Krall-Hahn Orthogonal Polynomials*

Manuel Domínguez de la Iglesia

Instituto de Matemáticas, UNAM

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OUTLINE

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Krall polynomials
SOME HISTORY

Families of orthogonal polynomials (OPs) \((p_n)_n\) satisfying

\[
D(p_n(x)) = \lambda_n p_n(x), \quad \text{where} \quad D = \sum_{l=1}^{k} f_l(x) \frac{d^l}{dx^l}
\]

where \(f_l, l = 1, \ldots, k,\) are polynomials of degree at most \(l\) independent of \(n.\)

- \(k = 4: \) H. L. Krall (1939). \(k\) must be even and there are 3 NEW examples:
  \[\chi[-1,1] + M(\delta_{-1} + \delta_1), \quad e^{-x} + M\delta_0 \quad \text{and} \quad (1 - x)^\lambda \chi[0,1] + M\delta_0\]

- \(k \geq 6: \) A. M. Krall, L. Littlejohn, Koekoek’s, Zhedanov, Kwon, Lee, Grünbaum-Haine (Darboux transformation), Iliev, etc.

Typically all examples are orthogonal with respect to a (measure of the form

\[
\omega(x) \quad \text{where} \quad \omega \quad \text{is a classical weight and} \quad x_0 \quad \text{is an endpoint of the sup}(\omega).\]
Families of discrete OPs \((p_n)_n\) satisfying difference equations

\[
D(p_n(x)) = \lambda_n p_n(x), \quad \text{where} \quad D = \sum_{l=s}^{r} f_l(x) s_l
\]

where \(f_l, l = 1, \ldots, k\), are polynomials independent of \(n\) and \(s_l(p) = p(x + l)\).

Here \(r, s \in \mathbb{Z}, r \geq s\), and the order is defined by \(r - s \geq 0\).

- **\(r = -s = 1\)**: Chebychev (1858).
  - Charlier, Meixner, Krawtchouk and Hahn (20th century).

- **\(r = -s \geq 2\)**: The trick of adding deltas at the endpoints of the support does not work here (Bavinck-van Haeringen-Koekoek, 1994). However it works if one considers \(q\)-difference equations (Grunbaum-Hame, 1996 and Vinet-Zhedanov, 2001).

It was not until very recently (Durán, 2012) where the first examples appeared (he also proved that \(s = -r\)). Then the order must be even.

Typically now the measures are of the form

\[
\omega(x) = \prod_{f \in F} (x - f),
\]

where \(\omega\) is a classical discrete weight and \(F\) is a finite set of real numbers.

This is also called a Christoffel transform of \(\omega\) (related to the Geronimus transform).
Let \( \omega_a(x) = \sum_{x \in \mathbb{N}} \frac{a^x}{x!} \delta_x, \ a > 0 \) be the Charlier weight

- \( F = \{1\} \): \( \omega^F(x) = (x - 1)\omega_a(x) = -\delta_0 + \sum_{x=2}^{\infty} \frac{a^x}{x(x - 2)!} \delta_x, \ a > 0 \)

Then, the corresponding OPs are eigenfunctions of a \textbf{fourth-order} difference operator

\[
D = x(x - 3)\mathfrak{S}_{-2} - 2x(x - 2)\mathfrak{S}_{-1} + x(x + 2a - 1)\mathfrak{S}_0 - 2xa\mathfrak{S}_1 + a^2\mathfrak{S}_2, \ a \neq 1, 2, \ldots
\]

- \( F = \{1, 2\} \): \( \omega^F(x) = (x - 1)(x - 2)\omega_a(x) = 2\delta_0 + \sum_{x=3}^{\infty} \frac{a^x}{x(x - 3)!} \delta_x, \ a > 0 \)

Then, the corresponding OPs are eigenfunctions of a \textbf{sixth-order} difference operator

\[
D = 2x \left( \frac{x - 4}{2} \right) \mathfrak{S}_{-3} - 6x \left( \frac{x - 3}{2} \right) \mathfrak{S}_{-2} + 3x(x - 3)(x + a - 2)\mathfrak{S}_{-1} \\
- x(x - 2)(x + 6a - 1)\mathfrak{S}_0 + 3ax(x + a - 1)\mathfrak{S}_1 - 3a^2x\mathfrak{S}_2 + a^3\mathfrak{S}_3
\]

The same can be done with Meixner, Krawtchouk and Hahn families.
FIRST EXAMPLES

If we call $p_F(x) = \prod_{f \in F} (x - f)$, then it is possible to generate examples of OPs with respect to $\omega^F = p_F \omega$ satisfying higher-order difference equations in the following situations:

$p_F(x)$, $p_{-c-F}(x)$, $p_{N-1-F}(x)$, $p_{\alpha+F}(x)$

where $\tilde{p}(x) = p(-x)$, $c, \alpha, N \in \mathbb{R}$ and $F$ is a finite set of positive integers.

Positivity: If $F = \bigcup F_i$ where $F_i$ contains consecutive nonnegative integers, then $\omega^F$ is positive if and only if the cardinal of each $F_i$ is even for all $i$.

In all the examples gave by Durán in 2012, the sets $F$ contains consecutive positive integers. A number of conjectures were proposed stating that this can always be done for any set $F$ (not necessarily consecutive). Moreover, the order of the higher-order difference operator can be explicitly computed.

Some of these conjectures (for the Charlier, Meixner and Krawtchouk families) were recently proved by Durán-MdI in 2014. In these cases the corresponding eigenvalue of the second-order difference equation is linear in $n$.

More recently, in 2015, Durán-MdI worked out the Hahn case, where now the difficulty comes in the fact that the corresponding eigenvalue of the second-order difference equation is nonlinear in $n$ (quadratic).
Krall-Hahn polynomials. An example
Hahn polynomials

For \( a, a+b+1, a+b+N+1 \neq -1, -2, \ldots \) we consider the Hahn polynomials defined by

\[
h_{n}^{a,b,N}(x) = \sum_{j=0}^{n} \frac{(-x)_{j}(N-n+1)_{n-j}(a+b+1)_{j+n}}{(2+a+b+N)_{n}(a+1)_{j}(n-j)!j!}, \quad n \geq 0
\]

Hahn polynomials are eigenfunctions of the second-order difference operator

\[
D_{a,b,N} = x(x-b-N-1)\mathcal{S}_{-1} - [(x+a+1)(x-N) + x(x-b-N-1)]\mathcal{S}_{0} + (x+a+1)(x-N)\mathcal{S}_{1}
\]

That is

\[
D_{a,b,N}(h_{n}^{a,b,N}) = \theta_{n}h_{n}^{a,b,N}, \quad \theta_{n} = n(n+a+b+1), \quad n \geq 0
\]

They satisfy the following three-term recurrence formula (TTRR) \((h_{-1}^{a,b,N} = 0)\)

\[
xh_{n}^{a,b,N} = a_{n+1}h_{n+1}^{a,b,N} + b_{n}h_{n}^{a,b,N} + c_{n}h_{n-1}^{a,b,N}, \quad n \geq 0,
\]

where

\[
a_{n} = -\frac{n(n+a)(n+a+b+N+1)}{(2n+a+b-1)(2n+a+b)},
\]

\[
b_{n} = \frac{N(a+1)(a+b) + n(2N+b-a)(n+a+b+1)}{(2n+a+b)(2n+a+b+2)},
\]

\[
c_{n} = -\frac{(n+a+b)(n+b)(N-n+1)}{(2n+a+b)(2n+a+b+1)}
\]
HANH POLYNOMIALS

When \( N+1 \) is a positive integer, \( a, b \neq -1, -2, \ldots, -N \), and \( a+b \neq -1, -2, \ldots, -2N-1 \), the first \( N+1 \) Hahn polynomials are orthogonal with respect to the Hahn measure

\[
\rho_{a,b,N} = N! \sum_{x=0}^{N} \frac{\Gamma(a+x+1)\Gamma(N-x+b+1)}{x!(N-x)!} \delta_x
\]

The discrete measure \( \rho_{a,b,N} \) is positive only when \( a, b > -1 \) or \( a, b < -N \).

We also need the so-called dual Hahn polynomials, \( a \neq -1, -2, \ldots \)

\[
R_{n}^{a,b,N}(x) = \sum_{j=0}^{n} \frac{(-1)^j (-n)_j (-N+j)_{n-j}}{(a+1)_j j!} \prod_{i=0}^{j-1} [x - i(i + a + b + 1)], \quad n \geq 0
\]

Observe that \( (-1)^j \prod_{i=0}^{j-1} (x(x + a + b + 1) - i(a + b + 1 + i)) = (-x)_j (x + a + b + 1)_j \), therefore we have the duality

\[
R_{n}^{a,b,N}(n(n + a + b + 1)) = \frac{(-1)^n n!(N + a + b + 2)_n (-N)_x}{(a + b + 1)_n (-N)_n} h_{n}^{a,b,N}(x), \quad x, n \geq 0
\]
AN EXPLICIT EXAMPLE

Let $F = \{1, 3\}$ and consider the discrete weight

$$\rho_{a,b,N}^F(x) = (x - 1)(x - 3)\rho_{a,b,N}(x)$$

Now consider the change of variables

$$a \rightarrow a - 4, \quad b \rightarrow b, \quad N \rightarrow N + 4, \quad x \rightarrow x + 4$$

The new weight is given by

$$\tilde{\rho}_{a,b,N}^F(x) = (x + 4)^2(x - 3)(x - 1)\rho_{a - 4,b,N + 4}(x + 4)$$

The key to prove that $\langle \tilde{\rho}_{a,b,N}^F, x^j q_n \rangle \equiv 0$ for $0 \leq j \leq n - 1$ and $\langle \tilde{\rho}_{a,b,N}^F, x^n q_n \rangle \neq 0$ is

$$\langle \tilde{\rho}_{a,b,N}^F, h_{a,b,N}^{a,b,N}(x) \rangle = A_1(n, F)Y_1(\theta_n) + A_2(n, F)Y_2(\theta_n), \quad n \geq 0$$

For this new weight we try to write its OPs $(q_n)_n$ as the following Casorati determinant

$$0 = B_1(F)Y_1(\theta_{-1}) + B_2(F)Y_1(\theta_{-1})$$

for certain sequences $A_i(n, F)$ and constants $B_i(F), C_i(F), Y_1(\theta_{-1}), Y_2(\theta_{-1})$.

This happens when $Y_1, Y_2$ satisfy the following recurrence relations (as a function of $n$)

$$\xi_{n,1} \xi_{n,1} Y_1(\theta_n) + \xi_{n,2} \xi_{n,2} Y_2(\theta_n) = \xi_{n,1} \xi_{n,2} Y_1(\theta_n)$$

where $\xi_{x,j} = (\varepsilon_n + 1)^{a(x + y - 1, F + 1)}/(x - j + a + 1)_j$ and $b_n Y_1(n) + c_n Y_2(n - 1) = (j + 1)Y_j(n)$

where $\varepsilon_n = -\frac{n + b}{n + a}$ and $a_n, b_n, c_n$ are the coefficients of the TTRR for the Hahn family.

The solutions are given by the dual Hahn polynomials $R_{j}^{-a,-b,-2-N}(n + a + b)$.
AN EXPLICIT EXAMPLE

The right choice \( Y_1, Y_2 \) comes from the following involution of the set \( F = \{1, 3\} \):

\[
I(F) = \{1, 2, \ldots, \max F\} \setminus \{\max F - f, f \in F\}
\]

In this case \( I(F) = \{1, 3\} \). That means that

\[
Y_1(x) = R_1^{a,-b,-2-N}(x + a + b), \quad Y_2(x) = R_3^{a,-b,-2-N}(x + a + b)
\]

The OPs \((q_n)_n\) with respect to \( \tilde{\rho}^F \) are then given by

\[
q_n = \begin{vmatrix}
 h_{n-1}^{a,b,N}(x) & -h_{n-1}^{a,b,N}(x) & h_{n-2}^{a,b,N}(x) \\
 \xi_{n,2}R_1^{a,-b,-2-N}(\theta_n + a + b) & \xi_{n-1,1}R_1^{a,-b,-2-N}(\theta_{n-1} + a + b) & R_1^{a,-b,-2-N}(\theta_{n-2} + a + b) \\
 \xi_{n,2}R_3^{a,-b,-2-N}(\theta_n + a + b) & \xi_{n-1,1}R_3^{a,-b,-2-N}(\theta_{n-1} + a + b) & R_3^{a,-b,-2-N}(\theta_{n-2} + a + b)
\end{vmatrix}
\]

Now that we have the idea of constructing the polynomials \((q_n)_n\) let us compute the explicit expression of the higher-order difference operator for which the polynomials \((q_n)_n\) are eigenfunctions and the corresponding order.
AN EXPLICIT EXAMPLE

The higher-order difference operator can be written as

\[
D_q = \frac{1}{2} P(D_{a,b,N}) + \tilde{M}_1(D_{a,b,N}) D R_1^{-a,-b,-2-N}(D_{a,b,N}) \\
+ \tilde{M}_2(D_{a,b,N}) D R_3^{-a,-b,-2-N}(D_{a,b,N})
\]

Here \( P, \tilde{M}_1 \) and \( \tilde{M}_2 \) are certain polynomials in \( \theta_x \) (we will explain later how to calculate them) and \( D \) is what is called a \( D \)-operator for the Hahn family:

\[
D = \frac{a+b+1}{2} I + (x - b - N - 1) \nabla
\]

where \( \nabla(f) = (\mathfrak{S}_0 - \mathfrak{S}_{-1}) f = f(x) - f(x-1) \).

For instance, if we fix the parameters \( a = 1/2, b = -1/2 \) and \( N = 9 \), then we have

\[
P(x) = \frac{2}{70785} x(2x^3 + 192x^2 + 3925x + 34656)
\]

\[
\tilde{M}_1(x) = -\frac{4}{2145} x^2 - \frac{84}{715} x - \frac{12}{11}
\]

\[
\tilde{M}_2(x) = \frac{4}{11}
\]

and the OPs \( (q_n)_n \) are eigenfunctions of \( D_q \), i.e. \( D_q(q_n) = \lambda_n q_n \), where

\[
\lambda_n = \frac{2}{70785} n(n+1)(2n^6 + 6n^5 + 198n^4 + 386n^3 + 4117n^2 + 3925n + 34656)
\]

order\((D_q) = 8\)
Krall-Hahn polynomials. General case
GENERAL CASE

In order to study the Krall-Hahn OPs $(q_n)_n$ we will follow the following guideline:

1. **Computation of the higher-order difference operator**
   This result will be valid for any family of polynomials $(p_n)_n$ (not necessarily orthogonal) eigenfunctions of certain operator, i.e. $D(p_n) = \theta_n p_n$. It is based on the abstract concept of $D$-operator and there will be $Y_1, \ldots, Y_m$ arbitrary polynomials.

2. **Orthogonality**
   Only for a convenient choice of the polynomials $Y_1, \ldots, Y_m$, the polynomials $(q_n)_n$ are also orthogonal with respect to a measure. We already have a clue on how to construct this measure in terms of Christoffel transforms of the Hahn weight. The right choice of the polynomials $Y_1, \ldots, Y_m$ will be certain families of dual Hahn polynomials.
D-OPERATORS

Let $\mathcal{A}$ be an algebra of difference operators acting in the linear space of polynomials $\mathbb{P}$

$$\mathcal{A} = \left\{ \sum_{l=s}^{r} h_l \delta_l : h_l \in \mathbb{P}, l = s, \ldots, r, s \leq r \right\}$$

We will work with the Hahn polynomials $(h_{n}^{a,b,N})_n$ for which we know that there exists a second-order difference operator $D_{a,b,N} \in \mathcal{A}$ such that

$$D_{a,b,N}(h_{n}^{a,b,N}) = \theta_n h_{n}^{a,b,N}, \quad \theta_n = n(n + a + b + 1)$$

Given two sequences of numbers $(\varepsilon_n)_n$ and $(\sigma_n)_n$, a $\mathcal{D}$-operator (of type 2) associated to the algebra $\mathcal{A}$ and the Hahn polynomials is defined as

$$\mathcal{D}(h_{n}^{a,b,N}) = -\frac{1}{2} \sigma_{n+1} h_{n}^{a,b,N} + \sum_{j=1}^{n} (-1)^{j+1} \sigma_{n-j+1} \varepsilon_{n-j+1} \varepsilon_{n-j+1} h_{n-j}^{a,b,N}, \quad n \geq 0$$

We then say that $\mathcal{D}$ is a $\mathcal{D}$-operator if $\mathcal{D} \in \mathcal{A}$. 
D-OPERATORS

There are four different $D$-operators for the Hahn polynomials. They are defined by the sequences $(\varepsilon_{n,h})_n$ and $(\sigma_n)_n$, $h = 1, 2, 3, 4$, given by

\[
\varepsilon_{n,1} = -\frac{n - N + 1}{n + a + b + N + 1},
\]

\[
\varepsilon_{n,2} = \frac{(n + b)(n - N + 1)}{(n + a)(n + a + b + N + 1)},
\]

\[
\varepsilon_{n,3} = 1,
\]

\[
\varepsilon_{n,4} = -\frac{n + b}{n + a},
\]

\[
\sigma_n = -(2n + a + b - 1),
\]

These sequences define four $D$-operators:

\[
D_1 = \frac{a + b + 1}{2}I + x\nabla,
\]

\[
D_2 = \frac{a + b + 1}{2}I + (x - N)\Delta,
\]

\[
D_3 = \frac{a + b + 1}{2}I + (x + a + 1)\Delta,
\]

\[
D_4 = \frac{a + b + 1}{2}I + (x - b - N - 1)\nabla,
\]

where $\Delta(f) = f(x + 1) - f(x)$, $\nabla(f) = f(x) - f(x - 1)$.

From now on we will restrict our attention ONLY to the fourth $D$-operator, in which case we will use the notation $\varepsilon_n, \sigma_n$ and $D$. All the results can be generalize to include all the $D$-operators for the Hahn polynomials.
**THEOREM (DURÁN-MDI, 2015)**

Let \( \xi_{x,j} = \prod_{h=0}^{j-1} \varepsilon_{x-h} = (-1)^j \frac{(x-j+b+1)_j}{(x-j+a+1)_j} \) and \( Y_1, \ldots, Y_m \) arbitrary polynomials such that

\[
\Omega(n) = \det (\xi_{n-j,m-j} Y_l(\theta_{n-j}))_{l,j=1}^{m} \neq 0, \quad n \geq 0
\]

Consider the sequence of polynomials \((q_n)_n\) defined by

\[
q_n(x) = \begin{vmatrix}
 h_n^{a,b,N}(x) & -h_{n-1}^{a,b,N}(x) & \cdots & (-1)^m h_{n-m}^{a,b,N}(x) \\
\xi_{n,m} Y_1(\theta_n) & \xi_{n-1,m-1} Y_1(\theta_{n-1}) & \cdots & Y_1(\theta_{n-m}) \\
 & \vdots & & \vdots \\
\xi_{n,m} Y_m(\theta_n) & \xi_{n-1,m-1} Y_m(\theta_{n-1}) & \cdots & Y_m(\theta_{n-m})
\end{vmatrix}
\]

**REMARK:** \( q_n(x) \) is a linear combination of \( m + 1 \) consecutive Hahn polynomials. Define the rational function \( S(x) \) given by

\[
S(x) = -\sigma_{x-\frac{m-1}{2}}^{m-1} \left[ (x-m+a+1)_{m-1} \right]_m \frac{[(x-m+b+1)_{m-1}]^m}{q(x) \prod_{i=1}^m (x-m+b+1)(x-m+i+a)_{m-i}}
\]

where \( q(x) = (-1)^{\frac{m(m-1)}{2}} \prod_{p=1}^{m-1} \left( \prod_{s=1}^{p} \sigma_{x-m+s+p+1} \right) \).
THEOREM (DURÁN-MDI, 2015)

With this choice of $S(x)$ for the Hahn polynomials we get the following properties:

1. $S(x)\Omega(x)$ is a polynomial in $x$
   
   Then, define the polynomial (eigenvalue) $\lambda_x$ by
   
   $$\lambda_x - \lambda_{x-1} = S(x)\Omega(x)$$

2. There exist $\tilde{M}_1, \ldots, \tilde{M}_m$, polynomials in $x$ such that $M_h(x) = \sigma_{x+1} \tilde{M}_h(\theta_x)$, where
   
   $$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi_{x,m-j} S(x + j) \det(\xi_{x+j-r,m-r} Y_l(\theta_{x+j-r}))_{l \in I_h, r \in I_j}$$

   where $I_h = \{1, 2, \ldots, m\} \setminus \{h\}$.

3. There exists a polynomial $P_S$ such that $P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^{m} Y_h(\theta_x) M_h(x)$

   $P_S$ also satisfies $P_S(\theta_x) - P_S(\theta_{x-1}) = S(x)\Omega(x) + S(x + m)\Omega(x + m)$.

Then there exists an operator $D_{q,S} \in \mathcal{A}$ such that $D_{q,S}(q_n) = \lambda_n q_n$, $n \geq 0$, where the operator $D_{q,S}$ is defined by

$$D_{q,S} = \frac{1}{2} P_S(D_{a,b,N}) + \sum_{h=1}^{m} \tilde{M}_h(D_{a,b,N}) D Y_h(D_{a,b,N})$$
ORTHOGONALITY

Consider the \( m \)-tuple \( G \) of \( m \) positive integers, \( G = (g_1, \ldots, g_m) \) and define

\[
q^G_n(x) = \begin{vmatrix}
    h_n^{a,b,N}(x) & -h_n^{a,b,N}(x) & \cdots & (-1)^m h_n^{a,b,N}(x) \\
    \xi_{n,m} Z_{g_1}(\theta_n) & \xi_{n-1,m-1} Z_{g_1}(\theta_{n-1}) & \cdots & Z_{g_1}(\theta_{n-m}) \\
    \vdots & \vdots & \ddots & \vdots \\
    \xi_{n,m} Z_{g_m}(\theta_n) & \xi_{n-1,m-1} Z_{g_m}(\theta_{n-1}) & \cdots & Z_{g_m}(\theta_{n-m})
\end{vmatrix}
\]

The polynomials \( (q^G_n)_n \) are orthogonal with respect to certain measure \( \tilde{\rho} \) if

\[
\langle \tilde{\rho}, h_n^{a,b,N} \rangle = \sum_{i=1}^{m} A_n(G) Z_{g_i}^i(\theta_n), \quad n \geq 0,
\]

\[
0 = \sum_{i=1}^{m} B_n(G) Z_{g_i}^i(\theta_n), \quad 1 - m \leq n < 0,
\]

\[
0 \neq \sum_{i=1}^{m} C(G) Z_{g_i}^i(\theta_{-m}),
\]

This will always be possible if \( (Z_j(n))_{n \in \mathbb{Z}} \) satisfies

\[
\varepsilon_{n+1} a_{n+1} Z_j(n+1) - b_n Z_j(n) + \frac{c_n}{\varepsilon_n} Z_j(n-1) = (j+1) Z_j(n), \quad n \in \mathbb{Z}
\]

where \( (a_n)_n \), \( (b_n)_n \) and \( (c_n)_n \) are the coefficients of the TTRR for the Hahn polynomials. In this case we have that this recurrence relation is satisfied by the dual Hahn polynomial

\[
R_j^{-a,-b,-2-N}(x + a + b), \quad j \geq 0
\]
ORTHOGONALITY

For any set $F$ of positive integers, call $\max F = f_M$, and consider

$$G = \{1, 2, \ldots, f_M\} \setminus \{f_M - f, f \in F\} = (g_1, \ldots, g_m)$$

and the measure

$$\rho_{a,b,N}^F = \prod_{f \in F} (x + f_M + 1 - f)\rho_{a-f_M-1,b,N+f_M+1}(x + f_M + 1)$$

Then, for $0 \leq n \leq N + m$, the polynomials

$$q_n(x) = \begin{cases} (-1)^{j-1} h_{n+1-j}^{a,b,N}(x) & j=1,\ldots,m+1 \\ [\xi_{n-j,m-j} R_{g}^{-a,-b,-2-N}(\theta_{n-j} + a + b) & g \in G \\ g \in G \\ \right] 

are orthogonal with respect to $\rho_{a,b,N}^F$ and they are eigenfunctions of a higher-order difference operator $D_{q,S}$ of order

$$\text{ord}(D_{q,S}) = 2 \left( \sum_{f \in F} f - \binom{\#F}{2} + 1 \right)$$

**REMARK:** After a proper change of variables the measure $\rho_{a,b,N}^F$ can be transformed into the measure $\rho_{a,b,N}^F = \prod_{f \in F} (x - f)\rho_{a,b,N}$ and the conjecture of A. Durán is proved.
COMBINATION OF ALL D-OPERATORS

It is possible to combine the four $D$-operators and get the same result:

\[
q_n(x) = \begin{cases}
(-1)^{j-1} h^{a,b,N}_{n+1-j}(x) & j=1,\ldots,m+1 \\
\xi^1_{n-j,m-j} R^{b-a,a+b+N}(\theta_{n-j} + a + b) & g \in G_1 \\
\xi^2_{n-j,m-j} R^{a-b,a+b+N}(\theta_{n-j} + a + b) & g \in G_2 \\
\xi^3_{n-j,m-j} R^{b-a,-2-N}(\theta_{n-j} + a + b) & g \in G_3 \\
\xi^4_{n-j,m-j} R^{a-b,-2-N}(\theta_{n-j} + a + b) & g \in G_4
\end{cases}
\]

where $m = m_1 + m_2 + m_3 + m_4$, and $m_i$ denotes de cardinal of $G_i$.

Then, the associate measure is, for certain sets of positive integers $F_i, i = 1, 2, 3, 4$

\[
\tilde{\rho}_{a,b,N} = \prod_{f \in F_1} (b + N + 1 - f - x) \prod_{f \in F_2} (x + a + 1 - f) \\
\times \prod_{f \in F_3} (N + f - x) \prod_{f \in F_4} (x + f_{4,M} + 1 - f)\rho_{\tilde{a},\tilde{b},\tilde{N}}(x + f_{4,M} + 1)
\]

where $f_{i,M} = \max F_i$ and

\[
\tilde{a} = a - f_{2,M} - f_{4,M} - 2, \quad \tilde{b} = b - f_{1,M} - f_{3,M} - 2, \quad \tilde{N} = N + f_{3,M} + f_{4,M} + 2
\]
THANK YOU