New phenomena on examples of orthogonal matrix polynomials satisfying differential equations

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Outline

Historical review

- The theory of orthogonal matrix polynomials (OMP) was introduced in [Krein, 1949].
- Systematically studied in the last 15 years.
- In [Durán, 1997] it is raised the problem of characterizing OMP satisfying *second order differential equations*.
- Scalar case: [Bochner, 1929] $\Rightarrow$ Hermite, Laguerre and Jacobi.
- New (non-trivial) matrix examples:
  [Durán and Grünbaum, 2004] and
  [Grünbaum, Pacharoni and Tirao, 2003].

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Richer behavior of the algebra of differential operators having a fixed family of OMP as eigenfunctions (usually non-commutative).

Background I

- Given a self adjoint positive definite matrix weight function W we define a skew symmetric bilinear form (inner product):

$$(P, Q) = \int_a^b P(t)\, W(t)\, Q^*(t)\, dt$$

- This leads to a family of OMP $\{P_n\}_{n\geq 0}$ with deg $P_n = n$, non–singular leading coefficient and $(P_n, P_m) = \Theta, n \neq m$.
- We say that a weight matrix W_1 reduces to scalar weights if there exists a nonsingular constant matrix T with $W_1 = TW_2T^*$ and W_2 diagonal.
- Examples considered here satisfy

$$\ell_2(P_n) \equiv P_n''(t)A_2(t) + P_n'(t)A_1(t) + P_n(t)A_0 = \Gamma_n P_n(t)$$

- ℓ_2 is symmetric if $(\ell_2(P), Q) = (P, \ell_2(Q))$.

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Background II

Generating examples:

- [Durán and Grünbaum, 2004]:

Symmetry Equations

$$A_2 W = W A_2^*$$

$$2(A_2 W)' = A_1 W + W A_1^*,$$

$$(A_2 W)'' - (A_1 W)' + A_0 W = W A_0^*$$

Boundary Conditions

$$\lim_{t \to x} A_2(t) W(t) = 0 = \lim_{t \to x} (A_1(t) W(t) - W(t) A_1^*(t)), \quad \text{for } x = a, b.$$

- Matrix spherical functions associated with $P_N(\mathbb{C}) = \mathrm{SU}(N+1)/\mathrm{U}(N)$. [Grünbaum, Pacharoni and Tirao, 2003].

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The family of examples

The Matrix Weight

$$W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^* t}, \ \alpha > -1, \ t > 0$$

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{C}, \ J = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Factorization

$W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^\alpha e^{-t} T(t) T^*(t)$, where

$$T'(t) = \frac{1}{2}\left(A + \frac{J}{t}\right) T(t), \quad \mathrm{ad}_A J = [A, J] = -A.$$

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Second order differential operators

First differential operator

$$\ell_{2,1} = D^2 tI + D^1[(\alpha+1)I + J + t(A - I)] + D^0[(J + \alpha I)A - J]$$

Second differential operator

Assuming

$$i(N-i)|\nu_{N-1}|^2 = (N-1)|\nu_i|^2 + (N-i-1)|\nu_i|^2|\nu_{N-1}|^2, \ i = 1, \cdots, N-2.$$

$\Rightarrow \ell_{2,2} = D^2 A_2 + D^1 A_1 + D^0 A_0$, where

$$A_2 = t(J - At),$$
$$A_1 = (1 + \alpha + J)J + Y - t(J + (\alpha+2)A + Y^* - \mathrm{ad}_A Y),$$
$$A_0 = \frac{N-1}{|\nu_{N-1}|^2}[J - (\alpha + J)A],$$

where $(Y)_{i,i-1} = \frac{i(N-i)}{\nu_i}$, $i = 2, \cdots, N$, and $(Y)_{i,j} = 0$ otherwise.

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The algebra of differential operators

The Weight Matrix for $N = 2$

$$W_{\alpha,a}(t) = t^{\alpha} e^{-t} \begin{pmatrix} t(1+|a|^2 t) & at \\ \bar{a}t & 1 \end{pmatrix}, \quad \alpha > -1, t > 0$$

The algebra of differential operators

Given a fixed family of OMP $(\mathcal{P}_n)_n$, we study the algebra over $\mathbb{C}$

$$\mathcal{D}(W_{\alpha,a}) = \left\{ \ell = \sum_{i=0}^{k} D^i A_i(t) : \ell(\mathcal{P}_n(t)) = \Gamma_n(\ell)\mathcal{P}_n(t), \; n \geq 0 \right\}$$

Why do we study this algebra?

There is a rich and deep connection between algebras of differential operators having a common set of eigenfunctions (not necessarily orthogonal polynomials) and certain algebraic-geometric object. See [Burchnall and Chaundy].

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Scalar case

If $\mathcal{L}$ is the corresponding second order differential operator of the classical orthogonal polynomials (Hermite, Laguerre or Jacobi):

$$H_n(t)'' - 2tH_n(t)' = -2nH_n(t),$$

$$tL_n^{\alpha}(t)'' + (\alpha + 1 - t)L_n^{\alpha}(t)' = -nL_n^{\alpha}(t), \quad \text{and}$$

$$t(1-t)P_n^{(\alpha,\beta)}(t)'' + (\alpha + 1 - (\alpha+\beta+2)t)P_n^{(\alpha,\beta)}(t)' =$$

$$-n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(t),$$

then

$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{L}^i,$$

where $c_i \in \mathbb{C}$ and $\mathcal{U}$ is an even order differential operator.

Then $\mathcal{D} = \mathbb{C}[\mathcal{L}]$

The algebra $\mathcal{D}(W_{\alpha,a})$

NEW linearly independent differential operators

order	0	1	2	3	4	5	6	7	8
dimension	1	0	2	2	2	2	2	2	2

Basis for the second order differential operators

$$\mathcal{L}_1 = D^2 tI + D^1 \begin{pmatrix} \alpha + 2 - t & at \\ 0 & \alpha + 1 - t \end{pmatrix} + D^0 \begin{pmatrix} -\frac{1+|a|^2}{|a|^2} & (1+\alpha)a \\ 0 & -\frac{1}{|a|^2} \end{pmatrix}$$

$$\mathcal{L}_2 = D^2 \begin{pmatrix} t & -2at^2 \\ 0 & -t \end{pmatrix} + D^1 \begin{pmatrix} \alpha + 2 + t & -\frac{(2+|a|^2(2\alpha+5))t}{\bar{a}} \\ \frac{2}{a} & -t - \alpha - 1 \end{pmatrix} + D^0 \begin{pmatrix} \frac{1+|a|^2}{|a|^2} & -\frac{(1+\alpha)(2+|a|^2)}{\bar{a}} \\ 0 & -\frac{1}{|a|^2} \end{pmatrix}$$

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Third order differential operators

Basis for the non-commutative third order differential operators

$$\mathcal{L}_3 = D^3 A_{3,1}(t) + D^2 A_{2,1}(t) + D^1 A_{1,1}(t) + A_{0,1}$$

$$\mathcal{L}_4 = D^3 A_{3,2}(t) + D^2 A_{2,2}(t) + D^1 A_{1,2}(t) + A_{0,2},$$

where the leading coefficients are given by:

$$A_{3,1}(t) = t\left(At - \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{ad}_A^n A^*\right) = \begin{pmatrix} -|a|^2t^2 & at^2(1+|a|^2t) \\ -\bar{a}t & |a|^2t^2 \end{pmatrix}$$

$$A_{3,2}(t) = t\left(At + \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{ad}_A^n A^*\right) = \begin{pmatrix} |a|^2t^2 & at^2(-1+|a|^2t) \\ \bar{a}t & -|a|^2t^2 \end{pmatrix}$$

The corresponding eigenvalues

Eigenvalues

$$\mathcal{L}_1 = -\frac{1}{|a|^2}\begin{pmatrix}\gamma_{n+1,a} & 0\\ 0 & \gamma_{n,a}\end{pmatrix},$$

$$\mathcal{L}_2 = \frac{1}{|a|^2}\begin{pmatrix}\gamma_{n+1,a} & 0\\ 0 & -\gamma_{n,a}\end{pmatrix},$$

$$\mathcal{L}_3 = \frac{1}{|a|^2}\begin{pmatrix}0 & a(1+\alpha+n)\gamma_{n,a}\gamma_{n+1,a}\\ \bar{a} & 0\end{pmatrix},$$

$$\mathcal{L}_4 = \frac{1}{|a|^2}\begin{pmatrix}0 & -a(1+\alpha+n)\gamma_{n,a}\gamma_{n+1,a}\\ \bar{a} & 0\end{pmatrix}.$$

where $\gamma_{n,a} = 1 + n|a|^2$.

A sample of relations I

Four quadratic relations

$$\mathcal{L}_1^2 = \mathcal{L}_2^2, \qquad \mathcal{L}_3^2 = -\mathcal{L}_4^2,$$
$$\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1, \quad \mathcal{L}_3\mathcal{L}_4 = -\mathcal{L}_4\mathcal{L}_3.$$

Four permutational relations

$$\mathcal{L}_1\mathcal{L}_3 - \mathcal{L}_2\mathcal{L}_4 = 0, \quad \mathcal{L}_2\mathcal{L}_3 - \mathcal{L}_1\mathcal{L}_4 = 0,$$
$$\mathcal{L}_3\mathcal{L}_2 + \mathcal{L}_4\mathcal{L}_1 = 0, \quad \mathcal{L}_3\mathcal{L}_1 + \mathcal{L}_4\mathcal{L}_2 = 0.$$

Four more quadratic relations

$$\mathcal{L}_3 = \mathcal{L}_1\mathcal{L}_4 - \mathcal{L}_4\mathcal{L}_1, \quad \mathcal{L}_4 = \mathcal{L}_1\mathcal{L}_3 - \mathcal{L}_3\mathcal{L}_1$$
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Cubic relations

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A sample of relations II

Small basis

Considering $\mathbb{C}\langle \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 \rangle$ and $\mathcal{L}_1 \prec \mathcal{L}_2 \prec \mathcal{L}_3 \prec \mathcal{L}_4$ where $\prec$ is the graded lex order and the two-side ideal generated by the 14 previous relations, we have that the following 9 relations form a small basis (minimal set of relations that generate the whole ideal):

$$\begin{array}{rl} \mathcal{L}_2^2 - \mathcal{L}_1^2, & \mathcal{L}_4^2 + \mathcal{L}_3^2, \\ \mathcal{L}_2\mathcal{L}_1 - \mathcal{L}_1\mathcal{L}_2, & \mathcal{L}_4\mathcal{L}_3 + \mathcal{L}_3\mathcal{L}_4, \\ \mathcal{L}_2\mathcal{L}_4 - \mathcal{L}_1\mathcal{L}_3, & \mathcal{L}_2\mathcal{L}_3 - \mathcal{L}_1\mathcal{L}_4, \\ \mathcal{L}_4\mathcal{L}_1 - \mathcal{L}_1\mathcal{L}_4 + \mathcal{L}_3, & \mathcal{L}_3\mathcal{L}_1 - \mathcal{L}_1\mathcal{L}_3 + \mathcal{L}_4, \checkmark \\ \mathcal{L}_3\mathcal{L}_2 + \mathcal{L}_1\mathcal{L}_4 - \mathcal{L}_3 & \end{array}$$

A sample of relations III

$\mathcal{L}_2$ in terms of $\mathcal{L}_1$ and $\mathcal{L}_3$

$$
\begin{aligned}
\left[|a|^2(2+\alpha)-1\right]\left[|a|^2(\alpha-1)-1\right]\mathcal{L}_2 &= 2|a|^2\left[|a|^2(2\alpha+1)-2\right]\mathcal{L}_1 \\
&+\left[|a|^4(\alpha^2+\alpha-5)-|a|^2(2\alpha+1)+1\right]\mathcal{L}_1^2 \\
&-2|a|^2\left[|a|^2(2\alpha+1)-2\right]\mathcal{L}_1^3+3|a|^4\mathcal{L}_1^4 \\
&-\frac{1}{2}\left[|a|^2(2\alpha+1)-2\right]\mathcal{L}_3^2+\frac{15}{2}|a|^2\mathcal{L}_3^2\mathcal{L}_1-\frac{9}{2}|a|^2\mathcal{L}_3\mathcal{L}_1\mathcal{L}_3
\end{aligned}
$$

Conjecture

$$\mathcal{D}(W_{\alpha,a})=\mathbb{C}\langle\mathcal{L}_1,\mathcal{L}_3\rangle$$

Note

For the exceptional values of $\alpha=1+\frac{1}{|a|^2}$ or $\alpha=-2+\frac{1}{|a|^2}$ ("cusps")

$\Rightarrow$ Conjecture: $\mathcal{D}(W_{\alpha,a})=\mathbb{C}\langle\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3\rangle$

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Closing remarks and future directions

- It is remarkable that weight matrices corresponding to $N = 3$ and 4 yield to fifth and seventh order (respectively) symmetric differential operators, but no third order.
- This is the first example of weight matrix which does not reduce to scalar weights having symmetric odd order differential operators.
- We hope to find theoretical proofs of these new phenomena and illustrate new ones in future publications.
- The matrix case is much richer than the scalar one.

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Possible Applications

Quantum mechanics

[Durán and Grünbaum, 2006] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006).

Random walks

[Grünbaum, 2007] *Random walks and orthogonal polynomials: some challenges*, arXiv: math.PR/0703375v1, (2007).

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