\mathcal{E} \subset E^n \subset P^n$ be an ellipsoid. Then every point $O \in P^n - \partial\mathcal{E}$ is a pole of $\mathcal{E}$ and every non-tangent hyperplane of $\mathcal{E}$ is a polar of $\mathcal{E}$.

Let $O \in P^n - \partial K$ and let $A \in \partial K$. We say that $B \in \partial K$ is a *O-antipode* of A if the line L through O and A is such that $L \cap \partial K = \{A, B\}$. Suppose now that $O \in P^n - \partial K$ is a pole of K with polar hyperplane H and let A, B be two different O-antipodes of K. Then, the supporting hyperplanes of K through A and B intersect in a (n-2)-plane contained in H. In fact:

$$\{\Gamma / \Gamma \subset H - K \text{ is a } (n-2)\text{-plane}\} = \tag{2}$$

$$\{H_A \cap H_B \;/\; H_A \text{ and } H_B \text{ are supporting hyperplanes of } K \text{ at } O\text{-antipode points } A \text{ and } B \text{ of } \partial K\}.$$

The converse is essentially true (see [5;16.17]), that is:

Theorem 2.1. *Let H be a hyperplane of P^n with $H \cap K = \phi$. Suppose that $O \in$ intK is such that the supporting hyperplanes of K through O-antipodal points intersect in a (n-2)-plane contained in H, then O is a pole or projective center of symmetry of K with polar hyperplane H.*

If $O \in P^n - K$, then a similar result is not true. Counterexamples can easily be constructed. In this direction we have the following straightforward result:

Lemma 2.2. *Let U be an open set of S^{n-1} and let $f, g : S^{n-1} \to R^+$ be two smooth functions. Let $V_f = \{f(u)u \in R^n / u \in U\}$ and $V_g = \{g(u)u \in R^n / u \in U\}$. Then, there is a constant k such that $f = kg$ if and only if for every $u \in U$ the tangent hyperplanes of V_f at $f(u)u$ and the tangent hyperplane of V_g at $g(u)u$ are parallel (intersect in a (n-2)-plane contained in the hyperplane at infinity).*

Let $K \subset P^n$ be such that $\pi(K) \subset E^n \subset P^n$ is a strictly convex body for some projective isomorphism $\pi : P^n \to P^n$. Let $O \in P^n - \partial K$ and let $H \subset P^n$ be a hyperplane. We say that *O is a pole of K with polar hyperplane H* if $\pi(O)$ is a pole of $\pi(K)$ with polar hyperplane $\pi(H)$.

Let $C_\Omega \subset P^n$ be a convex cone with apex the point $\Omega \in P^n$. Let H be a hyperplane and L a line of P^n through Ω. We say that *L is a pole of C_Ω with polar hyperplane H* if given a hyperplane Γ that does not pass through Ω, the point $L\cap\Gamma$ is a pole of $C_\Omega \cap \Gamma$ with polar hyperplane $H \cap \Gamma$. Note that our definition does not depend on the choice of Γ. Note also that if we identify the set of lines of P^n through Ω with P^{n-1} and also C_Ω and H with the set of lines through Ω contained in C_Ω and H, respectively, our definitions of poles and polars of C_Ω coincide.

Theorem 2.3. ***Characterization of Poles and Polars.*** *Let $K \subset E^n$ be a convex body, $O \in P^n - \partial K$ and let $H \subset P^n$ be a hyperplane. The point O is a pole of K with polar hyperplane H if and only if for every $\Omega \in H - K$ the line through Ω and O is a pole of $C_\Omega(K)$ with polar hyperplane H.*

Proof. Suppose first O is a pole of K with polar H. Let $\Omega \in H - K$ and let Δ be a hyperplane that doesn't contain Ω. We shall prove that $\pi_\Omega(\Delta, O)$ is a pole of $\pi_\Omega(\Delta, K)$ with polar hyperplane $\Delta \cap H$. Let $\ell^{'} \subset \Delta$ be any line through $O^{'} = \pi_\Omega(\Delta, O)$ and suppose that $\ell^{'} \cap \partial\pi_\Omega(\Delta, K) = \{A^{'}, B^{'}\}$, and $\ell^{'} \cap H = Q^{'}$. We must prove that $[A^{'}, B^{'}; O^{'}, Q^{'}] = -1$. Let $A, B \in \partial K$ be such that $\pi_\Omega(\Delta, A) = A^{'}$ and $\pi_\Omega(\Delta, B) = B^{'}$ and let Π be the plane generated by Ω and $\ell^{'}$. Hence, $\{\Omega, Q^{'}, O^{'}, A^{'}, B^{'}, A, B, O\} \subset \Pi$, furthermore, O is a pole of $K\cap\Pi$ with corresponding polar the line $\Pi\cap H$ that passes through Ω and $Q^{'}$. Since the supporting lines of $K \cap \Pi$ at A and B meet at Ω, then by (2), A and B are O-antipodes and therefore there is a line $\ell \subset \Pi$ such that $\{A, B, O\} \subset \ell$. Therefore, $[A, B; O, \ell\cap H] = -1$. Consequently, since $\pi_\Omega(\ell^{'}, A) = A^{'}, \pi_\Omega(\ell^{'}, B) = B', \pi_\Omega(\ell^{'}, O) = O^{'}$, and $\pi_\Omega(\ell^{'}, \ell\cap H) = Q^{'}$, then $[A^{'}, B^{'}; O^{'}, Q^{'}] = -1$.

For the converse, let ℓ be a line through O and suppose $\ell\cap\partial K = \{A, B\}$. We shall prove that $[A, B; O, \ell \cap H] = -1$. Let H_A and H_B be the supporting hyperplanes of K at A and B, respectively and let $\Omega \in H_A \cap H_B \cap H \neq \phi$. Furthermore, let Γ be a hyperplane through ℓ that does not contain Ω. Then, O is a pole of $\pi_\Omega(\Gamma, K)$ with polar hyperplane $H \cap \Gamma$. Since $A, B \in \partial\pi_\Omega(\Gamma, K)$, then $[A, B; O, \ell \cap H] = -1$. This concludes the proof of the theorem. ∎

Remark 2.1. *The above theorem is the dual version of the following trivial fact: The point O is a pole of K with polar hyperplane H if and only if for every hyperplane Γ through O that intersects the interior of K, the point O is a pole of $\Gamma \cap K$ with polar hyperplane $\Gamma \cap H$.*

3. Polarity and Characterization of Ellipsoids

A classic characterization of ellipsoids [5;16.13] states that *if K is a convex body with the property that for every direction d, the middle point of all chords of K, parallel to d, lies in a hyperplane of P^n, then K is an ellipsoid.*

From our point of view, this well known characterization can be restated as follows:

Theorem 3.1. *Let $K \subset E^n$ be a convex body and let H be a hyperplane of P^n such that $H \cap K = \phi$. If every point of H is a pole of K, then K is an ellipsoid.*

Essentially the same ideas as in the classic proof of the above theorem may be used to prove the following sort of dual theorem:

Theorem 3.2. *Let $K \subset E^n$ be a convex body and let $P \in intK$ be a point. If every hyperplane through P is a polar hyperplane of K, then K is an ellipsoid.*

Our next purpose is to prove an analogous characterization of ellipsoids for hyperplanes H that intersect the interior of K. Related results may be found in [13] and Theorem 2 of [2].

Theorem 3.3. *Let $K \subset E^n$ be a convex body and let H be a hyperplane of P^n such that $H\cap$ $intK \neq \phi$. Then, each one of the following conditions implies that K is an ellipsoid:*
a) Every point of $H\cap$ $intK$ is a pole of K.
b) Every point of $H - K$ is a pole of K.

Proof. By Remark 2.1 and Burton's Theorem [6], it is enough to prove the theorem for n=2.
Proof of a). Let $K \subset E^2$ be a compact convex figure and let L be a line with the property that $L\cap$ int$K \neq \phi$ and for every point $O \in L\cap$ int$K$, O is a pole of K. Let $\{A, B\} = L \cap \partial K$ and L_A, L_B the supporting lines of K through A and B, respectively. Let $\{P\} = L_A \cap L_B \in P^n$.

We shall start proving that the polars of the poles in $L\cap$ int$K$ all pass through P. This is so because if $O \in L\cap$ int$K$, then A, B are O-antipodal points and hence, by (2), $L_A \cap L_B = \{P\}$ is contained in the polar of the pole O.

Next, note that P must be a pole of K with corresponding polar L, because for every line ℓ that intersects int$K$ and is such that $\ell \cap \partial K = \{X, Y\}$, we have that $[X, Y; \ell \cap L, P] = -1$. Let $\ell \cap L = \{O\}$, L_X and L_Y the supporting lines of K through X and Y, respectively, and L_O the polar of the pole O. Then, by (2), since P is a pole of K with polar L and since X and Y are P-antipodal points, we have that $L_X \cap L_Y \subset L$. Furthermore, O is a pole of K with polar L_O and since X and Y are also O-antipodal points, we have that $L_X \cap L_Y = L \cap L_O$.

Let now $\mathcal{E}$ be an ellipse with the property that $\{A, B\} \subset \partial\mathcal{E}$ and the supporting lines of $\mathcal{E}$ through A and B are, respectively, L_A and L_B. Note first that the polar of $\mathcal{E}$ with respect to the pole P is L. Furthermore, note that $O \in L\cap$ int$K$ is a pole of $\mathcal{E}$ with polar L_O, because $[A, B; O, L \cap L_O] = -1$. Thus, if $\{X_1, Y_1\} = \ell \cap \partial\mathcal{E}$ and L_{X_1}, L_{Y_1} are the supporting lines of $\mathcal{E}$ through X_1 and Y_1, respectively, then, $L_{X_1} \cap L_{Y_1} = L \cap L_O = L_X \cap L_Y$. Consequently, by Lemma 2.2, sending L to the line at infinity, we have that K is a projective image of the ellipse $\mathcal{E}$, which implies that K is an ellipse.
Proof of b). Let $K \subset E^2$ be a compact convex figure and let L be a line with the property that $L\cap$ int$K \neq \phi$ and for every point $O \in L - K$, O is a pole of K. We shall follow the proof of a). Let $\{A, B\} = L \cap \partial K$ and L_A, L_B the supporting lines of K through A and B, respectively. Let $\{P\} = L_A \cap L_B \in P^n$.

We shall start proving that the polars of the poles in $L - K$ all pass through P. This is so because if $O \in L - K$, then A, B are O-antipodal points and hence, by (2), $L_A \cap L_B = \{P\}$ is contained in the polar of the pole O.

Let ℓ be the polar of K corresponding to the pole O, $\{Z\} = L\cap\ell$, $\ell\cap\partial K = \{X, Y\}$ and L_X, L_Y the supporting lines of K through X and Y, respectively. Hence $P \in \ell$ and, by (1), $L_X \cap L_Y = \{O\}$.

Let now $\mathcal{E}$ be an ellipse with the property that $\{A, B\} \subset \partial\mathcal{E}$ and the supporting lines of $\mathcal{E}$ through A and B are, respectively, L_A and L_B. The point O is a pole of $\mathcal{E}$, furthermore, by (2), the polar of $\mathcal{E}$ with respect the pole O pass through

P and since $[A, B; O, Z] = -1$, the polar of $\mathcal{E}$ with respect the pole O is ℓ. Let $\{X_1, Y_1\} = \ell \cap \partial\mathcal{E}$ and let L_{X_1} , L_{Y_1} be the supporting lines of $\mathcal{E}$ through X_1 and Y_1, respectively. Hence, by (1), $L_{X_1} \cap L_{Y_1} = \{O\}$. Consequently, by Lemma 2.2, sending L to the line at infinity, we have that K is a projective image of the ellipse $\mathcal{E}$, which implies that K is an ellipse. This concludes the proof of the theorem. ∎

The following sort of dual theorem can be proved using the same ideas.

Theorem 3.4. *Let $K \subset E^n$ be a convex body and let $P \in P^n - K$ be a point. Then, each one of the following conditions implies that K is an ellipsoid:*
a) Every hyperplane through P that does not intersect K is a polar.
b) Every hyperplane through P that intersects the interior of K is a polar.

Our next purpose is to state a characterization in the spirit of Hobinger-Burton-Larman's Theorem [8]. The proof follows exactly the ideas in [2], but now using our Theorems 3.1 and 3.3. Before stating the theorem, we require some definitions.

A *slab* in E^n , in the direction $u \in S^{n-1}$, is a set of the form $\{x \in E^n \ / \ \delta \leq \langle x, u\rangle \leq \alpha\}$. The slab is said to be *degenerate* if $\delta = \alpha$ and in this case it is just a hyperplane. The slab is *transversal* to K if every hyperplane in the slab intersects the interior of K.

Theorem 3.5. *Let $K \subset E^n$ be a convex body, $n \geq 3$, and let $\sum_i$, $i = 1, ..., n-1$, be a collection of slabs transversal to K [slabs that do not intersect K, respectively] in linearly independent directions. Suppose that every hyperplane in these slabs is a polar hyperplane of K and at least one of the slabs is non-degenerated. Then K is an ellipsoid.*

4. k-Polarity and Shadow Boundaries

Let Γ be a k-plane of P^n and let Λ be a $(n-k-1)$-plane of P^n. $0 \leq k \leq n-1$. Suppose that $\Gamma \cap \Lambda = \phi$ and $\Gamma \cup \Lambda$ is not contained in a hyperplane of P^n. We say that Γ is a *polar k-plane of a convex body K with dual polar the $(n-k-1)$-plane* Λ if for every line L that meets Γ, Λ and int$K$, we have that

$$[A, B; P, Q] = -1,$$

where $L \cap \partial K = \{A, B\}$, $L \cap \Gamma = \{P\}$ and $L \cap \Lambda = \{Q\}$. If this is so, then for every $(k+1)$-plane Δ through Γ, $\Delta \cap \Lambda$ consists of a single point which is a pole of $\Delta \cap K$ with polar hyperplane Γ.

Note that if Γ is a *polar k-plane of a convex body K with dual polar the $(n-k-1)$-plane* Λ and $\Gamma \cap K = \phi$, then

$$\Lambda \cap \partial K = S\partial(K, \Gamma).$$

Let $K \subset P^n$ be such that $\pi(K) \subset E^n \subset P^n$ is a strictly convex body for some projective isomorphism $\pi : P^n \to P^n$. Let $\Gamma \subset P^n$ be a k-plane. We say that Γ *is a polar k-plane of K* if $\pi(\Gamma)$ is a polar k-plane of $\pi(K)$.

Let $C_\Omega \subset P^n$ be a convex cone with apex the point $\Omega \in P^n$. Let H be a k-plane of P^n through Ω. We say that H *is a polar* k-plane *of* C_Ω if there is a $(n-k)$-plane Λ through Ω such that $\Lambda \cap H = \{\Omega\}$ and with the property that given a hyperplane Γ, transversal to H, that does not pass through Ω, then $\Gamma \cap H$ is a polar $(k-1)$-plane of $C_\Omega \cap \Gamma = C_\Omega(\Gamma, K)$ with dual polar the $(n-k-1)$-plane $\Gamma \cap \Lambda$. Note that our definition does not depend on the choice of Γ.

Theorem 4.1. ***Characterization of Polar k-Planes.*** *Let $K \subset E^n$ be a convex body and let H be a k-plane of P^n. $1 \leq k \leq n-1$, $n \geq 3$. The k-plane H is a polar k-plane of K if and only if for every $\Omega \in H - K$, H is a polar k-plane of $C_\Omega(K)$.*

Proof. Suppose first H *is a polar* k-plane *of* K. Let $\Omega \in H - K$ be any point and let Γ be a hyperplane, transversal to H, that does not contain Ω. We shall prove that the $(k-1)$-plane $\Gamma \cap H$ is a polar $(k-1)$-plane of $\pi_\Omega(\Gamma, K)$. Let Λ be the dual $(n-k-1)$-polar plane of K to H. By definition, for any $(k+1)$-plane Π through H, we have that $\Pi \cap \Lambda$ consists of a single point P. Furthermore, if Π intersects int$K$, then P is a pole of $\Pi \cap K$ with polar H.

The projection $\pi_\Omega(\Gamma, \Pi)$ is a k-plane with the property that $\pi_\Omega(\Gamma, \Pi) \cap H = H \cap \Gamma$. Furthermore, $\pi_\Omega(\Gamma, \Pi \cap K) = \pi_\Omega(\Gamma, \Pi) \cap \pi_\Omega(\Gamma, K)$. Since P is a pole of $\Gamma \cap K$ with polar H and Ω is a point of $H - (K \cup \Gamma)$, then by Theorem 2.3, $\pi_\Omega(\Gamma, P)$ is a pole of $\pi_\Omega(\Gamma, \Pi \cap K) = \pi_\Omega(\Gamma, \Pi) \cap \pi_\Omega(\Gamma, K)$ with polar $\pi_\Omega(\Gamma, \Lambda) \cap H = \Gamma \cap H$. Thus, H is a polar k-plane of $C_\Omega(K)$ with dual the $(n-k)$-plane generated by Λ and Ω.

For the converse, let us prove first the case $k = n-1$. Let us take a hyperplane Γ transversal to H, that is $H \cap \Gamma$ is a $(n-2)$-plane. Then, for every $\Omega \in H - (K \cup \Gamma)$, $H \cap \Gamma$ is a polar hyperplane of $\pi_\Omega(\Gamma, K)$ with pole $c_\Omega \in \Gamma$. We shall prove that the lines L_Ω through Ω and c_Ω are concurrent in a point $Q \notin H \cup K$. For that purpose, it is easy to see that it is enough to show that for every line $L \subset H - K$, there is a plane Δ through L that contains $\bigcup_{\Omega \in L} L_\Omega$. To see this, let $\Omega_0 = L \cap \Gamma$ and let us consider Ξ the set of all supporting 2-planes of K through L. Then $\{\Pi \cap \Gamma \ / \ \Pi \in \Xi\} = \{\ell \subset \Gamma \ / \ \ell$ is a supporting line of $\pi_P(\Gamma, K)$ through $\Omega_0\}$, where P is any point of L. Thus, $C_{\Omega_0}(\pi_P(\Gamma, K))$ is independent of the point $P \in L$. By hypothesis $H \cap \Gamma$ is a polar hyperplane of $\pi_P(\Gamma, K)$ with pole c_P, hence by Theorem 2.3, $H \cap \Gamma$ is a polar hyperplane of the cone $C_{\Omega_0}(\pi_P(\Gamma, K))$ with pole the line through c_P and Ω_0. Therefore $\{c_P \ / \ P \in L\}$ is contained in a line that passes through Ω_0. Consequently, every line $L \subset H - K$, $\bigcup_{\Omega \in L} L_\Omega$ is contained in a 2-plane and therefore the lines L_Ω through Ω and c_Ω are concurrent in a point $Q \notin H \cup K$. Note that since for every $\Omega \in H - (K \cup \Gamma)$, $H \cap \Gamma$ is a polar hyperplane of $\pi_\Omega(\Gamma, K)$ with pole $c_\Omega = \pi_\Omega(\Gamma, Q)$, then by Theorem 2.3, H is a polar hyperplane of K with pole Q.

Suppose now $k < n-1$. For every $(k+1)$-plane Λ through H that intersects int$K$ and every $\Omega \in H - K$, H is a polar k-plane of $C_\Omega(K)$ but also of $C_\Omega(K \cap \Lambda)$. By the proof of this theorem for the case $k = n-1$, H is a polar hyperplane of $K \cap \Lambda$ with pole $c_\Lambda \in \Lambda$. The proof will be complete if we show that the set Θ of all points c_Λ, for all $(k+1)$-planes Λ through H that intersect int$K$, is contained in a $(n-k-1)$-plane, but this is so because $\pi_\Omega(\Gamma, \Theta)$ is contained in a $(n-k-1)$-plane, for every $\Omega \in H - K$. This finishes the proof of the theorem. ∎

Remark 4.1. *For Γ parallel to H and $k = n-1$, the hypothesis of Theorem 4.1 is the hypothesis of the classical Hobinger's Conjecture [8].*

The following characterizations of ellipsoids are generalizations of our previous results.

Theorem 4.2. *Let $K \subset E^{n+1}$ be a convex body and let $H \subset P^{n+1}$ be a non-supporting hyperplane of K. Let $0 \leq k \leq n-1$. If every k-plane $\Delta \subset H - K$ is a polar k-plane of K, then K is an ellipsoid.*

Proof. The proof of the theorem follows by induction on k. Theorem 3.4 b) takes care of the case $k = 0$. If $k > 0$, let $\Omega \in H - K$ be any point and let Γ be a hyperplane that does not contain Ω. By Theorem 4.1, every $(k-1)$-plane $L \subset (\Gamma \cap H) - \pi_\Omega(\Gamma, K)$ is a polar $(k-1)$-plane of $\pi_\Omega(\Gamma, K)$, which implies by induction that $\pi_\Omega(\Gamma, K)$ is an ellipsoid and also that the cone $C_\Omega(K)$ is an elliptic cone. The theorem follows now by the dual version of Burton's Theorem [6] that states that if there is a hyperplane H with the property that for every point $\Omega \in H - K$, the cone $C_\Omega(K)$ is an elliptic cone then K is an ellipsoid. ∎

As a consequence of the above results we have the following important characterization of ellipsoids in terms of planar shadow boundaries.

Theorem 4.3. *Let $K \subset E^n$ be a convex body and let $H \subset P^n$ be a non-supporting hyperplane of K, $n \geq 3$. Let $0 \leq k \leq n-3$. The body K is an ellipsoid if and only if for every k-plane $\Gamma \subset H - K$, the shadow boundary $S\partial(K, \Gamma)$ is contained in a $(n-k-1)$-plane.*

Proof. For every k-plane $\Gamma \subset H - K$, let $H(\Gamma)$ be the $(n-k-1)$-plane containing $S\partial(K, \Gamma)$. For every $(k+1)$-plane $L \subset H - K$, $S\partial(K, L) = S\partial(K, \Gamma_1) \cap S\partial(K, \Gamma_2)$, where Γ_1 and Γ_2 are two k-planes that generate L. Therefore $S\partial(K, L) = (H((\Gamma_1) \cap H(\Gamma_2)) \cap \partial K$, where $H((\Gamma_1) \cap H(\Gamma_2)$ is a linear plane of P^n whose dimension is $n-k-2$ because $S\partial(K, L)$ has dimension $n-k-3$. Thus, for every $(k+1)$-plane $L \subset H - K$ let $H(L)$ be the $(n-k-2)$-plane containing $S\partial(K, L)$. Note that $\Gamma \subset L$ if and only if $H(L) \subset H(\Gamma)$. Furthermore, if Λ is a $(n-k-1)$-plane through $H(L)$, then it is easy to see that there is a k-plane $\Gamma \subset L$ such that $H(\Gamma) = \Lambda$. We shall prove that every $(k+1)$-plane $L \subset H - K$ is a polar $(k+1)$-plane of K with dual polar $(n-k-2)$-plane $H(L)$. For that purpose, it will be enough to prove that if Δ is a $(k+2)$-plane through L that meets $\operatorname{int}(K)$, then $O = H(L) \cap \Delta$ is a pole of $\Delta \cap K$ with polar L. To see this last, it is enough, by Theorem 2.1, to prove that if A and B are O-antipodes in $\Delta \cap K$ and L_A, L_B are the supporting $(k+1)$-planes of $\Delta \cap K \subset \Delta$ at A and B, respectively, then $L_A \cap L_B \subset L$. This is so because if Λ is the $(n-k-1)$-plane through $H(L)$ containing A and B, then there is a k-plane $\Gamma \subset L$ such that $H(\Gamma) = \Lambda$. That is, $\partial K \cap \Lambda = S\partial(K, \Gamma)$ and therefore, L_A is the $(k+1)$-plane generated by Γ and A, L_B is the $(k+1)$-plane generated by Γ and B and then $L_A \cap L_B = \Gamma$. Consequently, by Theorem 4.1, K is an ellipsoid. ∎

Remark 4.2. *Theorem 4.2, when $H \cap K = \phi$ and $k = 0$, is a classic characterization of ellipsoids (see for example [16] and [5;16.14]). It is interesting to note that Theorem 4.2, for $k = 0$, concerning planar shadow boundaries, is easily proved to be equivalent to Theorem 4.1, when $k = 1$. Note also that Theorem 4.1, for $k = 0$, is equivalent to the classic characterization of ellipsoids concerning middle points of chords [5;16.13].*

Theorem 4.4. *Let $K \subset E^n$ be convex body and let $P \in P^n - \partial K$. Let $1 \leq k \leq n-1$. If every k-plane Γ through P that intersects the interior of K is a polar k-plane of K, then K is an ellipsoid.*

Proof. The proof is by induction on $n-1-k$. Theorem 3.4 b) takes care of the case $n-1-k = 0$. Suppose now that $k < n-1$. Let Δ be a hyperplane through P that intersects the interior of K. For every k-plane $\Gamma \subset \Delta$ through P that intersects the interior of K, we have that Γ is a polar k-plane of $\Delta \cap K$. By induction, $\Delta \cap K$ is an ellipsoid and hence, by Burton's Theorem [6], K is an ellipsoid.

Theorem 4.5. *Let $K \subset E^n$ be a convex body and let $P \in P^n - \partial K$, $n \geq 3$. Let $1 \leq k \leq n-2$. The body K is an ellipsoid if and only if for every $(k+1)$-plane Γ through P, there is a $(n-k-2)$-plane Δ such that $S\partial(K,\Delta) = \Gamma$.*

Proof. For every $(k+1)$-plane Γ through P, let $H(\Gamma)$ be the $(n-k-2)$-plane such that $S\partial(K, H(\Gamma)) = \Gamma \cap \partial K$. For every k-plane L through P, let Γ_1 and Γ_2 be two $(k+1)$-planes such that $\Gamma_1 \cap \Gamma_2 = L$ and let $H(L)$ be the linear plane generated by $H(\Gamma_1)$ and $H(\Gamma_2)$. Thus $S\partial(K, H(L)) = S\partial(K, H(\Gamma_1)) \cap S\partial(K, H(\Gamma_2))$. Therefore $S\partial(K, H(L)) = (\Gamma_1 \cap \Gamma_2) \cap \partial K = L \cap \partial K$. Note that $H(L)$ is a linear plane of P^n whose dimension is $(n-k-1)$ because $S\partial(K, H(L))$ has dimension $k-1$. Thus, for every k-plane L through P let $H(L)$ be the $(n-k-1)$-plane such that $S\partial(K, H(L)) = L$. It is easy to see now that $\{H(\Gamma) \ / \ \Gamma$ is a $(k+1)$-plane Γ through $L\} = \{\Delta \ / \ \Delta$ is a $(n-k-2)$-plane contained in $H(L)\}$. Exactly, as in the proof of Theorem 4.2, this implies that every k-plane L through P is a polar plane with dual polar plane the $(n-k-1)$-plane $H(L)$. Consequently, by Theorem 4.3, K is an ellipsoid. ∎

5. The False Plane of Symmetry; A Conjecture.

Let $K \subset E^n$ be a convex body and let $H \subset E^n$ be a hyperplane that intersects the interior of K. We say that H is a *hyperplane of symmetry* of K if H is a polar hyperplane of K whose pole is a point at infinity. Note that, if this is the case, then there is a direction with the property that the middle point of all chords of K in this direction lies in H.

Conjecture 5.1. *Let $K \subset E^n$ be a convex body and let $H \subset E^n$ be a hyperplane, $n \geq 3$. Suppose that for every hyperplane Γ that intersects the interior of K and is orthogonal to H, we have that $\Gamma \cap K$ has a $(n-2)$-hyperplane of symmetry parallel to H. Then, either K has a hyperplane of symmetry parallel to H, in the direction orthogonal to H or K is an ellipsoid.*

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