Properties of matrix orthogonal polynomials via their Riemann-Hilbert characterization[1]

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[1] joint work with F. A. Grünbaum and A. Martínez-Finkelshtein

Outline

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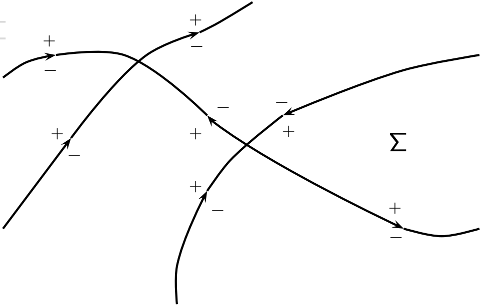
Riemann-Hilbert problem

Let $\Sigma \subset \mathbb{C}$ be an oriented contour and $\Sigma^0 = \Sigma \setminus \{\text{points of self-intersection of } \Sigma\}$.
Suppose that there exists a matrix-valued smooth map $\mathbf{G} : \Sigma^0 \to GL(m, \mathbb{C})$.
The Riemann-Hilbert problem (RHP) determined by a pair $(\Sigma, \mathbf{G})$ consists of finding an $m \times m$ matrix-valued function $\mathbf{Y}(z)$ s.t.

1. $\mathbf{Y}(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
2. $\mathbf{Y}_+(z) = \mathbf{Y}_-(z)\mathbf{G}(z)$ when $z \in \Sigma$
   $\mathbf{Y}_\pm(z) = \lim_{z' \to z, \pm\text{side}} \mathbf{Y}(z)$
3. $\mathbf{Y}(z) \to \mathbf{I}_m$ as $z \to \infty$



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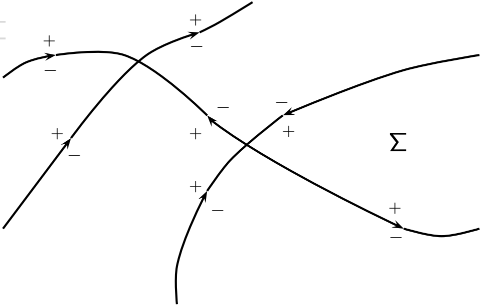

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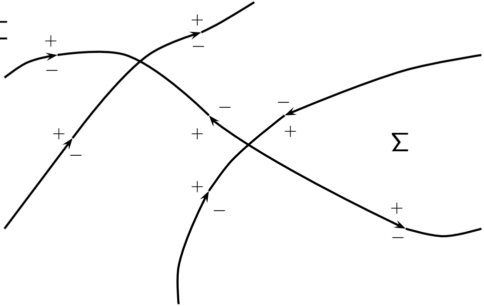

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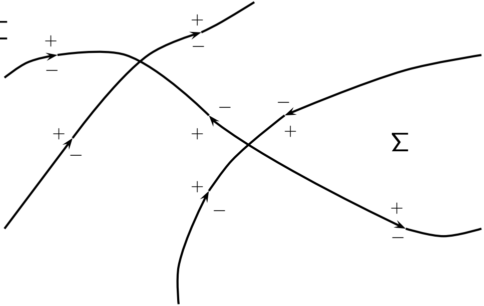
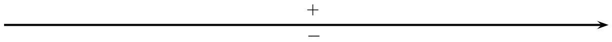


Scalar and additive RHP ($m = 1$)

Let $\omega : \mathbb{R} \to \mathbb{R}$ be a $L^1(\mathbb{R})$ and Hölder continuous function. The Riemann-Hilbert problem determined by $(\mathbb{R}, \omega)$ consists of finding a function $f : \mathbb{C} \to \mathbb{C}$ such that

1. $f(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $f_+(x) = f_-(x) + \omega(x)$ when $x \in \mathbb{R}$
3. $f(z) = \mathcal{O}(1/z)$ as $z \to \infty$



$\Sigma = \mathbb{R}$

The unique solution of this RHP is the Stieltjes or Cauchy transform of ω

$$f(z) = C(\omega)(z) \doteq \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(t)}{t - z} dt$$

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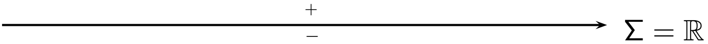

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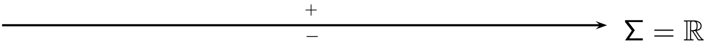

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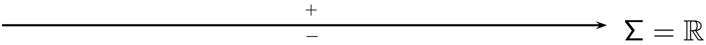

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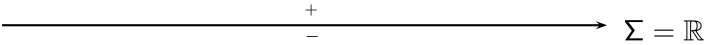

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Applications

- Integrable models. Inverse scattering of some nonlinear differential and difference equations like the nonlinear Schrödinger equation, the Korteweg-de Vries equation or the Toda equations.
- Orthogonal polynomials and random matrices. The distribution of eigenvalues of random matrices in several ensembles is reduced to computations involving orthogonal polynomials.
- Combinatorial probability. On the distribution of the length of the longest increasing subsequence of a random permutation.

Advantages for orthogonal polynomials

1. Algebraic properties: three term recurrence relation, ladder operators, second order differential equation
2. Uniform asymptotics: *steepest descent analysis for RHP* (Deift-Zhou,1993).

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Orthogonal polynomials

Let $d\mu$ be a positive Borel measure supported on $\mathbb{R}$.
We will assume $d\mu(x) = \omega(x)dx$, $\omega \geq 0$ and $x^i\omega, x^j\omega' \in L^1(\mathbb{R})$.
We can then construct a family of orthonormal polynomials $(p_n)_n$ s.t.

$$(p_n, p_m)_\omega = \int_{\mathbb{R}} p_n(x)p_m(x)\omega(x)dx = \delta_{n,m}, \quad n, m \geq 0$$

$$p_n(x) = \kappa_n(x^n + a_{n,n-1}x^{n-1} + \cdots) = \kappa_n \widehat{p}_n(x)$$

The monic polynomials $\widehat{p}_n(x)$ satisfy a three-term recurrence relation

$$x\widehat{p}_n(x) = \widehat{p}_{n+1}(x) + \alpha_n \widehat{p}_n(x) + \beta_n \widehat{p}_{n-1}(x)$$

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Solution of the RHP for orthogonal polynomials ($m = 2$)

We try to find a 2×2 matrix-valued function $\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2\times 2}$ such that

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_2 + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RHP above is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{p}_n(z) & C(\widehat{p}_n\omega)(z) \\ -2\pi i\gamma_{n-1}\widehat{p}_{n-1}(z) & -2\pi i\gamma_{n-1}C(\widehat{p}_{n-1}\omega)(z) \end{pmatrix}$$

where $C(f)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}dt$ and $\gamma_n = \kappa_n^2$ (Fokas-Its-Kitaev, 1990).
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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

$$\mathbf{Y}^{n+1}(z) = \underbrace{\begin{pmatrix} z-\alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i\gamma_n & 0 \end{pmatrix}}_{\mathbf{E}_n(z)} \mathbf{Y}^n(z)$$

$$\frac{d}{dz}\mathbf{Y}^n(z) = \underbrace{\begin{pmatrix} -\mathfrak{B}_n(z) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathfrak{A}_n(z) \\ 2\pi i\,\mathfrak{A}_{n-1}(z)\gamma_{n-1} & \mathfrak{B}_n(z) \end{pmatrix}}_{\mathbf{F}_n(z)} \mathbf{Y}^n(z)$$

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Compatibility conditions

Cross-differentiating the Lax pair yield

$$\mathbf{E}_n'(z) + \mathbf{E}_n(z)\mathbf{F}_n(z) = \mathbf{F}_{n+1}(z)\mathbf{E}_n(z)$$

also known as string equations. In our situation, the compatibility conditions entry-wise are

$$1 + (z - \alpha_n)(\mathfrak{B}_{n+1}(z) - \mathfrak{B}_n(z)) = \beta_{n+1}\mathfrak{A}_{n+1}(z) - \beta_n\mathfrak{A}_{n-1}(z)$$
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Transformation of the RHP

Consider the transformation

$$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

We observe that X^n is invertible and that

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That means that $\mathbf{X}^n$ has a constant jump

$\Rightarrow$ $\mathbf{E}_n(z)$ and $\mathbf{F}_n(z)$ are completely determined by their behavior at $z \to \infty$.

Transformation of the RHP

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Example: Hermite polynomials

The solution $\mathbf{Y}^n(z)$ of the RHP

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2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_2 + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{H}_n(z) & C(\widehat{H}_n e^{-t^2})(z) \\ -2\pi i \gamma_{n-1} \widehat{H}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\widehat{H}_{n-1} e^{-t^2})(z) \end{pmatrix}$$

where $(\widehat{H}_n)_n$ is the family of monic Hermite polynomials.

The Lax pair and compatibility conditions

$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} e^{-z^2/2} & 0 \\ 0 & e^{z^2/2} \end{pmatrix}$ satisfies the following Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z), \quad \frac{d}{dz}\mathbf{X}^n(z) = \begin{pmatrix} -z & -\frac{1}{\pi i}\gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & z \end{pmatrix} \mathbf{X}^n(z)$$

The difference equation gives (using $\beta_n = \gamma_n/\gamma_{n+1}$) the TTRR

$$x\widehat{H}_n(x) = \widehat{H}_{n+1}(x) + \alpha_n \widehat{H}_n(x) + \beta_n \widehat{H}_{n-1}(x),$$

while the differential equation gives the ladder operators

$$\widehat{H}_n'(x) = 2\beta_n \widehat{H}_{n-1}(x), \quad \widehat{H}_n'(x) - 2z\widehat{H}_n(x) = -2\widehat{H}_{n+1}(x).$$

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$$\alpha_n = 0, \quad \beta_{n+1} - \beta_n = \frac{1}{2} \Rightarrow \beta_n = \frac{n}{2}$$

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Outline

Matrix orthogonal polynomials

The theory of matrix orthogonal polynomials on the real line (MOP) was introduced by Krein in 1949.

A $N \times N$ matrix polynomial on the real line is

$$\mathbf{P}(x) = \mathbf{A}_n x^n + \mathbf{A}_{n-1} x^{n-1} + \cdots + \mathbf{A}_0, \quad x \in \mathbb{R} \quad \mathbf{A}_i \in \mathbb{C}^{N \times N}$$

Let $\mathbf{W}$ be a $N \times N$ a matrix of measures or weight matrix.
We will assume $d\mathbf{W}(x) = \mathbf{W}(x)dx$ and $\mathbf{W}$ smooth and positive definite on $\mathbb{R}$.
We can construct a family of MOP with respect to the inner product

$$(\mathbf{P}, \mathbf{Q})_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{P}(x)\mathbf{W}(x)\mathbf{Q}^*(x)dx \in \mathbb{C}^{N \times N}$$

such that

$$(\mathbf{P}_n, \mathbf{P}_m)_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{P}_n(x)\mathbf{W}(x)\mathbf{P}_m^*(x)dx = \delta_{n,m}\mathbf{I}_N, \quad n, m \geq 0$$

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Solution of the RHP for MOP ($m = 2N$)

$\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2N\times 2N}$ such that

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} \mathbf{I}_N & \mathbf{W}(x) \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_{2N} + \mathcal{O}(1/z)) \begin{pmatrix} z^n\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & z^{-n}\mathbf{I}_N \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RH problem above is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{\mathbf{P}}_n(z) & C(\widehat{\mathbf{P}}_n\mathbf{W})(z) \\ -2\pi i\gamma_{n-1}\widehat{\mathbf{P}}_{n-1}(z) & -2\pi i\gamma_{n-1}C(\widehat{\mathbf{P}}_{n-1}\mathbf{W})(z) \end{pmatrix}$$

where $C(\mathbf{F})(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mathbf{F}(t)}{t-z}dt$ and $\gamma_n = \boldsymbol{\kappa}_n^*\boldsymbol{\kappa}_n$.

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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

$$\mathbf{Y}^{n+1}(z)=\underbrace{\begin{pmatrix} z-\alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i\gamma_n & 0 \end{pmatrix}}_{\mathbf{E}_n(z)}\mathbf{Y}^n(z),$$

$$\frac{d}{dz}\mathbf{Y}^n(z)=\underbrace{\begin{pmatrix} -\mathfrak{B}_n(z) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathfrak{A}_n(z) \\ 2\pi i\,\mathfrak{A}_{n-1}(z)\gamma_{n-1} & \mathfrak{B}_n^*(z) \end{pmatrix}}_{\mathbf{F}_n(z)}\mathbf{Y}^n(z),$$

where

$$\mathfrak{A}_n(z)=-\gamma_n\int_{\mathbb{R}}\frac{\widehat{\mathbf{P}}_n(t)\mathbf{W}'(t)\widehat{\mathbf{P}}_n^*(t)}{t-z}dt,\ \mathfrak{B}_n(z)=-\left(\int_{\mathbb{R}}\frac{\widehat{\mathbf{P}}_n(t)\mathbf{W}'(t)\widehat{\mathbf{P}}_{n-1}^*(t)}{t-z}dt\right)\gamma_{n-1}$$

Compatibility conditions

Cross-differentiating the Lax pair yield

$$\mathbf{E}_n'(z) + \mathbf{E}_n(z)\mathbf{F}_n(z) = \mathbf{F}_{n+1}(z)\mathbf{E}_n(z)$$

also known as string equations. In our situation, the compatibility conditions entry-wise are

$$\mathbf{I}_N + \mathfrak{B}_{n+1}(z)(z\mathbf{I}_N - \alpha_n) - (z\mathbf{I}_N - \alpha_n)\mathfrak{B}_n(z) = \mathfrak{A}_{n+1}^*(z)\beta_{n+1} - \beta_n\mathfrak{A}_{n-1}^*(z)$$

$$\mathfrak{B}_{n+1}(z) + \gamma_n^{-1}\mathfrak{B}_n^*(z)\gamma_n = (z\mathbf{I}_N - \alpha_n)\mathfrak{A}_n^*(z)$$

Problem. Again, the coefficients $\mathfrak{A}_n(z)$ and $\mathfrak{B}_n(z)$ are difficult to obtain. We need to tranform the RHP in another RHP with constant jump.

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Transformation of the RHP

Goal: obtain an invertible transformation $\mathbf{Y}^n \to \mathbf{X}^n$ such that $\mathbf{X}^n$ has a constant jump across $\mathbb{R}$. Consider $\mathbf{X}^n(z) = \mathbf{Y}^n(z)\mathbf{V}(z)$ where

$$\mathbf{V}(z) = \begin{pmatrix} \mathbf{T}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-*}(z) \end{pmatrix}$$

where $\mathbf{T}$ is an invertible $N \times N$ smooth matrix function.

This motivates to consider a factorization of the weight in the form

$$\mathbf{W}(x) = \mathbf{T}(x)\mathbf{T}^*(x), \quad x \in \mathbb{R}.$$

This factorization is not unique since

$$\mathbf{T}(x) = \widehat{\mathbf{T}}(x)\mathbf{S}(x), \quad x \in \mathbb{R},$$

where $\widehat{\mathbf{T}}(x)$ is an upper triangular matrix and $\mathbf{S}(x)$ is an arbitrary smooth and unitary matrix.

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Transformation of the RHP II

We additionally assume

$$\mathbf{T}'(z) = \mathbf{G}(z)\mathbf{T}(z),$$

where **G** is a matrix polynomial of degree m (most of our examples)

Then

$$\frac{d}{dz}\mathbf{X}^n(z) = \mathbf{F}_n(z;\mathbf{G})\mathbf{X}^n(z),$$

$$\mathbf{F}_n(z;\mathbf{G}) = \begin{pmatrix} -\mathcal{B}_n(z;\mathbf{G}) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathcal{A}_n(z;\mathbf{G}) \\ 2\pi i\,\mathcal{A}_{n-1}(z;\mathbf{G})\gamma_{n-1} & \mathcal{B}_n^*(z;\mathbf{G}) \end{pmatrix}$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are matrix polynomials of degree $m-1$ and m respectively.

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Transformation of the RHP III

If there exists a non-trivial matrix-valued function $\mathbf{S}$, non-singular on $\mathbb{C}$, smooth and unitary on $\mathbb{R}$, s.t.

$$\mathbf{H}(z) = \mathbf{T}(z)\mathbf{S}'(z)\mathbf{S}^*(z)\mathbf{T}^{-1}(z)$$

is also a polynomial, then $\widetilde{\mathbf{T}} = \mathbf{T}\mathbf{S}$ satisfies

$$\mathbf{W}(x) = \widetilde{\mathbf{T}}(x)\widetilde{\mathbf{T}}^*(x), \quad x \in \mathbb{R}, \qquad \widetilde{\mathbf{T}}'(z) = \widetilde{\mathbf{G}}(z)\widetilde{\mathbf{T}}(z), \quad z \in \mathbb{C},$$

with $\widetilde{\mathbf{G}}(z) = \mathbf{G}(z) + \mathbf{H}(z)$ and the matrix $\mathbf{X}^n$ satisfies

$$\frac{d}{dz}\mathbf{X}^n(z) = \left(\mathbf{F}_n(z;\mathbf{G}) + \mathbf{F}_n(z;\mathbf{H})\right)\mathbf{X}^n(z) - \mathbf{X}^n(z)\begin{pmatrix} \chi(z) & \mathbf{0} \\ \mathbf{0} & -\chi^*(z) \end{pmatrix}$$

with $\chi(z) = \mathbf{S}'(z)\mathbf{S}^*(z)$.

Consequences: We have a class of ladder operators.

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$$\mathbf{H}(z) = \mathbf{T}(z)\mathbf{S}'(z)\mathbf{S}^*(z)\mathbf{T}^{-1}(z)$$

is also a polynomial, then $\widetilde{\mathbf{T}} = \mathbf{TS}$ satisfies

$$\mathbf{W}(x) = \widetilde{\mathbf{T}}(x)\widetilde{\mathbf{T}}^*(x), \quad x \in \mathbb{R}, \qquad \widetilde{\mathbf{T}}'(z) = \widetilde{\mathbf{G}}(z)\widetilde{\mathbf{T}}(z), \quad z \in \mathbb{C},$$

with $\widetilde{\mathbf{G}}(z) = \mathbf{G}(z) + \mathbf{H}(z)$ and the matrix $\mathbf{X}^n$ satisfies

$$\frac{d}{dz}\mathbf{X}^n(z) = \left(\mathbf{F}_n(z;\mathbf{G}) + \mathbf{F}_n(z;\mathbf{H})\right)\mathbf{X}^n(z) - \mathbf{X}^n(z)\begin{pmatrix} \chi(z) & \mathbf{0} \\ \mathbf{0} & -\chi^*(z) \end{pmatrix}$$

with $\chi(z) = \mathbf{S}'(z)\mathbf{S}^*(z)$.

Consequences: We have a class of ladder operators.

Example: Hermite type MOP

Let us consider $\mathbf{T}(x) = e^{-x^2/2} e^{\mathbf{A}x}$ and

$$\mathbf{W}(x) = e^{-x^2} e^{\mathbf{A}x} e^{\mathbf{A}^* x}, \quad \mathbf{A} \in \mathbb{C}^{N \times N}, \quad x \in \mathbb{R}.$$

Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z\mathbf{I}_N - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z)$$

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Compatibility conditions

$$\alpha_n = (\mathbf{A} + \gamma_n^{-1}\mathbf{A}^*\gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = \mathbf{A}\alpha_n - \alpha_n\mathbf{A} + \mathbf{I}_N$$

Example: Hermite type MOP

Let us consider $\mathbf{T}(x) = e^{-x^2/2} e^{\mathbf{A}x}$ and

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$$\widehat{\mathbf{P}}_n'(x) + \widehat{\mathbf{P}}_n(x)\mathbf{A} - \mathbf{A}\widehat{\mathbf{P}}_n(x) = 2\beta_n \widehat{\mathbf{P}}_{n-1}(x),$$

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Combining them we get a second order differential equation

Second order differential equation

$$\widehat{\mathbf{P}}_n''(x) + 2\widehat{\mathbf{P}}_n'(x)(\mathbf{A} - x\mathbf{I}_N) + \widehat{\mathbf{P}}_n(x)(\mathbf{A}^2 - 2x\mathbf{A})$$
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The matrix $\mathbf{H}$ can be written as

$$\mathbf{H}(x) = e^{\mathbf{A}x}\chi e^{-\mathbf{A}x} = \chi + \mathrm{ad}_{\mathbf{A}}(\chi)x + \mathrm{ad}_{\mathbf{A}}^{2}(\chi)\frac{x^2}{2} + \cdots,$$

where $\chi(x) = \mathbf{S}'(x)\mathbf{S}^*(x)$ is skew-Hermitian on $\mathbb{R}$.

This matrix equation was considered already by Durán-Grünbaum (2004), when χ is a constant matrix.

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First case $\mathbf{A} = \mathbf{L}$

New compatibility conditions

$$\mathbf{J}\alpha_n - \alpha_n\mathbf{J} + \alpha_n = \mathbf{L} + \frac{1}{2}(\mathbf{L}^2\alpha_n - \alpha_n\mathbf{L}^2), \quad \mathbf{J} - \gamma_n^{-1}\mathbf{J}\gamma_n = \mathbf{L}\alpha_n + \alpha_n\mathbf{L} - 2\alpha_n^2$$

New ladder operators (0-th order)

$$\widehat{\mathbf{P}}_n(x)\mathbf{J} - \mathbf{J}\widehat{\mathbf{P}}_n(x) - x(\widehat{\mathbf{P}}_n(x)\mathbf{L} - \mathbf{L}\widehat{\mathbf{P}}_n(x)) + 2\beta_n\widehat{\mathbf{P}}_n(x) - n\widehat{\mathbf{P}}_n(x) = 2(\mathbf{L} - \alpha_n)\beta_n\widehat{\mathbf{P}}_{n-1}(x)$$

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Sturm-Liouville type differential equation

Finally, something remarkable happens. Combining the second and the first order differential equation will give surprisingly

Sturm-Liouville type differential equation

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This is a second-order differential equation of Sturm-Liouville type satisfied by the MOP, already given by Durán-Grünbaum (2004)

Conclusions

1. The ladder operators method gives more insight about the differential properties of MOP and new phenomena
2. This method works for every weight matrix $\mathbf{W}$. The corresponding MOP satisfy differential equations, but not necessarily of Sturm-Liouville type

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