TOPOLOGICAL TOMOGRAPHY IN CONVEXITY

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Abstract. The main theorem of this paper generalizes a classic Aumann's characterization of compact convex sets, via the acyclicity of their hypersections, to arbitrary weakly closed subsets of a locally convex linear space.

1. Introduction

Throughout this paper we will consider homology and cohomology with compact supports and coefficients in a field. We shall say that a subset X, of locally convex linear space, is *homologically acyclic* if for every $\lambda \geq 0$, $H_\lambda(X) = 0$. Similarly, X is *cohomologically acyclic* if for every $\lambda \geq 0$, $H^\lambda(X) = 0$. In our case, for fields as coefficients, both notions coincide [2], so we shall just say that X is *acyclic*. Furthermore, we shall say that a subset X, of locally convex linear space, is *tomographically acyclic* if all its sections by hyperplanes are acyclic.

After this definition we can formulate Aumann's Theorem [1] .

Aumann's Theorem. *Every tomographically acyclic compact subset of a finite dimensional linear space is convex.*

The main result of the paper is the following theorem.

Closure Theorem. *Let E be a locally convex linear space with the weak topology. Then the closure of every acyclic and tomographically acyclic subset of E is convex.*

To understand that the noncompact case, in our result, is very specific, let us consider Kosinski's Theorem [4], which is a *local version* of Aumann's theorem.

Kosinski's Theorem. *Let C be a compact subset of a finite dimensional linear space E. If a point $x \in E$ belongs to the convex hull of C and does not belong to C, then there is a hyperplane passing through x whose intersection with C is not acyclic.*

The following example shows that Kosinski's theorem is not valid for arbitrary closed sets.

Example. Let H^+ be a half-space of R^n with boundary H, a hyperplane through the origin, let S be the unit hemisphere in R^n given by $S = H^+ \cap S^{n-1}$ and let p be a point on the boundary of S, that is, $p \in H \cap S^{n-1} \subset S$. Note that for every hyperplane Γ of R^n through the origin O, the section $\Gamma \cap (S - \{p\})$ is acyclic. Let us consider now a projective isomorphism π that sends only the point p to infinity and let $C = \pi(S - \{p\}) \subset R^n$. Hence, C is a closed subset of R^n. Furthermore, $\pi(O) \in R^n$ is a point that belongs to the convex hull of C and does not belong to C, but the intersection of every hyperplane of R^n through $\pi(O)$ with C is acyclic.

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Our next theorem is the following generalization. We say that *a map* $f : X \to R^n$ *is tomographically acyclic* if the preimage of every hyperplane is acyclic.

Mapping Closure Theorem. *The closure of the image of a tomographically acyclic map is convex.*

Finally, as a corollary of our techniques, it is easy to derive the following:

Open Set Theorem. *Tomographically acyclic open subsets of Euclidean space are convex.*

For the infinite dimensional case, this theorem is true when the set is a regular open set (i.e. coincides with the interior of its closure in the strong topology and has the same closure in weak and strong topologies). Another form of this result is the following: the weak closure of every tomographically acyclic strongly open set is convex. This form presents a strengthening of the Closure Theorem because the condition of global acyclicity of the set is omitted. In fact, we conjecture that the Closure Theorem is true without the global acyclicity hypothesis.

It is impossible to characterize closed sets only by support sections as it is done in [3]. For example the complement to an open ball is a subset which does not have supporting hyperplanes.

2. The Proof of the Theorems.

The ϵ-neighborhood, $O_\epsilon H$, of a plane H, will be called a *thick plane*.

Thick Plane Intersection Lemma. *If a set S is acyclic in dimension k and tomographically acyclic, then the intersection of S with $O_\epsilon H$ is acyclic in dimension k for any hyperplane H.*

Proof. Here we will work with cohomology. Let H be a hyperplane, and let $H^\pm$ be two half-spaces with $H = H^+ \cap H^-$ and $R^n = H^+ \cup H^-$. Since the intersection $S \cap H = (S \cap H^+) \cap (S \cap H^-)$ is acyclic, using the cohomological Mayer-Vietoris exact sequence for the decomposition $S = (S \cap H^+) \cup (S \cap H^-)$, the isomorphism $H^*(S) \cong H^*(S \cap H^+) \oplus H^*(S \cap H^-)$ follows. Furthermore, from the fact that $H^*(S) = 0$, one concludes that $H^*(S \cap H^\pm) = 0$. Therefore, the intersection of S with any closed half-space is cohomologically acyclic and hence, homologically acyclic. Since every open half-space is the direct limit of closed subspaces, this implies that intersections of S with open subspaces are homologically acyclic. For every hyperplane H, its open ϵ-neighborhood is the intersection of two open half-spaces. As the intersections of S with both half-spaces is acyclic, the same consideration, but now using the homological Mayer-Vietoris exact sequence, leads to the conclusion that the intersection of S with $O_\epsilon H$ is acyclic, for every H. ∎

Cycle Decomposition Lemma. *Let $X = U \cup V$, where U and V are open. Then, every cycle $z \in H_*(U \cap V)$ which is trivial in X can be presented as the sum $z = z_1 + z_2$, where z_1 bounds in U and z_2 bounds in V.*

Proof. Let us consider the Mayer-Vietoris exact sequence for pair U, V modulo the intersection $U \cap V$.

$$\to H_k(U \cap V, U \cap V) \to H_k(U, U \cap V) \oplus H_k(V, U \cap V) \to H_k(X, U \cap V) \to \\ H_{k-1}(U \cap V, U \cap V) \to$$

Since $H_k(U\cap V, U\cap V)=0$, for all k, one obtain isomorphisms $H_k(U, U\cap V)\oplus H_k(V, U\cap V)=H_k(X, U\cap V)$.

Now, suppose a cycle $z\in H_k(U\cap V)$ is the boundary of a chain z' in X. The chain z' represents a cycle in $H_{k+1}(X, U\cap V)$, but the above isomorphism implies that $z'=z'_1-z'_2$, where $z'_1\in H_{k+1}(U, U\cap V)$ and $z'_2\in H_{k+1}(V, U\cap V)$. In this case, $z=\partial z'=\partial z'_1+\partial z'_2$ is the desired decomposition of z. ∎

Intersection Bounding Lemma. *Let $X=U\cup V$, where U and V are open. If $H_{k+1}X=0$, then every cycle $z\in H_k(U\cap V)$ which bounds in U as well as in V is trivial.*

Proof. By virtue of the condition $H_{k+1}X=0$, from the Mayer-Vietoris sequence one obtains that the homomorphism $H_k(U\cap V)\to H_k(U)\oplus H_k(V)$ is a monomorphism. This immediately implies our conclusion. ∎

Lemma on Banach Spaces. *Let $H\supset L$ be two planes in a Banach space such that $\dim H=\dim L+1$. Let H^+ and H^- be the closed half-planes on which L divides H. Then for every ϵ one has the equality $O_\epsilon H^+\cap O_\epsilon H^-=O_\epsilon L$.*

Proof. If $x\in O_\epsilon H^+\cap O_\epsilon H^-$, then there are points $y^+\in H^+$ and $y^-\in H^-$ such that $\|x-y^+\|\le\epsilon$ and $\|x-y^-\|\le\epsilon$. The segment $[y^-, y^+]$ intersects L in a point $z=ty^++(1-t)y^-$ for some $t\in[0,1]$. For this point we have that $\|z-x\|=\|ty^++(1-t)y^--(tx-(1-t)x)\|\le\|t(y^+-x)\|+\|(1-t)(y^--x)\|\le t\epsilon+(1-t)\epsilon=\epsilon$. So, $x\in O_\epsilon L$. ∎

The following lemma is a crucial one. It works for any exact homology theory with compact supports. In particular, it works for the Steenrod-Sitnikov homology with any coefficients.

Rotational Lemma. *Let S be a subset of a finite dimensional Banach space, let L be a codimension ≥ 2 plane and let ϵ be a positive number. If for every plane H containing L, such that $\dim H=\dim L+1$, the inclusion induced homomorphism $H_k(O_\epsilon(L)\cap S)\to H_k(O_\epsilon(H)\cap S)$ is trivial and $H_{k+1}(\ O_\epsilon(H)\cap S)=0$, then $H_k(O_\epsilon(L)\cap S)=0$.*

Proof. A plane which contains L and has dimension equal to $\dim L+1$ will be called an L-plane. Let us fix some orientation α on L. Any L-plane H is divided by L into two half-planes. For every orientation β of H, let us denote by H^β the half-plane of H which induces on L the orientation α.

Let H be an oriented L-plane with an orientation β. A cycle $z\in H_k(O_\epsilon(L)\cap S)$ will be called H^β-bounded if it bounds in $O_\epsilon(H^\beta)\cap S$. If this is so, the plane H with the orientation β will be called a z-bounding oriented plane. Since the set of all oriented L-planes is connected, it is sufficient to prove that the set consisting of all z-bounding oriented planes is open and closed.

It is easy to see that the set of all z-bounding oriented planes is open. Indeed, if z bounds in $O_\epsilon(H^\beta)$, then there is a compact set $C\subset O_\epsilon(H^\beta)$ in which z bounds, but C is also contained in $O_\epsilon(H_1^{\beta'})$, if H_1 is a L-plane with the orientation β' sufficiently close to the plane H with the orientation β.

Let us prove that the set of z-bounding oriented planes is closed. Suppose z is $H_i^{\beta_i}$-bounded for some sequence of planes H_i with orientation β_i convergent to a plane H with orientation β. Since z bounds in $O_\epsilon(H)\cap S$, bearing in mind that $O_\epsilon(H) = O_\epsilon(H^\beta) \cup O_\epsilon(H^{-\beta})$ and $O(_\epsilon L) = O_\epsilon\beta) \cap O(_\epsilon H^{-\beta})$, from the Cycle Decomposition Lemma, one obtains that $z = z_1 + z_2$, where z_1 bounds in $O_\epsilon(H^\beta) \cap S$ and z_2 bounds in $O_\epsilon(H^{-\beta}) \cap S$. By the openness of the z_i-bounding oriented planes, one obtains that the planes H_i with orientation β_i are z_1-bounding and z_2-bounding for all but finitely many i's. For every such i, one has the following situation: z_1 bounds in $O_\epsilon(H^{\beta_i})$, z_2 bounds in $O_\epsilon(H^{-\beta_i})$ and $z_1 + z_2$ bounds in $O_\epsilon(H^{\beta_i})$. Hence, z_2 also bounds in $O_\epsilon(H^{\beta_i})$. In this case, from the Intersection Bounding Lemma, one gets that z_2 bounds in $O_\epsilon(L) \cap S$. Therefore, z_2 bounds in $O_\epsilon(H^\beta)$ and $z = z_1 + z_2$ also bounds in it. The closedness is proved.

Now let us consider any cycle $z \in H_k(O_\epsilon(L) \cap S)$. Let H be an L-plane with an orientation β. By the Cycle Decomposition Lemma, $z = z_1 + z_2$, where z_1 bounds in $O_\epsilon(H^\beta)$ and z_2 bounds in $O_\epsilon(H^{-\beta})$. But z_1 has to bound in $O_\epsilon(H^{-\beta})$ as well. Therefore, by the Intersection Bounding Lemma, z_1 is trivial in $O_\epsilon(L)$. The same is true for z_2. Therefore, z bounds in $O_\epsilon(L)$. ∎

Functional Rotational Lemma. *Let S be a subset of a finite dimensional Banach space. Let $f : S \to B$ be a continuous mapping into a Banach space B, let L be a codimension ≥ 2 plane of B and let ϵ be a positive number. If for every plane H containing L such that* $\dim H = \dim L + 1$, *the inclusion induced homomorphism $H_k(f^{-1}(O_\epsilon L)) \to H_k(f^{-1}(O_\epsilon H))$ is trivial and $H_{k+1}(f^{-1}(O_\epsilon(H))) = 0$, then $H_k(f^{-1}(O_\epsilon L)) = 0$.*

Proof. To get the proof of the lemma it is sufficient to follow the proof of the Rotation Lemma changing all intersections with S to preimages under f^{-1}. ∎

Lemma on the Convexity of the Closure. *Let S be a subset of a topological linear space E. If for every pair of points $x, y \in S$ the segment $[x, y]$ belongs to the closure of S, then the closure of S is convex.*

Proof. Let x, y be a pair of points in the closure of S and let z be any point of the segment $[x, y]$. Then $z = \lambda x + \mu y$, where λ and μ are nonnegative and $\lambda + \mu = 1$. Let U be any neighborhood of z. The mapping $p : E \times E \to E$ given by formula $p(u, v) = \lambda u + \mu v$ is continuous. Therefore, there are neighborhoods Ox, Oy of x and y, respectively, such that $p(Ox \times Oy) \subset U$. As the intersections of Ox and Oy with S are nonempty, choose $x' \in Ox \cap S$ and $y' \in Oy \cap S$. Then $z' = \lambda x' + \mu y' \in U$. But z' is known to belong to the closure of S. Hence S intersects U which implies that z belongs to this closure too. ∎

Finite Dimensional Theorem. *Let B be a finite dimensional Banach space, and let S be an acyclic and tomographically acyclic subset. Then the closure of S is convex.*

Proof. First, by virtue of the Lemma on Thick Intersections, one can conclude that the intersection of S with any thick hyperplane is acyclic. By the Rotational Lemma, one obtains, by induction on the codimension of the planes, that the intersection of S with any thick plane of any dimension is acyclic. In particular, one obtains that the intersection of S with any thick line is acyclic and hence connected. Let x, y be two different points of S. Let us prove that the segment $[x, y]$ is contained in the closure of S. If not, there is an open ball U that intersects the segment but does not intersect S. In this case, x and y lie in different components of the intersection of S with a thick line, which is a contradiction. Hence the segment $[x, y]$ belongs to the closure of S, for any $x, y \in S$. If x, y belong to the closure of S, then there are two sequences $\{x_i\}$, $\{y_i\}$ such that $x_i, y_i \in S$, for all i, and $\lim x_i = x$, $\lim y_i = y$. In this case, all segments $[x_i, y_i]$ belongs to the closure of S, but the closure of these segments contain $[x, y]$. Therefore, $[x, y]$ belongs to the closure of S too. ∎

The same proof, applying instead of the Rotational Lemma the corresponding functional version, allow us to prove the following generalization.

Finite Dimensional Functional Theorem. *Let $f : S \to B$ be a mapping of an acyclic space into a finite-dimensional Banach space. If the preimage of any hyperplane under f is acyclic, then the closure of $f(S)$ is convex.*

Now we are ready to prove our main theorem.

Proof of the Closure Theorem. Let S be an acyclic and tomographically acyclic subset of a locally convex linear space E. Let $p : E \to B$ be a linear mapping onto finite dimensional Banach space B. In this case, the restriction of p to S satisfies all the conditions of the Functional Rotational Lemma. Therefore, the closure of the image of p is convex. To prove the convexity of the closure of S (in the weak topology) by the Lemma on the Convexity of the Closure, it is enough to prove that for every pair $x, y \in S$ the segment $[x, y]$ belongs to the closure of S. If not, there is a point $z \in [x, y]$ that does not belong to the weak closure of S. Hence there is a linear mapping $p : E \to B$ into a finite dimensional Banach space for which $p(z)$ does not belong to the closure of $p(S)$, but this contradicts our previous result. ∎

References

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