# Discrete Krall Polynomials*

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*Joint work with Antonio J. Durán

# OUTLINE

1. Krall-polynomials

2. *D*-operators and orthogonality

3. Some explicit examples

# Krall polynomials

# SOME HISTORY

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Families of orthogonal polynomials (OPs) $(p_n)_n$ satisfying

$$D(p_n(x)) = \lambda_n p_n(x), \quad \text{where} \quad D = \sum_{l=1}^{k} f_l(x) \frac{d^l}{dx^l}$$

where $f_l, l = 1, \ldots, k$, are polynomials of degree at most $l$ independent of $n$.

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**Über Sturm-Liouvillesche Polynomsysteme.**

Von

S. Bochner in München.

Wir betrachten irgendeine Differentialgleichung der Form

(1) $$p_0(x)y'' + p_1(x)y' + p_2(x)y + \lambda y = 0.$$

Die Koeffizienten $p_0(x)$, $p_1(x)$, $p_2(x)$ sind irgendwelche reell- oder komplexwertige Funktionen der Variablen $x$; von denen wir in erster Linie nur anzunehmen brauchen, daß sie in einem gemeinsamen Intervall $J$ der $x$-Achse definiert sind; und $\lambda$ bedeutet einen Parameter, der aller komplexen Werte fähig ist.

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  $\chi_{[-1,1]} + M(\delta_{-1} + \delta_1)$, $e^{-x} + M\delta_0$ and $(1-x)^\alpha \chi_{[0,1]} + M\delta_0$

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- $k \geq 6$: A. M. Krall, L. Littlejohn, Koekoek's, Zhedanov, Kwon, Lee, Grünbaum-Haine (Darboux transformation), Iliev, etc.

  Typically all examples are orthogonal with respect to a measure of the form

$$\omega(x) + \sum_{j=0}^{m-1} a_j, \delta_{x_0}^{(j)}, \quad a_j \in \mathbb{R}$$

  where $\omega$ is a classical weight and $x_0$ is an endpoint of the $\sup(\omega)$.

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where $f_l, l = 1, \ldots, k$, are polynomials independent of $n$ and $\mathfrak{s}_l(p) = p(x+l)$.
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**ORTHOGONAL POLYNOMIALS DEFINED BY DIFFERENCE EQUATIONS.*†**

By Otis E. Lancaster.

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**1. Introduction:** Many analogous properties of differential and difference equations have been studied. Here these analogies are extended to include some ideas relative to orthogonal solutions of difference equations.

Although some general theorems are given, the main study is confined to polynomial solutions of difference equations of the form

$$(1) \qquad (ax^2 + bx + c)\underset{h}{\Delta^2} y(x) + (dx + f)\underset{h}{\Delta} y(x) + \lambda y(x+h) = 0$$

where $h > 0$ is the interval of difference, $a$, $b$, $c$, $d$, and $f$ are constants and $\lambda$ is a parameter which is determined so as to insure polynomial solutions.[1]

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  It was not until very recently (Durán, 2012) where the first examples appeared (he also proved that $s = -r$. Then the order must be even).
  Typically now the measures are of the form

$$\omega^F(x) = \prod_{f \in F} (x - f)\omega(x)$$

  where $\omega$ is a classical discrete weight and $F$ is a finite set of real numbers.
  This is also called a Christoffel transform of $\omega$ (related to the Geronimus transform).

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Then, the corresponding OPs are eigenfunctions of a fourth-order difference operator

$$D = x(x-3)\mathfrak{S}_{-2} - 2x(x-2)\mathfrak{S}_{-1} + x(x+2a-1)\mathfrak{S}_0 - 2xa\mathfrak{S}_1 + a^2\mathfrak{S}_2, \quad a \neq 1, 2, \ldots$$

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The same can be done with Meixner, Krawtchouk and Hahn families.

Positivity: If $F = \bigcup F_i$ where $F_i$ contains consecutive nonnegative integers, then $\omega^F$ is positive if and only if the cardinal of each $F_i$ is even for all $i$.

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In all the examples gave by Durán in 2012, the sets $F$ contains consecutive positive integers. A number of conjectures were proposed stating that this can always be done for any set $F$ (not necessarily consecutive), which have been proved by Durán-MdI in 2014 and 2015.

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- Let $\omega_a$ be the Charlier weight

$$\omega_a(x) = \sum_{x=0}^{\infty} \frac{a^x}{x!}\delta_x, \quad a > 0$$

and for a finite set $F$ of positive integers consider

$$\omega_a^F = \prod_{f\in F}(x-f)\omega_a$$

Then the OPs $(q_n)_n$ with respect to $\omega_a^F$ are eigenfunctions of a higher-order difference operator of the form $D = \sum_{l=s}^{r} f_l(x)\mathfrak{S}_l$ with $-s = r = r_F$, where

$$r_F = \sum_{f\in F} f - \binom{\#(F)}{2} + 1$$

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- Let $\omega_{a,c}$ be the Meixner weight

$$\omega_{a,c}(x) = \Gamma(c)(1-a)^c \sum_{x=0}^{\infty} \frac{(c)_x a^x}{x!} \delta_x, \quad 0 < a < 1, \quad c > 0$$

and for two finite sets $F_1, F_2$ of positive integers consider

$$\omega_{a,c}^{F_1,F_2} = \prod_{f \in F_1} (x+c+f) \prod_{f \in F_2} (x-f) \omega_{a,c}$$

Then the OPs $(q_n)_n$ with respect to $\omega_a^{F_1,F_2}$ are eigenfunctions of a higher-order difference operator with $-s = r = r_{F_1} + r_{F_2} - 1$, where $r_F$ is as before.

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- Let $\omega_{a,N}$ be the Krawtchouk weight

$$\omega_{a,N}(x) = \frac{1}{(1+a)^{N-1}} \sum_{x=0}^{N-1} \binom{N-1}{x} a^x \delta_x, \quad a > 0$$

  and for two finite sets $F_1, F_2$ of positive integers consider

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and for four finite sets $\mathcal{F}=(F_1,F_2,F_2,F_4)$ of positive integers consider

$$\omega_{a,b,N}^{\mathcal{F}} = \prod_{f\in F_1}(b+N+1+f-x)\prod_{f\in F_2}(a+1+f+x)\prod_{f\in F_3}(N-f-x)\prod_{f\in F_4}(x-f)\omega_{a,b,N}$$

Then the OPs $(q_n)_n$ with respect to $\omega_{a,b,N}^{\mathcal{F}}$ are eigenfunctions of a higher-order difference operator with $-s=r=r_{F_1}+r_{F_2}+r_{F_3}+r_{F_4}-3$, where $r_F$ is as before.

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In order to study discrete Krall OPs $(q_n)_n$ we will follow the following guideline:

**1.** Computation of the higher-order difference operator

This result will be valid for any family of polynomials $(p_n)_n$ (not necessarily orthogonal) eigenfunctions of certain operator, i.e. $D(p_n) = \theta_n p_n$. It is based on the abstract concept of $\mathcal{D}$-operator and there will be $Y_1, \ldots, Y_m$ arbitrary polynomials.

**2.** Orthogonality

Only for a convenient choice of the polynomials $Y_1, \ldots, Y_m$, the polynomials $(q_n)_n$ are also orthogonal with respect to a measure. The right choice for the polynomials $Y_1, \ldots, Y_m$ will be classical discrete polynomials of the same type but with different parameters (Charlier, Meixner and Krawtchouk) or dual Hahn polynomials (Hahn).

# $D$-operators and orthogonality

# *D*-OPERATORS OF TYPE I

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Let $\mathcal{A}$ be an algebra of difference operators acting in the linear space of polynomials $\mathbb{P}$

$$\mathcal{A} = \left\{ \sum_{l=s}^{r} f_l(x)\mathfrak{S}_l : f_l(x) \in \mathbb{P}, l = s, \ldots, r, s \leq r \right\}$$

Assume we have a family of polynomials $(p_n)_n$ such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$, i.e. with linear eigenvalue.

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Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n}(-1)^{j+1}\varepsilon_n \cdots \varepsilon_{n-j}p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

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We say that $\mathcal{D}$ is a $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

- Charlier: $\varepsilon_n = 1 \Rightarrow \mathcal{D} = \nabla$.
- Meixner: $\varepsilon_n^1 = \dfrac{a}{1-a} \Rightarrow \mathcal{D}_1 = \dfrac{a}{1-a}\Delta$, $\quad \varepsilon_n^2 = \dfrac{1}{1-a} \Rightarrow \mathcal{D}_2 = \dfrac{1}{1-a}\nabla$.
- Krawtchouk: $\varepsilon_n^1 = \dfrac{1}{1-a} \Rightarrow \mathcal{D}_1 = \dfrac{1}{1-a}\nabla$, $\quad \varepsilon_n^2 = -\dfrac{a}{1-a} \Rightarrow \mathcal{D}_2 = -\dfrac{a}{1-a}\Delta$.

where $\Delta f(x) = f(x+1) - f(x)$ and $\nabla f(x) = f(x) - f(x-1)$.

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Let $\xi^h_{x,j} = \prod_{i=0}^{j-1} \varepsilon^h_{x-i}, h = 1, \ldots, m$ and $Y_1, \ldots, Y_m$ arbitrary polynomials such that

$$\Omega(n) = \det \left( \xi^l_{n-j,m-j} Y_l(n-j) \right)_{l,j=1}^{m} \neq 0, \quad n \geq 0$$

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Define $M_1, \dots, M_m$, the following polynomials in $x$

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where $\mathbb{I}_h = \{1, 2, \ldots, m\} \setminus \{h\}$. $M_h$ are linear combinations of adjoint determinants of $\Omega(x)$. If we assume that $\Omega(x)$ and $M_h(x)$ are polynomials in $x$, then there exists $D_q \in \mathcal{A}$ with $D_q(q_n) = P(n) q_n$ and $P(x) - P(x-1) = \Omega(x)$, where

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$$\mathcal{D}(p_n) = -\frac{1}{2}\sigma_{n+1}p_n + \sum_{j=1}^{n}(-1)^{j+1}\sigma_{n-j+1}\varepsilon_n \cdots \varepsilon_{n-j+1}p_{n-j}, \quad n \geq 0$$

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There are four different $\mathcal{D}$-operators for the Hahn polynomials. Defining

$$\begin{aligned}
\varepsilon_n^1 &= -\frac{n-N+1}{n+a+b+N+1}, & \sigma_n &= -(2n+a+b-1),\\
\varepsilon_n^2 &= \frac{(n+b)(n-N+1)}{(n+a)(n+a+b+N+1)}, & \sigma_n &= -(2n+a+b-1),\\
\varepsilon_n^3 &= 1, & \sigma_n &= -(2n+a+b-1),\\
\varepsilon_n^4 &= -\frac{n+b}{n+a}, & \sigma_n &= -(2n+a+b-1).
\end{aligned}$$

then we have

$$\begin{aligned}
\mathcal{D}_1 &= \frac{a+b+1}{2}I + x\nabla, & \mathcal{D}_2 &= \frac{a+b+1}{2}I + (x-N)\Delta,\\
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For a rational function $S(x)$, we define the function $\lambda_x$ by

$$\lambda_x - \lambda_{x-1} = S(x)\Omega(x)$$

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Assume the following

1. $S(x)\Omega(x)$ is a polynomial in $x$
2. There exist $\tilde{M}_1, \ldots, \tilde{M}_m$, polynomials in $x$ such that $M_h(x) = \sigma_{x+1}\tilde{M}_h(\theta_x)$, where $M_h(x)$ are defined by

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi^h_{x,m-j} S(x+j) \det \left( \xi^l_{x+j-r,m-r} Y_l(\theta_{x+j-r}) \right)_{l \in \mathbb{I}_h; r \in \mathbb{I}_j}$$

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Then there exists an operator $D_{q,S} \in \mathcal{A}$ such that $D_{q,S}(q_n) = \lambda_n q_n, n \geq 0$, where the operator $D_{q,S}$ is defined by

$$D_{q,S} = \frac{1}{2} P_S(D_p) + \sum_{h=1}^{m} \tilde{M}_h(D_p) \mathcal{D}_h Y_h(D_p)$$

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However, if $(p_n)_n$ is one of the classical families it turns out that they have the bispectral property, meaning that they satisfy a higher-order recurrence relation of the form

$$Q(x)q_n(x) = \sum_{i=-s}^{s} \gamma_{n,i} q_{n+i}(x), \quad s \in \mathbb{N}, \quad \deg Q(x) = s$$

This has been proved so far for the Charlier case (Durán, 2019), the Laguerre and Jacobi cases (Durán-MdI, 2020,2022) and the Meixner case (Durán-Rueda, 2021).

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| Classical discrete family | $\mathcal{D}$-operators | $Y_j(x)$ |
|---|---|---|
| Charlier: $c_n^a(x)$ | $\nabla$ | $c_j^{-a}(-x-1)$ |
| Meixner: $m_n^{a,c}(x)$ | $\frac{a}{1-a}\Delta$ | $m_j^{1/a,2-c}(-x-1)$ |
| | $\frac{1}{1-a}\nabla$ | $m_j^{a,2-c}(-x-1)$ |
| Krawtchouk: $k_n^{a,N}(x)$ | $\frac{1}{1+a}\nabla$ | $k_j^{a,-N}(-x-1)$ |
| | $\frac{-a}{1+a}\Delta$ | $k_j^{1/a,-N}(-x-1)$ |
| Hahn: $h_n^{a,b,N}(x)$ | $\frac{a+b+1}{2}I + x\nabla$ | $R_j^{-b,-a,a+b+N}(x+a+b)$ |
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Given a set $G$ of $m$ positive integers, $G = \{g_1, \ldots, g_m\}$ we then define the sequence of polynomials $(q_n^G)_n$ by

$$q_n^G(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi^1_{n,m} Y_{g_1}(\theta_n) & \xi^1_{n-1,m-1} Y_{g_1}(\theta_{n-1}) & \cdots & Y_{g_1}(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^m_{n,m} Y_{g_m}(\theta_n) & \xi^m_{n-1,m-1} Y_{g_m}(\theta_{n-1}) & \cdots & Y_{g_m}(\theta_{n-m}) \end{vmatrix}$$

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The set $G$ will be identified with one of the sets $F$ associated with the corresponding Christoffel transformation of the discrete classical measure, in one of the following forms:

$$I(F) = \{1, 2, \dots, f_k\} \setminus \{f_k - f, f \in F\},$$
$$J_h(F) = \{0, 1, 2, \dots, f_k + h - 1\} \setminus \{f - 1, f \in F\}, \quad h \geq 1$$

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$$J_h(F) = \{0, 1, 2, \dots, f_k + h - 1\} \setminus \{f - 1, f \in F\}, \quad h \geq 1$$

where $f_k = \max F$ and $k = \#(F)$.

- Krawtchouk: $G_1 = I(F_1), G_2 = J_h(F_2)$

$$\omega_{a,N}^{F_1,F_2} = \prod_{f \in F_1} (x - f) \prod_{f \in F_2} (N - 1 - f - x)\omega_{a,N}$$

- Hahn: $G_1 = J_{h_1}(F_1), G_2 = J_{h_2}(F_2), G_3 = J_{h_3}(F_3), G_4 = I(F_4), \mathcal{F} = (F_1, F_2, F_2, F_4)$

$$\omega_{a,b,N}^{\mathcal{F}} = \prod_{f \in F_1} (b + N + 1 + f - x) \prod_{f \in F_2} (a + 1 + f + x) \prod_{f \in F_3} (N - f - x) \prod_{f \in F_4} (x - f)\omega_{a,b,N}$$

# Some explicit examples

# A KRALL-CHARLIER EXAMPLE

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Let $a = 1$, $F = \{1, 3\}$, $G = I(F) = \{1, 3\}$. The polynomials defined by

$$\frac{q_n^G(x)}{\Omega(n)} = c_n^1(x) + \beta_{n,1} c_{n-1}^1(x) + \beta_{n,2} c_{n-2}^1(x)$$

for some choice of sequences $\beta_{n,1}, \beta_{n,2}$ are orthogonal with respect to

$$\tilde{\omega}_1^F = (x+3)(x+1)\omega_1(x+4)$$

Here $\omega_a^F = a^{f_k+1} \tilde{\omega}_a^F(x - f_k - 1)$, where $f_k = \max F$ and $k = \#(F)$.

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The difference operator (of order 8) satisfying $D_q(q_n^G) = P(n) q_n^G$ is given by

$$D_q = P(D_a) + M_1(D_a)\nabla Y_1(D_a) + M_2(D_a)\nabla Y_2(D_a)$$

where $D_a$ is the Charlier second-order difference operator $(a = 1)$ and

$$Y_1(x) = -x, \qquad Y_2(x) = -\frac{1}{6}(x^3 + 3x^2 + 5x + 2)$$

$$M_1(x) = x^2 + 2x + 2, \qquad M_2(x) = -2$$

$$P(x) = -\frac{x}{12}(x^3 - 2x^2 - x - 2)$$

# A KRALL-HAHN EXAMPLE

Let $a = 1/2, b = -1/3, N = 9,\ F_4 = \{1,3\},\ G_4 = I(F_4) = \{1,3\}$. The polynomials

$$\frac{q_n^{G_4}(x)}{\Omega(n)} = h_n^{1/2,-1/2,9}(x) + \beta_{n,1} h_{n-1}^{1/2,-1/2,9}(x) + \beta_{n,2} h_{n-2}^{1/2,-1/2,9}(x)$$

for some choice of sequences $\beta_{n,1}, \beta_{n,2}$ are orthogonal with respect to

$$\tilde{\omega}_{a,b,N}^{F_4} = (x+3)(x+1)\omega_{a-4,b,N+4}(x+4)$$

If we change $a \to a+4, N \to N-4, x \to x-4$ then we obtain $\omega_{a,b,N}^{F_4} = (x-1)(x-3)\omega_{a,b,N}$.

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If we change $a \to a+4, N \to N-4, x \to x-4$ then we obtain $\omega_{a,b,N}^{F_4} = (x-1)(x-3)\omega_{a,b,N}$.

The difference operator (of order 8) satisfying $D_q(q_n^{G_4}) = \lambda_n q_n^{G_4}$ is given by

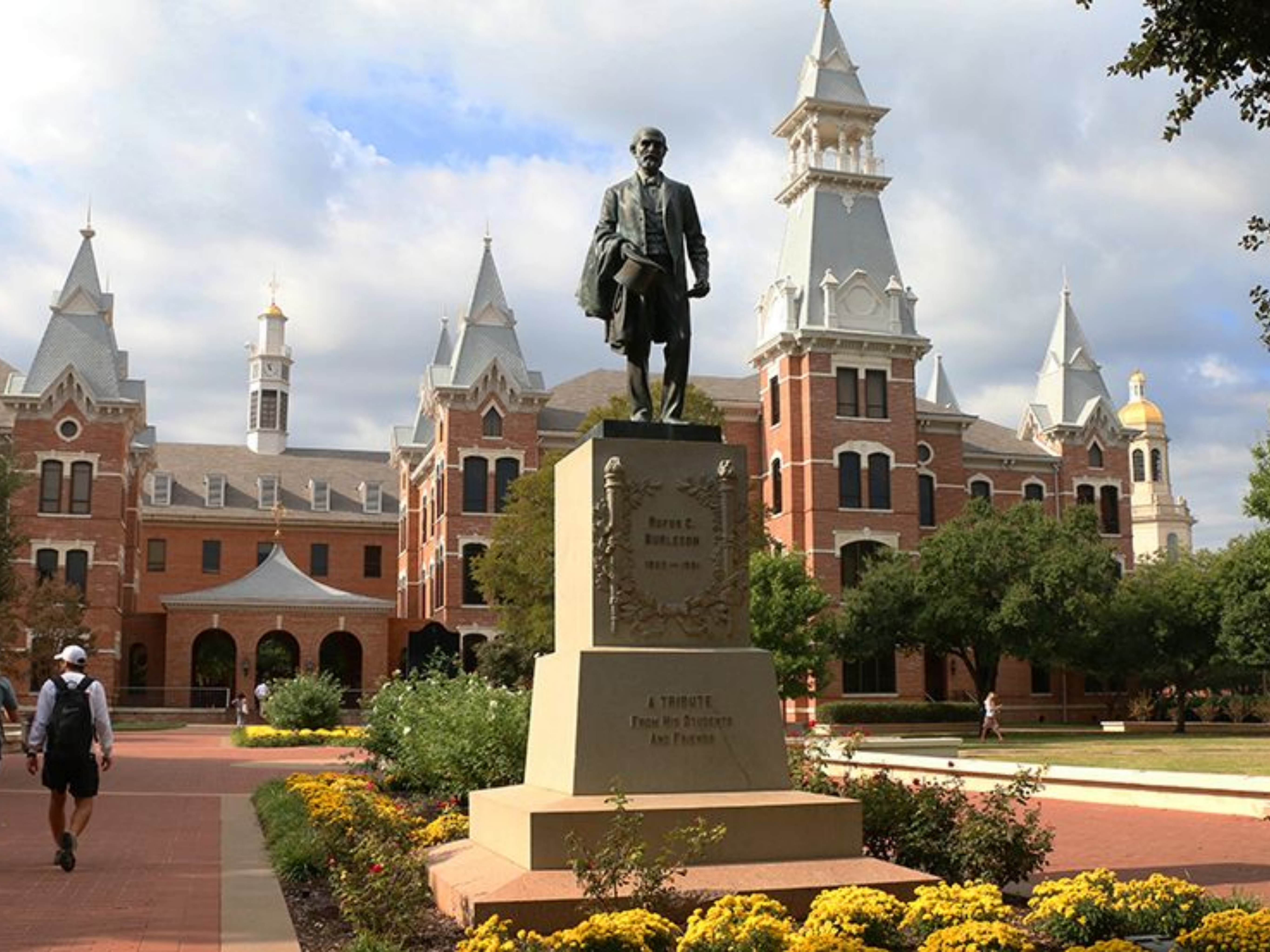
$$D_q = \frac{1}{2} P(D_{a,b,N}) + \tilde{M}_1(D_{a,b,N})\mathcal{D}_4 R_1^{-a,-b,-2-N}(D_{a,b,N}) + \tilde{M}_2(D_{a,b,N})\mathcal{D}_4 R_3^{-a,-b,-2-N}(D_{a,b,N})$$

where $D_{a,b,N}$ is the Hahn operator, $\mathcal{D}_4$ is the fourth $\mathcal{D}$-operator for the Hahn family and

$$P(x) = \frac{2}{70785} x(2x^3 + 192x^2 + 3925x + 34656)$$

$$\tilde{M}_1(x) = -\frac{4}{2145}x^2 - \frac{84}{715}x - \frac{12}{11}, \qquad \tilde{M}_2(x) = \frac{4}{11}$$

$$\lambda_n = \frac{2}{70785} n(n+1)(2n^6 + 6n^5 + 198n^4 + 386n^3 + 4117n^2 + 3925n + 34656)$$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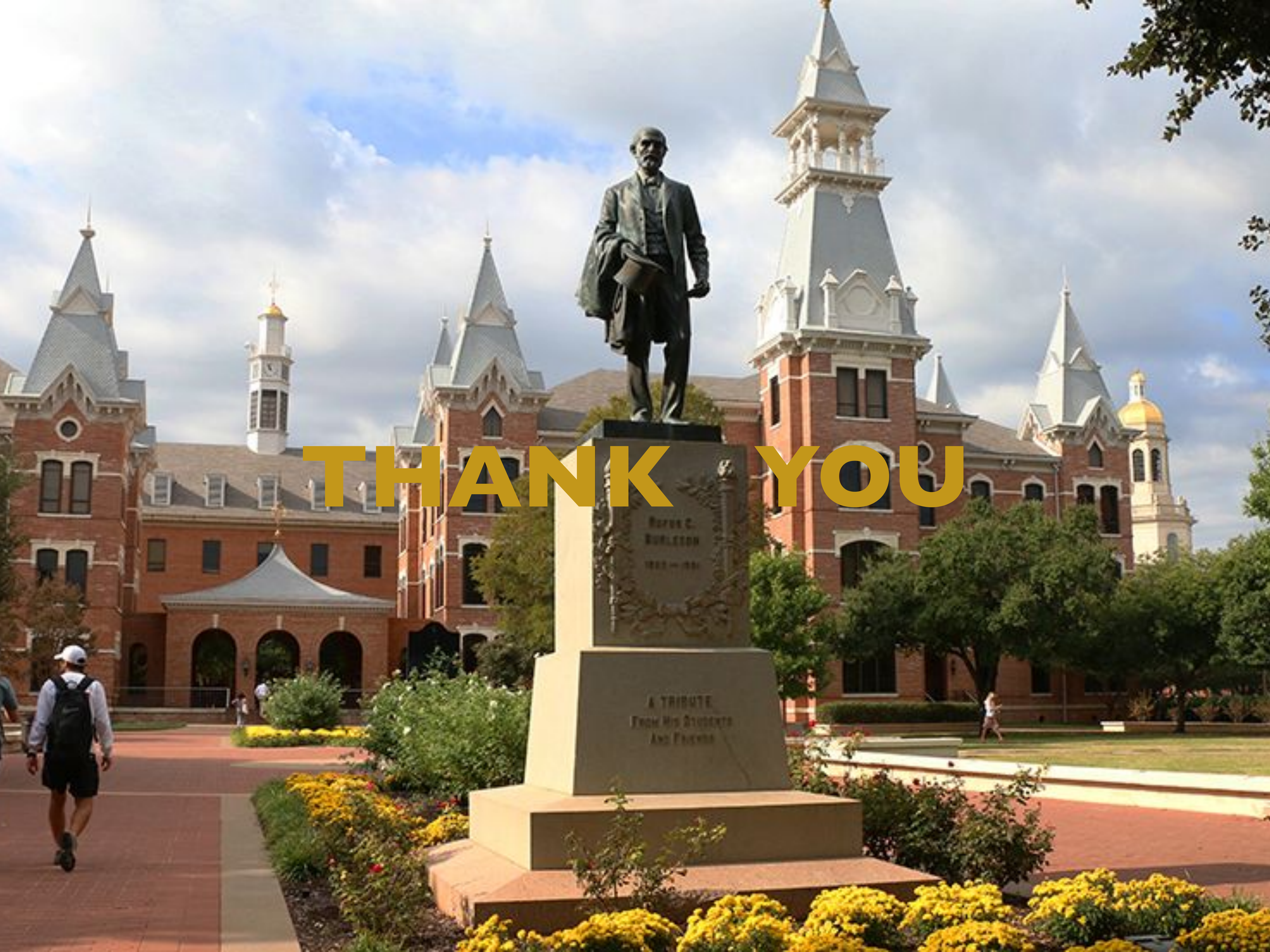
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