Properties of matrix orthogonal polynomials via their Riemann-Hilbert characterization

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Angers, April 17, 2012

\footnote{joint work with F. A. Grünbaum and A. Martínez-Finkelshtein}
Outline

1. What is a Riemann-Hilbert problem?
2. The Riemann-Hilbert problem for orthogonal polynomials
3. The Riemann-Hilbert problem for matrix orthogonal polynomials
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1. What is a Riemann-Hilbert problem?

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Riemann-Hilbert problem

Let $\Sigma \subset \mathbb{C}$ be an oriented contour and $\Sigma^0 = \Sigma \setminus \{\text{points of self-intersection of } \Sigma\}$. Suppose that there exists a matrix-valued smooth map $G : \Sigma^0 \to GL(m, \mathbb{C})$. The Riemann-Hilbert problem (RHP) determined by a pair $(\Sigma, G)$ consists of finding an $m \times m$ matrix-valued function $Y(z)$ s.t.

1. $Y(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
2. $Y_+(z) = Y_-(z)G(z)$ when $z \in \Sigma$
   
   $Y_\pm(z) = \lim_{z' \to z, \pm \text{side}} Y(z)$
3. $Y(z) \to I_m$ as $z \to \infty$
What is a RH problem?

The RH problem for OP

The RH problem for MOP

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Scalar and additive RHP \((m = 1)\)

Let \(\omega : \mathbb{R} \to \mathbb{R}\) be a \(L^1(\mathbb{R})\) and Hölder continuous function. The Riemann-Hilbert problem determined by \((\mathbb{R}, \omega)\) consists of finding a function \(f : \mathbb{C} \to \mathbb{C}\) such that

1. \(f(z)\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\)
2. \(f_+(x) = f_-(x) + \omega(x)\) when \(x \in \mathbb{R}\)
3. \(f(z) = \mathcal{O}(1/z)\) as \(z \to \infty\)

The unique solution of this RHP is the Stieltjes or Cauchy transform of \(\omega\)

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f(z) = C(\omega)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(t)}{t - z} dt\]
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Properties of MOP via their RH characterization
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\[\Sigma = \mathbb{R}\]

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\[f(z) = C(\omega)(z) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(t)}{t - z} dt\]
Applications

- **Integrable models.** Inverse scattering of some nonlinear differential and difference equations like the nonlinear Schrödinger equation, the Korteweg-de Vries equation or the Toda equations.

- **Orthogonal polynomials and random matrices.** The distribution of eigenvalues of random matrices in several ensembles is reduced to computations involving orthogonal polynomials.

- **Combinatorial probability.** On the distribution of the length of the longest increasing subsequence of a random permutation.

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Orthogonal polynomials

Let $d\mu$ be a positive Borel measure supported on $\mathbb{R}$. We will assume $d\mu(x) = \omega(x)dx$, $\omega \geq 0$ and $x^i\omega, x^j\omega' \in L^1(\mathbb{R})$.

We can then construct a family of orthonormal polynomials $(p_n)_n$ s.t.

$$(p_n, p_m)_{\omega} = \int_{\mathbb{R}} p_n(x)p_m(x)\omega(x)dx = \delta_{n,m}, \quad n, m \geq 0$$

$$p_n(x) = \kappa_n(x^n + a_{n,n-1}x^{n-1} + \cdots) = \kappa_n\hat{p}_n(x)$$

The monic polynomials $\hat{p}_n(x)$ satisfy a three-term recurrence relation

$$x\hat{p}_n(x) = \hat{p}_{n+1}(x) + \alpha_n\hat{p}_n(x) + \beta_n\hat{p}_{n-1}(x)$$
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Solution of the RHP for orthogonal polynomials \((m = 2)\)

We try to find a \(2 \times 2\) matrix-valued function \(Y^n : \mathbb{C} \to \mathbb{C}^{2 \times 2}\) such that

1. \(Y^n\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\)
2. \(Y^n_+(x) = Y^n_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}\) when \(x \in \mathbb{R}\)
3. \(Y^n(z) = (I_2 + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}\) as \(z \to \infty\)

For \(n \geq 1\) the unique solution of the RHP above is given by

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Y^n(z) = \begin{pmatrix} \hat{p}_n(z) & C(\hat{p}_n\omega)(z) \\ -2\pi i \gamma_{n-1} \hat{p}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\hat{p}_{n-1}\omega)(z) \end{pmatrix}
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where \(C(f)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} \, dt\) and \(\gamma_n = \kappa_n^2\) (Fokas-Its-Kitaev, 1990).

The existence and unicity is a consequence of the Morera’s theorem, Liouville’s theorem, the additive RHP and \(\det Y^n(z) = 1\).
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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

\[
Y^{n+1}(z) = \left( \begin{array}{cc}
    z - \alpha_n & \frac{1}{2\pi i} \gamma_n^{-1} \\
    -2\pi i \gamma_n & 0
\end{array} \right) Y^n(z)
\]

\[
\frac{d}{dz} Y^n(z) = \left( \begin{array}{cc}
    -\mathcal{B}_n(z) & -\frac{1}{2\pi i} \gamma_n^{-1} \mathcal{A}_n(z) \\
    2\pi i \mathcal{A}_{n-1}(z) \gamma_{n-1} & \mathcal{B}_n(z)
\end{array} \right) Y^n(z)
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where

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\mathcal{A}_n(z) = -\gamma_n \int_{\mathbb{R}} \frac{\hat{p}_n(t)\omega'(t)}{t-z} dt, \quad \mathcal{B}_n(z) = -\gamma_{n-1} \int_{\mathbb{R}} \frac{\hat{p}_n(t)\hat{p}_{n-1}(t)\omega'(t)}{t-z} dt
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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

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\]
Compatibility conditions

Cross-differentiating the Lax pair yield

\[ E'_n(z) + E_n(z)F_n(z) = F_{n+1}(z)E_n(z) \]

also known as string equations. In our situation, the compatibility conditions entry-wise are

\[ 1 + (z - \alpha_n)(\mathcal{B}_{n+1}(z) - \mathcal{B}_n(z)) = \beta_{n+1}\mathcal{A}_{n+1}(z) - \beta_n\mathcal{A}_{n-1}(z) \]
\[ \mathcal{B}_{n+1}(z) + \mathcal{B}_n(z) = (z - \alpha_n)\mathcal{A}_n(z) \]

Problem. Typically, the coefficients \( \mathcal{A}_n(z) \) and \( \mathcal{B}_n(z) \) are difficult to obtain. We can avoid that by transforming the RHP in another RHP with constant jump.
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Problem. Typically, the coefficients \( A_n(z) \) and \( \mathcal{B}_n(z) \) are difficult to obtain. We can avoid that by transforming the RHP in another RHP with constant jump.
Consider the transformation

$$X^n(z) = Y^n(z) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

We observe that $X^n$ is invertible and that

$$X^n_+(x) = Y^n_+(x) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} = Y^n_-(x) \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

$$= Y^n_-(x) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \begin{pmatrix} \omega^{-1/2} & 0 \\ 0 & \omega^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

$$= X^n_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

That means that $X^n$ has a constant jump

$$\Rightarrow E_n(z) \text{ and } F_n(z) \text{ are completely determined by their behavior at } z \to \infty.$$
Transformation of the RHP

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= Y^n_-(\chi) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & \omega^{1/2} \\ \omega^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \\
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Transformation of the RHP

Consider the transformation

\[ X^n(z) = Y^n(z) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \]

We observe that \( X^n \) is invertible and that

\[
X^n_+(x) = Y^n_+(x) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} = Y^n_-(x) \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \\
= Y^n_-(x) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \begin{pmatrix} \omega^{-1/2} & 0 \\ 0 & \omega^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \\
= X^n_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

That means that \( X^n \) has a constant jump

\[ \Rightarrow E_n(z) \text{ and } F_n(z) \text{ are completely determined by their behavior at } z \to \infty. \]
Example: Hermite polynomials

The solution $\mathbf{Y}^n(z)$ of the RHP

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$

2. $\mathbf{Y}_+(x) = \mathbf{Y}_-(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$

3. $\mathbf{Y}^n(z) = (\mathbf{I}_2 + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \hat{H}_n(z) & C(\hat{H}_n e^{-t^2})(z) \\ -2\pi i \gamma_n \hat{H}_{n-1}(z) & -2\pi i \gamma_n C(\hat{H}_{n-1} e^{-t^2})(z) \end{pmatrix}$$

where $(\hat{H}_n)_n$ is the family of monic Hermite polynomials.
The Lax pair and compatibility conditions

\[ \mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} e^{-z^2/2} & 0 \\ 0 & e^{z^2/2} \end{pmatrix} \]

satisfies the following Lax pair

\[ \mathbf{X}^{n+1}(z) = \begin{pmatrix} z - \alpha_n \\ -2\pi i \gamma_n \end{pmatrix} \begin{pmatrix} \mathbf{X}^n(z) \\ 0 \end{pmatrix}, \quad \frac{d}{dz} \mathbf{X}^n(z) = \begin{pmatrix} -z \\ 4\pi i \gamma_{n-1} \end{pmatrix} \begin{pmatrix} -\frac{1}{i} \gamma_n^{-1} \\ z \end{pmatrix} \mathbf{X}^n(z) \]

The difference equation gives (using \( \beta_n = \gamma_n / \gamma_{n+1} \)) the TTRR

\[ x \hat{H}_n(x) = \hat{H}_{n+1}(x) + \alpha_n \hat{H}_n(x) + \beta_n \hat{H}_{n-1}(x), \]

while the differential equation gives the ladder operators

\[ \hat{H}'_n(x) = 2\beta_n \hat{H}_{n-1}(x), \quad \hat{H}'_n(x) - 2z \hat{H}_n(x) = -2\hat{H}_{n+1}(x). \]

The compatibility conditions are

\[ \alpha_n = 0, \quad \beta_{n+1} - \beta_n = \frac{1}{2} \Rightarrow \beta_n = \frac{n}{2} \]
The Lax pair and compatibility conditions

\[ X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2} & 0 \\ 0 & e^{z^2/2} \end{pmatrix} \] satisfies the following Lax pair

\[ X^{n+1}(z) = \begin{pmatrix} z - \alpha_n & \frac{1}{2\pi i} \gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} X^n(z), \quad \frac{d}{dz} X^n(z) = \begin{pmatrix} -z & -\frac{1}{\pi i} \gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & z \end{pmatrix} X^n(z) \]

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The difference equation gives (using \( \beta_n = \gamma n / \gamma n+1 \)) the TTRR

\[ x \hat{H}_n(x) = \hat{H}_n(x+1) + \alpha_n \hat{H}_n(x) + \beta_n \hat{H}_n(x-1), \]

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Outline

1. What is a Riemann-Hilbert problem?

2. The Riemann-Hilbert problem for orthogonal polynomials

3. The Riemann-Hilbert problem for matrix orthogonal polynomials
What is a RH problem?
The RH problem for OP
The RH problem for MOP

Matrix orthogonal polynomials

The theory of matrix orthogonal polynomials on the real line (MOP) was introduced by Krein in 1949.

A \( N \times N \) matrix polynomial on the real line is

\[
P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0, \quad x \in \mathbb{R} \quad A_i \in \mathbb{C}^{N \times N}
\]

Let \( W \) be a \( N \times N \) a matrix of measures or weight matrix. We will assume \( dW(x) = W(x)dx \) and \( W \) smooth and positive definite on \( \mathbb{R} \).

We can construct a family of MOP with respect to the inner product

\[
(P, Q)_W = \int_{\mathbb{R}} P(x) W(x) Q^*(x) dx \in \mathbb{C}^{N \times N}
\]

such that

\[
(P_n, P_m)_W = \int_{\mathbb{R}} P_n(x) W(x) P_m^*(x) dx = \delta_{n,m} I_N, \quad n, m \geq 0
\]

\[
P_n(x) = \kappa_n (x^n + a_{n,n-1} x^{n-1} + \cdots) = \kappa_n \hat{P}_n(x)
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\]

\[
P_n(x) = \kappa_n(x^n + a_{n-1}x^{n-1} + \cdots) = \kappa_n \hat{P}_n(x)
\]
Solution of the RHP for MOP \((m = 2N)\)

\(Y^n : \mathbb{C} \to \mathbb{C}^{2N \times 2N}\) such that

1. \(Y^n\) is analytic in \(\mathbb{C} \setminus \mathbb{R}\)

2. \(Y_+(x) = Y_-(x) \begin{pmatrix} I_N & W(x) \\ 0 & I_N \end{pmatrix}\) when \(x \in \mathbb{R}\)

3. \(Y^n(z) = (I_{2N} + O(1/z)) \begin{pmatrix} z^n I_N & 0 \\ 0 & z^{-n} I_N \end{pmatrix}\) as \(z \to \infty\)

For \(n \geq 1\) the unique solution of the RH problem above is given by

\[
Y^n(z) = \begin{pmatrix} \hat{P}_n(z) & C(\hat{P}_nW)(z) \\ -2\pi i \gamma_{n-1} \hat{P}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\hat{P}_{n-1}W)(z) \end{pmatrix}
\]

where \(C(F)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(t)}{t-z} dt\) and \(\gamma_n = \kappa_n^* \kappa_n\).
Solution of the RHP for MOP \((m = 2N)\)

\[ Y^n : \mathbb{C} \to \mathbb{C}^{2N \times 2N} \text{ such that} \]

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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

$$
Y^{n+1}(z) = \left( \begin{array}{cc}
    z - \alpha_n & \frac{1}{2\pi i} \gamma_n^{-1} \\
    -2\pi i \gamma_n & 0
\end{array} \right) \underbrace{Y^n(z)}_{E_n(z)},
$$

$$
\frac{d}{dz} Y^n(z) = \left( \begin{array}{cc}
    -\mathcal{B}_n(z) & -\frac{1}{2\pi i} \gamma_n^{-1} \mathcal{A}_n(z) \\
    2\pi i \mathcal{A}_{n-1}(z) \gamma_{n-1} & -\mathcal{B}^*_n(z)
\end{array} \right) \underbrace{Y^n(z)}_{F_n(z)},
$$

where

$$
\mathcal{A}_n(z) = -\gamma_n \int_{\mathbb{R}} \frac{\hat{P}_n(t) W'(t) \hat{P}^*_n(t)}{t - z} dt, \quad \mathcal{B}_n(z) = -\left( \int_{\mathbb{R}} \frac{\hat{P}_n(t) W'(t) \hat{P}^*_{n-1}(t)}{t - z} dt \right) \gamma_{n-1}
$$
Compatibility conditions

Cross-differentiating the Lax pair yield

\[ E'_n(z) + E_n(z)F_n(z) = F_{n+1}(z)E_n(z) \]

also known as string equations. In our situation, the compatibility conditions entry-wise are

\[
\begin{align*}
I_N + \mathcal{B}_{n+1}(z)(zI_N - \alpha_n) - (zI_N - \alpha_n)\mathcal{B}_n(z) &= \mathcal{A}_{n+1}^*(z)\beta_{n+1} - \beta_n\mathcal{A}_{n-1}^*(z) \\
\mathcal{B}_{n+1}(z) + \gamma_{n}^{-1}\mathcal{B}_n^*(z)\gamma_n &= (zI_N - \alpha_n)\mathcal{A}_n^*(z)
\end{align*}
\]

Problem. Again, the coefficients \(\mathcal{A}_n(z)\) and \(\mathcal{B}_n(z)\) are difficult to obtain. We need to transform the RHP in another RHP with constant jump.
Compatibility conditions

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\[ \mathcal{B}_{n+1}(z) + \gamma_n^{-1}\mathcal{B}_n(z)\gamma_n = (zI_N - \alpha_n)\mathcal{A}_n^*(z) \]

**Problem.** Again, the coefficients \( \mathcal{A}_n(z) \) and \( \mathcal{B}_n(z) \) are difficult to obtain. We need to transform the RHP in another RHP with constant jump.
Transformation of the RHP

**Goal:** obtain an invertible transformation \( Y^n \to X^n \) such that \( X^n \) has a constant jump across \( \mathbb{R} \). Consider \( X^n(z) = Y^n(z)V(z) \) where

\[
V(z) = \begin{pmatrix} T(z) & 0 \\ 0 & T^{-*}(z) \end{pmatrix}
\]

where \( T \) is an invertible \( N \times N \) smooth matrix function.

This motivates to consider a factorization of the weight in the form

\[
W(x) = T(x)T^*(x), \quad x \in \mathbb{R}.
\]

This factorization is not unique since

\[
T(x) = \hat{T}(x)S(x), \quad x \in \mathbb{R},
\]

where \( \hat{T}(x) \) is an upper triangular matrix and \( S(x) \) is an arbitrary smooth and unitary matrix.
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Transformation of the RHP II

We additionally assume

\[ T'(z) = G(z)T(z), \]

where \( G \) is a matrix polynomial of degree \( m \) (most of our examples).

Then

\[
\frac{d}{dz}X^n(z) = F_n(z; G)X^n(z),
\]

\[
F_n(z; G) = \begin{pmatrix}
  -B_n(z; G) & -\frac{1}{2\pi i} \gamma_n^{-1} A_n(z; G) \\
  2\pi i A_{n-1}(z; G) \gamma_{n-1} & B^*_n(z; G)
\end{pmatrix}
\]

where \( A_n \) and \( B_n \) are matrix polynomials of degree \( m - 1 \) and \( m \) respectively.
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\end{pmatrix}
\]

where \( A_n \) and \( B_n \) are matrix polynomials of degree \( m - 1 \) and \( m \) respectively.
Transformation of the RHP III

If there exists a non-trivial matrix-valued function $S$, non-singular on $\mathbb{C}$, smooth and unitary on $\mathbb{R}$, s.t.

$$H(z) = T(z)S'(z)S^*(z)T^{-1}(z)$$

is also a polynomial, then $\tilde{T} = TS$ satisfies

$$W(x) = \tilde{T}(x)\tilde{T}^*(x), \quad x \in \mathbb{R}, \quad \tilde{T}'(z) = \tilde{G}(z)\tilde{T}(z), \quad z \in \mathbb{C},$$

with $\tilde{G}(z) = G(z) + H(z)$ and the matrix $X^n$ satisfies

$$\frac{d}{dz}X^n(z) = (F_n(z; G) + F_n(z; H))X^n(z) - X^n(z) \begin{pmatrix} \chi(z) & 0 \\ 0 & -\chi^*(z) \end{pmatrix}$$

with $\chi(z) = S'(z)S^*(z)$.

Consequences: We have a class of ladder operators.
Transformation of the RHP III

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with $\chi(z) = S'(z)S^*(z)$.

Consequences: We have a class of ladder operators.
Example: Hermite type MOP

Let us consider $T(x) = e^{-x^2/2}e^{Ax}$ and

$W(x) = e^{-x^2}e^{Ax}e^{A^*x}$, \( A \in \mathbb{C}^{N\times N}, \ x \in \mathbb{R} \).

Lax pair

$$
X^{n+1}(z) = \begin{pmatrix} zI_N - \alpha_n & 1/(2\pi^2)\gamma_n^{-1} \\ -2\pi i\gamma_n & 0 \end{pmatrix} X^n(z)
$$

$$
\frac{d}{dz} X^n(z) = \begin{pmatrix} -zI_N + A & -1/(\pi^2)\gamma_n^{-1} \\ 4\pi i\gamma_n^{-1} & zI_N - A^* \end{pmatrix} X^n(z)
$$

Compatibility conditions

$$
\alpha_n = (A + \gamma_n^{-1}A^*\gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I_N
$$
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\]
What is a RH problem?

The RH problem for OP

The RH problem for MOP

Ladder operators

\[ \hat{P}_n'(x) + \hat{P}_n(x)A - A\hat{P}_n(x) = 2\beta_n\hat{P}_{n-1}(x), \]
\[ -\hat{P}_n'(x) + 2x\hat{P}_n(x) + A\hat{P}_n(x) - \hat{P}_n(x)A - 2\alpha_n\hat{P}_n(x) = 2\hat{P}_{n+1}(x). \]

Combining them we get a second order differential equation

Second order differential equation

\[ \hat{P}''(x) + 2\hat{P}_n(x)(A - xI_N) + \hat{P}_n(x)(A^2 - 2xA) \]
\[ = (-2xA + A^2 - 4\beta_n)\hat{P}_n(x) + 2(A - \alpha_n)(\hat{P}_n'(x) + \hat{P}_n(x)A - A\hat{P}_n(x)). \]
Ladder operators

\[ \hat{P}_n'(x) + \hat{P}_n(x)A - A\hat{P}_n(x) = 2\beta_n\hat{P}_{n-1}(x), \]
\[ -\hat{P}_n'(x) + 2x\hat{P}_n(x) + A\hat{P}_n(x) - \hat{P}_n(x)A - 2\alpha_n\hat{P}_n(x) = 2\hat{P}_{n+1}(x). \]

Combining them we get a second order differential equation

Second order differential equation

\[ \hat{P}''_n(x) + 2\hat{P}_n'(x)(A - xI_N) + \hat{P}_n(x)(A^2 - 2xA) \]
\[ = (-2xA + A^2 - 4\beta_n)\hat{P}_n(x) + 2(A - \alpha_n)(\hat{P}_n'(x) + \hat{P}_n(x)A - A\hat{P}_n(x)). \]
In order to use the freedom in the matrix case by a unitary matrix function $S$ we have to impose additional constraints on the weight $W$.

The matrix $H$ can be written as

$$H(x) = e^{Ax} \chi e^{-Ax} = \chi + \text{ad}_A(\chi)x + \text{ad}_A^2(\chi)\frac{x^2}{2} + \cdots,$$

where $\chi(x) = S'(x)S^*(x)$ is skew-Hermitian on $\mathbb{R}$.

This matrix equation was considered already by Durán-Grünbaum (2004), when $\chi$ is a constant matrix.

- If $\deg H = 0$ then $\chi = i\alpha I_N$, $\alpha \in \mathbb{R}$ ⇒ No new ladder operators.
- If $\deg H = 1$ then
  - $A = L = \sum_{i=1}^{N-1} \nu_i E_i$ and $x = \nu_1 I_N + \cdots + \nu_{N-1} E_{N-1}$
    - $\Rightarrow \text{ad}_A(\chi) = -A$ and $S(x) = e^{\alpha I}$
  - $A = L (I_N + L)^{-1}$ and $x = \nu_1 I_N$
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First case $A = L$

New compatibility conditions

\[ J\alpha_n - \alpha_n J + \alpha_n = L + \frac{1}{2}(L^2\alpha_n - \alpha_nL^2), \quad J - \gamma_n^{-1}J\gamma_n = L\alpha_n + \alpha_nL - 2\alpha_n^2 \]

New ladder operators (0-th order)

\[ \hat{P}_n(x)J - J\hat{P}_n(x) - x(\hat{P}_n(x)L - L\hat{P}_n(x)) + 2\beta_n\hat{P}_n(x) - n\hat{P}_n(x) = 2(L - \alpha_n)\beta_n\hat{P}_{n-1}(x) \]
\[ \hat{P}_n(x)(J - xL) - \gamma_n^{-1}(J - xL^*)\gamma_n\hat{P}_n(x) + 2\beta_{n+1}\hat{P}_n(x) - (n+1)\hat{P}_n(x) = 2(\alpha_n - L)\hat{P}_{n+1}(x) \]

First-order differential equation

\[ (L - \alpha_n)\hat{P}_n'(x) + (L - \alpha_n + xI_N)(\hat{P}_n(x)L - L\hat{P}_n(x)) - 2\beta_n\hat{P}_n(x) = \hat{P}_n(x)J - J\hat{P}_n(x) - n\hat{P}_n(x) \]
What is a RH problem?
The RH problem for OP
The RH problem for MOP

First case \( A = L \)

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Sturm-Liouville type differential equation

Finally, something remarkable happens. Combining the second and the first order differential equation will give surprisingly

\[ \hat{P}_n''(x) + 2\hat{P}_n'(x)(L - xI_N) + \hat{P}_n(x)(L^2 - 2J) = (-2nI_N + L^2 - 2J)\hat{P}_n(x) \]

This is a second-order differential equation of Sturm-Liouville type satisfied by the MOP, already given by Durán-Grünbaum (2004)

Conclusions

1. The ladder operators method gives more insight about the differential properties of MOP and new phenomena
2. This method works for every weight matrix \( W \). The corresponding MOP satisfy differential equations, but not necessarily of Sturm-Liouville type
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