Shaken False Centre Theorem I

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Dedicated to Rolf Schneider

1 Introduction

Let K be a convex body and let p_0 be a point. Suppose that every section of K through p_0 is centrally symmetric, then Rogers proved in [7] that K is centrally symmetric, although p_0 may not be the centre of K. If this is the case, Aitchison, Petty and Rogers [1] and Larman [2] proved that K must be an ellipsoid. Suppose now that for every direction we can choose continuously a section of K that is centrally symmetric, if K is strictly convex, then Montejano [3] proved that K must by centrally symmetric. Consider now the following example: Let D be a solid sphere centered at the origin in which two symmetric caps are deleted. Then, D is centrally symmetric with respect the origin and has a lot of circular sections whose center is not the origin. In fact, we can choose continuously, for every direction, a section of D which is centrally symmetric in such a way that not all these sections pass through the origin. Nevertheless, no matter how we choose these sections, there is always many of them that necessarily pass through the origin. For those sections, of course, we have not impose really any condition which explain the fact that D is not a quadric elsewhere.

Let K be a convex body centrally symmetric and suppose that for every direction we can choose continuously a section of K which is centrally symmetric. The purpose of this paper is to prove that if in addition those sections through the centre are ellipses, then K is an ellipsoid.

2 Chairal Chords and Equichordal Theorems

In this section, let $K \subset \mathbb{R}^2$ be a convex figure with $0 \in \operatorname{int} K$. Let $\Sigma(K) = \operatorname{bd} K \cap \operatorname{bd} K$ be the symmetric part of the boundary of K.

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Let Δ be a subarc of $\mathbb{S}^1$ and let $\delta : \Delta \to \mathbb{R}$ be an even continuous map, that is, suppose that $\delta(v) = \delta(-v)$, whenever $\{v, -v\} \subset \Delta$. For every $v \in \Delta$, let us define the line $L_v = \{tv + \delta(v)v^{\perp} \mid t \in \mathbb{R}\}$ and if $L_v \cap K \neq \phi$, let $\tau_v = \max\{t \in \mathbb{R} \mid tv + \delta(v)v^{\perp} \in K\}$ and $\delta_v = \tau_v v + \delta(v)v^{\perp}$. So, $L_{\pm v} \cap K = [\delta_{-v}, \delta_v]$, whenever $\{v, -v\} \subset \Delta$

Definition 1 *We say that the chord $L_v \cap K$ is chairal if it has the same length of the chord $(-L_v) \cap K$.*

If this is the case, and $L_v \cap \operatorname{int} K \neq \phi$, let δ^*_v be the point of $(-L_v) \cap \operatorname{bd} K$ such that $\frac{\delta^*_v - \delta^*_{-v}}{|\delta^*_v - \delta^*_{-v}|} = v$. Hence $\delta_v - \delta_{-v} = \delta^*_v - \delta^*_{-v}$.

Lemma 1 *Suppose that $L_v \cap K$ is a nonempty chairal chord. Then $\delta_v \in \Sigma(K)$ if and only if $\delta_{-v} \in \Sigma(K)$.*

Proof. Since $\delta_v \in \Sigma(K)$, we have that $-\delta_v \in (-L_v) \cap \operatorname{bd} K$ and thus $-\delta_v = \delta^*_{-v}$. Furthermore, using the fact that $L_v \cap K$ is chairal, we have that $\delta_v - \delta_{-v} = \delta^*_v - \delta^*_{-v}$ and hence that $0 = \delta_v + \delta^*_{-v} = \delta^*_v + \delta_{-v}$. Consequently $-\delta_{-v} = \delta^*_v$, but $\delta^*_v \in \operatorname{bd} K$. ∎

For $0 < \epsilon < \pi$, let $S^1_\epsilon = \{(\cos\theta, \sin\theta) \mid -\epsilon \leq \theta \leq \epsilon\} \subset \mathbb{S}^1$ and $S^+_\epsilon = \{(\cos\theta, \sin\theta) \mid 0 \leq \theta \leq \epsilon\}$ and $S^-_\epsilon = \{(\cos\theta, \sin\theta) \mid \epsilon \leq \theta \leq 0\}$.

Lemma 2 *Let $K \subset \mathbb{R}^2$ be a convex figure with $0 \in \operatorname{int} K$ and such that $K \cap \{(t, 0) \mid t \in \mathbb{R}\} = [(-1, 0), (1, 0)]$. Let $\delta : S^+_\epsilon \to \mathbb{R}$ be a C^1 map,$0 < \epsilon < \pi$, with the property that the chord $L_v \cap K$ is chairal. Suppose that $\delta^{-1}(0) \cap S^+_\epsilon = \{(1, 0)\}$ and the limit of $\{L_{(1,0)} \cap L_v\}$, when $v \to (1, 0)$ exists and lies in the interior of K.Then there is $0 < \rho < \epsilon$such that $\{\delta_v \mid v \in S^+_\rho\} \subset \Sigma(K)$.*

Proof. Let $C^+_\epsilon = \{\delta_v \mid v \in S^+_\epsilon\} \subset \operatorname{bd} K$ and $C^-_\epsilon = \{\delta_{-v} \mid v \in S^1_\epsilon\} \subset \operatorname{bd} K$. By calculating the coordinates of $L_v \cap L_w$ in terms of δ (see [6]) we can see that: i) the limit of $\{L_w \cap L_v\}$, when $v \to w$ exist if and only if the derivative of δ at w exist and ii) The map $H : S^+_\epsilon \times S^+_\epsilon \to \mathbb{R}^2$ given by $H(w, v) = L_w \cap L_v$, if $v \neq w$ and $H(w, w) = \lim \{L_w \cap L_v \mid v \to w\}$ is continuous. Therefore, if the limit of $\{L_{(1,0)} \cap L_v\}$, when $v \to (0, 1)$, is in the interior of K, then we have that $\{L_v \cap L_w \mid v, w \in S^1_{\epsilon_0}\} \subset \operatorname{int} K$, for $0 < \epsilon_0 < \epsilon$. So, the map $\delta_{-v} \to \delta_v$ is a homeomorphism between $C^+_{\epsilon_0}$ and $C^-_{\epsilon_0}$. Furthermore, the map $v \to \delta_v$ is a homeomorphism between $S^1_{\epsilon_0}$ and $C^+_{\epsilon_0}$. Also, using the fact that $\delta^{-1}(0) \cap S^1_\epsilon = \{0\}$,we may assume, without loss of generality, that $\{L_{(1,0)} \cap L_v \mid v \in S^+_{\epsilon_0}\} \subset ((0, 0), (1, 0))$, otherwise change K by $-K$.

Let $\Psi : C^+_{\epsilon_0} \to C^+_{\epsilon_0}$ and $E : C^+_{\epsilon_0} \to \mathbb{R}^2$ be defined as follows. For every $\delta_v \in C^+_{\epsilon_0}$, let $E(\delta_v) = \delta_v + \delta^*_{-v}$. Furthermore, since $L_{(1,0)} \cap L_v \in ((0, 0), (1, 0))$,then $\delta^*_{-v} \in C^-_{\epsilon_0}$ and hence there is a unique $u \in S^1_{\epsilon_0}$, such that $\delta^*_{-v} = \delta_{-u}$. So define $\Psi(\delta_v) = \delta_u$.

Note that $\delta_v \in \Sigma(K)$ if and only if $-\delta_v = \delta^*_{-v}$ if and only if $E(\delta_v) = 0$. Note also that $-\delta^*_{-v} + E(\delta_v) \in$bd$K$. Finally, observe that $\delta_v \in \Sigma(K)$ if and only if $\Psi(\delta_v) \in \Sigma(K)$. We will prove that $-\delta^*_{-v} + E(\Psi(\delta_v)) \in \mathrm{bd}\, K$. In fact, since $\delta^*_{-v} = \delta_{-u}$ and $\delta_u - \delta_{-u} = \delta^*_u - \delta^*_{-u}$, we have that $E(\Psi(\delta_v) = E(\delta_u) = \delta_u + \delta^*_{-u} = \delta^*_u + \delta^*_{-v}$.

Since $L_{(1,0)} \cap L_v \in ((0,0),(1,0))$ and $K \cap \{(t,0) \mid t \in \mathbb{R}\} = [(-1,0),(1,0)]$,then δ^*_{-v} lies in the relative interior of subarc of $C^-_{\epsilon_0} \subset \mathrm{bd}\, K$ with extreme points δ_{-v} and $(-1,0)$. This implies that $\delta_u = \Psi(\delta_v)$ lies in the relative interior of subarc of $C^+_{\epsilon_0} \subset \mathrm{bd}\, K$ with extreme points δ_v and $(1,0)$. Consequently, if $\Psi^n(\delta_v) = \Psi(\Psi^{n-1}(\delta_v))$, then the sequence $\{\Psi^n(\delta_v)\}$ converges to a point $\delta_w \in C^+_{\epsilon_0}$ with the property that $\Psi(\delta_w) = \delta_w$, so $\{\Psi^n(\delta_v)\}$ converges to $(1,0)$.

Observe that for a chord of K, the property of being chairal is invariant under linear isomorphisms, so we may assume, without loss of generality, that $\{(1,t) \mid t \in \mathbb{R}\}$ is a support line of K at $(1,0)$. Let $0 < \rho < \epsilon_0$ be so small that every point of $C^+_\rho \subset \mathrm{bd}\, K$ has negative slope. From the fact that

$-\delta^*_{-v} + E(\delta_v)$, and $-\delta^*_{-v} + E(\Psi(\delta_v))$

it follows that for $v \in S^1_\rho$, $\mid \Pi(E(\delta_v)) \mid \geq \mid \Pi(E(\Psi(\delta_v))) \mid$, where $\Pi : \mathbb{R}^2 \to \mathbb{R}$ is the orthogonal projection onto the first factor. So, having in mind that $\{\mid \Pi(E(\Psi(\delta_v))) \mid\}$ converges to $\mid E((1,0)) \mid = 0$, we have that $E(\delta_v) = 0$ for every $v \in S^+_\rho$ and hence that $C^+_\rho \subset \Sigma(K)$. ■

Theorem 1 *Let $K \subset \mathbb{R}^2$ be a convex figure such that $0 \in \mathrm{int}\, K$. Let $\delta : \mathbb{S}^1 \to \mathbb{R}$ be a even continuous map with the property that δ is C^1 in a neighborhood of $\delta^{-1}(0)$ and for every $w \in \delta^{-1}(0)$, $\mathrm{cl}\{L_w \cap L_v \mid v \in \mathbb{S}^1\} \subset \mathrm{int}\, K$. Suppose that for every $v \in \mathbb{S}^1$, either $\nu \in \delta^{-1}(0)$ and 0 is the midpoint of the chord $K \cap L_v$ or $\nu \notin \delta^{-1}(0)$ and the chord $K \cap L_v$ is chairal. Then K is centrally symmetric with respect to the origin.*

Proof. For every point $P \in \mathrm{bd}\, K$, let $P^* \in \mathrm{bd}\, K$ be such that the origin lies in the interval with extreme points P and P^*. Note that $P \in \Sigma(K)$ if and only if $P^* \in \Sigma(K)$. Let now $v, u \in \mathbb{S}^1$ and let C be the subarc of $\mathbb{S}^1$ between v and u that does not contain $-v$ and $-u$. Suppose that $\delta^{-1}(0) \cap C = \{u, v\}$.Without loss of generality, we may assume that there is $0 < \epsilon < \pi$ such that $v = (1,0), u = (\cos\epsilon, \sin\epsilon)$ and $C = S^+_\epsilon$. Moreover, assume that for every $w \in S^+_\epsilon$, $L_{(1,0)} \cap L_w \in [(0,0),(1,0)]$, otherwise change u by $-u$ and v by $-v$. The fact that the derivative of the function δ at the point $(1,0)$ exists implies that the limit of $\{L_{(1,0)} \cap L_v\}$, when $v \to (1,0)$ exists and clearly lies in the interior of K. By Lemma 2, there is $0 < \rho < \epsilon$, such that $C^+_\rho = \{\delta_v \mid v \in S^+_\rho\} \subset \Sigma(K)$.

Let $\Omega = \{(x,y) = r(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid r \in \mathbb{R},\ 0 \leq \theta \leq \epsilon\}$. Note that since $\delta^{-1}(0) \cap S^+_\epsilon = \{(1,0), (\cos\epsilon, \sin\epsilon)\}$, then for every $Q \in \Omega$ there is $w \in S^+_\epsilon$ such that Q lies in L_w. We will prove that $\mathrm{bd}\, K \cap \Omega \subset \Sigma(K)$. For that purpose let $P_0 = \delta_{w_0} \in \mathrm{bd}\, K \cap \Omega \cap \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$, with $w_0 \in S^+_\epsilon$. Since $P^*_0 \in \Omega$, then there is $w_1 \in S^+_\epsilon$ such that $P^*_0 \in L_{w_1}$ and hence $P^*_0 = \delta_{-w_1}$. Define $P_1 = \delta_{w_1}$. Observe that $P_0 \in \Sigma(K)$ if and only if $P_1 \in \Sigma(K)$.

Since $L_{w_1} \cap L_{(1,0)} \in [(0,0),(1,0)]$, we have that P_1 lies in the subarc of $C_\epsilon^+ \subset \mathrm{bd}K$ between P_0 and $(1,0)$. Analogously, define inductively from $P_i = \delta_{w_i} \in \mathrm{bd}\, K \cap \Omega \cap \{(x,y) \in R^2 \mid y \geq 0\}$, with $w_i \in S_\epsilon^+$, the point $P_{i+1} = \delta_{w_{i+1}}$, with $w_{i+1} \in S_\epsilon^+$ in such a way that P_{i+1} lies in the subarc of $C_\epsilon^+ \subset \mathrm{bd}K$ between P_i and $(1,0)$ and $P_i \in \Sigma(K)$ if and only if $P_{i+1} \in \Sigma(K)$.

The sequence of points $\{P_i\}$ converges to $P \in C_\epsilon^+$. We will prove that $P = (1,0)$. Let $\{w_{i_j}\}$ be a subsequence of $\{w_i\}$ that converge to $w \in S_\epsilon^+$. So, the sequence of lines $\{L_{w_{i_j}}\}$ converge to the line L_w that contains P. By construction, the distance between the line L_{w_i+1} and the line that passes through P_i and P_i^* tends to cero as i tends to infinity, thus the origin lies in L_w. This implies that $w = 0$ and hence that $P = (1,0)$.

By the above, for i sufficiently large, $P_i \in C_\rho^+ \subset \Sigma(K)$,and therefore $P_0 \in \Sigma(K)$.This implies that $\mathrm{bd}\, K \cap \Omega \subset \Sigma(K)$. In order to prove the theorem, let $P \in \mathrm{bd}\, K$. If $P \in \delta^{-1}(0)$, then $P \in \Sigma(K)$. If $P \notin \delta^{-1}(0)$, then there are $v, u \in S^1$ such that if C is the subarc of S^1 between v and u that does not contain $-v$ and $-u$, then $\delta^{-1}(0) \cap C = \{u, v\}$ and P is in the relative interior of the cone generated by C, but in this case again $P \in \Sigma(K)$. This concludes the proof of the theorem. ■

3 Shaken False Centre Theorem

The purpose of this section is to prove our main theorem but first we need some definitions.

Let L be a line through the origin. Define $T\partial(K,L)$ as the union of all tangent lines of K parallel to L. The *shadow boundary* of K in the direction of L is defined as $S\partial(K,L) = T\partial(K,L) \cap K$. If $v \neq 0$, denote by $S\partial(K,v) = S\partial(K,L)$ and $T\partial(K,v) = T\partial(K,L)$, where L is the line through the interval $[0,v]$. Blaschke's classic characterization of ellipsoids states that K is an ellipsoid if and only if every shadow boundary of K is planar.

Remember that the empty section is a centrally symmetric section.

Theorem 2 *Let $K \subset \mathbb{R}^3$ be a convex body centrally symmetric with respect the origin. Let $\delta : \mathbb{S}^2 \to \mathbb{R}$ be an even continuous map which is C^1 at a neighborhood of $\delta^{-1}(0)$. Suppose that for every $v \in \mathbb{S}^2$, either $\nu \in \delta^{-1}(0)$ and $K \cap H_v$ is an ellipse or $\nu \notin \delta^{-1}(0)$ and the section $K \cap H_v$ is centrally symmetric. Then K is an ellipsoid.*

Proof. For the proof, we shall use a variation of Blaschke's characterization of ellipsoids (Proposition 2 of [5]) which states that if for every plane H through the origin there is a line L_H such that $\mathrm{bd}(H \cap K) \subset S\partial(K, L_H)$, then K is an ellipsoid.

For every $v \in \mathbb{S}^2$ let c_v be the centre of the section $K \cap H_v$. We start proving that for every $v \notin \delta^{-1}(0)$, $\mathrm{bd}(v^\perp \cap K) \subset S\partial(K, c_v)$, where $v^\perp$ is the plane through the origin orthogonal to v. For this purpose, it will be enough to prove that, for $t > 0$ sufficiently small, the section $(v^\perp + tc_v) \cap K$ is centrally symmetric with respect tc_v, because, if this is the case, for $t > 0$ sufficiently small, we have that $(v^\perp + tc_v) \cap K - 2tc_v = (v^\perp - tc_v) \cap K$, which implies that $\mathrm{bd}(v^\perp \cap K) + \{\lambda c_v \mid \lambda \in \mathbb{R}\} = T\partial(K, v_H)$ and hence that $\mathrm{bd}(v^\perp \cap K) \subset S\partial(K, c_v)$.

In order to prove that $(v^\perp + tc_v) \cap K$ is centrally symmetric with respect tc_v we shall use our Shaken Equichordal Theorem 1. For every $w \in (c_v)^\perp \cap \mathbb{S}^2$, let $L_w = H_w \cap (v^\perp + tc_v)$. Note that the pedal function of the system of lines $\{L_w \mid w \in (c_v)^\perp \cap \mathbb{S}^2\}$ in the plane $(v^\perp + tc_v)$, with respect to the point tc_v, is basically the restriction of δ to $(c_v)^\perp \cap \mathbb{S}^2$. Define, for every $w \in (c_v)^\perp \cap \mathbb{S}^2$, the chord $I_w = L_w \cap K$ of the convex figure $(v^\perp + tc_v) \cap K$. Consequently, in order to apply our Shaken Equichordal Theorem 1, we need to prove the following facts. For $t > 0$, sufficiently small and:

i) for every $w_0 \in (c_v)^\perp \cap \mathbb{S}^2$ such that $c_v \in L_{w_0}$, we have that c_v is the midpoint of I_{w_0},

ii) for every $w_0 \in (c_v)^\perp \cap \mathbb{S}^2$ such that $c_v \in L_{w_0}$, we have that $\mathrm{cl}\{L_{w_0} \cap L_w \mid w \in (c_v)^\perp \cap S^2\} \subset \mathrm{relint}((v^\perp + tc_v) \cap K)$, and

iii) for every $w \in (c_v)^\perp \cap \mathbb{S}^2$, let ℓ_w be the line of the plane $(v^\perp + tc_v)$ symmetric to the line L_w with respect the point c_v and let $J_w = \ell_w \cap K$ be the corresponding chord of the convex figure $(v^\perp + tc_v) \cap K$. Then, the length of the chords I_w and J_w coincide.

Proof of i). If $w_0 \in (c_v)^\perp \cap \mathbb{S}^2$ is such that $c_v \in L_{w_0}$, then $0 \in H_{w_0}$ and hence $K \cap H_{w_0}$ is an ellipse with centre at 0. Furthermore, $H_{w_0} \cap H_v \cap K$ is a chord of $H_v \cap K$ through the centre c_v and hence c_v is the midpoint of the chord $H_{w_0} \cap H_v \cap K$ of the ellipse $H_{w_0} \cap K$. So, the line $\{tc_v \mid t \in R\}$ is a diametral line of the ellipse $H_{w_0} \cap K$, but hence tc_v is the midpoint of the chord I_{w_0},because $H_{w_0} \cap H_v \cap K$ and $I_{w_0} = L_{w_0} \cap K$ are parallel chords of the ellipse $K \cap H_{w_0}$.

Proof of ii). If $w_0 \in (c_v)^\perp \cap \mathbb{S}^2$ is such that $c_v \in L_{w_0}$, then $0 \in H_{w_0}$ and hence $K \cap H_{w_0}$ is an ellipse with centre at 0. If $w \in (c_v)^\perp \cap \mathbb{S}^2$, then $H_{w_0} \cap H_w$ is a line parallel to the diametral line $\{tc_v \mid t \in \mathbb{R}\}$ of the ellipse $H_{w_0} \cap K$. By hypothesis, $H_{w_0} \cap H_w$ intersects the interior of K, So $\mathrm{cl}\{H_{w_0} \cap H_w \cap v^\perp \mid w \in (c_v)^\perp \cap S^2\} \subset \mathrm{relint}(v^\perp \cap K)$. So, for $t > 0$, sufficiently small, $\mathrm{cl}\{L_{w_0} \cap L_w \mid w \in (c_v)^\perp \cap S^2\} \subset \mathrm{relint}((v^\perp + tc_v) \cap K)$.

Proof of iii). For every $w \in (c_v)^\perp \cap \mathbb{S}^2$, $M_w = H_v \cap H_w \cap K$ is a chord of $H_v \cap K$ and a chord of $H_w \cap K$. Furthermore $M_w^* = H_v \cap (-H_w) \cap K$ is a chord of $H_v \cap K$ which is symmetric to M_w with respect the centre v_H and consequently both chords have the same length. Moreover, $-M_w^* = (-H_v) \cap H_w \cap K$ is a chord of $H_w \cap K$ with the same length that M_w. So, one of the following two situations hold:

a) the centre of $H_w \cap K$ lies in $v^\perp$, or

b) for $-1 \leq \lambda \leq 1$, the length of the chords $H_w \cap (v^\perp + \lambda c_v) \cap K$ is the length of the chord M_w.

In the first case, since $H_w \cap K$ is centrally symmetric with centre at the plane $v^\perp$, we have that the length of I_w coincide with the length of the chord $H_w \cap (v^\perp - tc_v) \cap K$ and by symmetry of K that the length of I_w coincide with the length of the chord $(-H_w) \cap (v^\perp + tc_v) \cap K$ which is precisely J_w, for $0 \leq t \leq 1$. Similarly, in the second case, the length of the chords I_w and J_w also coincide.

Using all the above and the Shaken Equichordal Theorem 1.1, we have that, for $t > 0$ sufficiently small, the section $(v^\perp + tc_v) \cap K$ is centrally symmetric with respect tc_v and consequently that for every $v \notin \delta^{-1}(0)$, $\mathrm{bd}(v^\perp \cap K) \subset S\partial(K, c_v)$.

Let now $v \in \mathrm{int}\delta^{-1}(0)$. We will prove next that $\mathrm{bd}(v^\perp \cap K) = S\partial(K, w)$, for some $w \in \mathbb{S}^2$.This follows inmedately from the following well known fact: Let K be a convex body centrally symmetric with respect the origin. Let U be an open set of S^2 with the property that $u \in S^2$ if and only if $-u \in \mathbb{S}^2$, and let $u_0 \in U$. Suppose that for every $v \in \mathbb{S}^2$, the section $v^\perp \cap K$ is an ellipse. Then there is an open neighborhood V of $u_0^\perp \cap \mathrm{bd}\, K$ in $\mathrm{bd}\, K$ which is an ellipsoid and consecuently $\mathrm{bd}(u_0^\perp \cap K) = S\partial(K, w)$, for some $w \in \mathbb{S}^2$.

With this we have proven that there is a dense, open subset Ω of $\mathbb{S}^2$ with the property that for every $v \in \Omega$,$\mathrm{bd}(v^\perp \cap K) = S\partial(K, w)$, for some $w \in \mathbb{S}^2$. Let $\{v_i\} \subset \Omega$ such that $v_i \to v$. Then, $\lim_{i\to\infty} \mathrm{bd}(v_i^\perp \cap K) = \mathrm{bd}(v^\perp \cap K)$. Suppose $\mathrm{bd}(v_i^\perp \cap K) = S\partial(K, w_i)$, with $w_i \in \mathbb{S}^2$ and without loss of generality suppose that $w_i \to w$. Then $\lim_{i\to\infty} S\partial(K, w_i) \subset S\partial(K, w)$. So, $\mathrm{bd}(v^\perp \cap K) = S\partial(K, w)$. Therefore, we may assume $\Omega = \mathbb{S}^2$ and therefore by Proposition 2 of [5] we have that K is an ellipsoid. ■

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