Krall-Hahn Orthogonal Polynomials*

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OUTLINE

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Krall polynomials

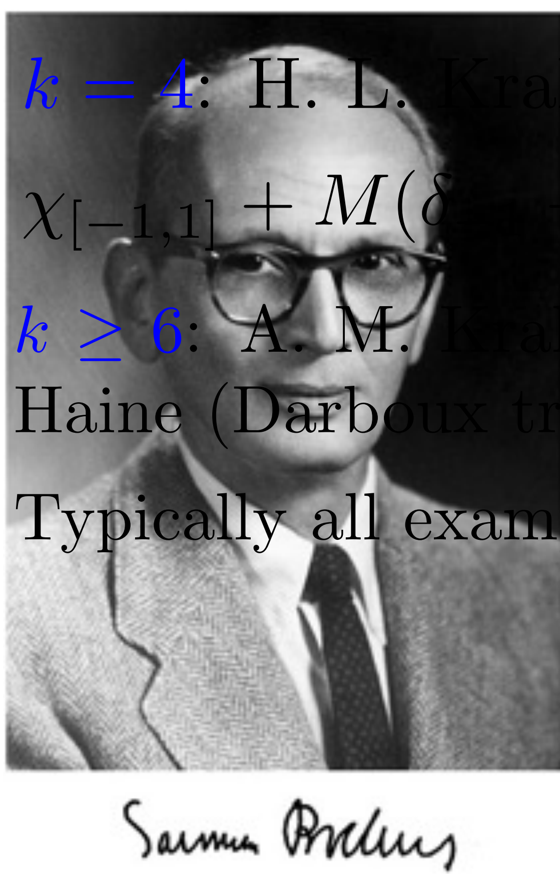
SOME HISTORY

Families of orthogonal polynomials (OPs) $(p_n)_n$ satisfying

$$D(p_n(x)) = \lambda_n p_n(x), \quad \text{where} \quad D = \sum_{l=1}^{k} f_l(x) \frac{d^l}{dx^l}$$

where $f_l, l = 1, \ldots, k$, are polynomials of degree at most l independent of n.

- $k = 2$: Legendre (18th century). Jacobi, Hermite and Laguerre (19th century). S. Bochner (1929): complete classification.



- $k = 4$: H. L. Krall (1939). k must be even and there are 3 NEW examples: $\chi_{[-1,1]} + M(\delta_{-1} + \delta_1)$, $e^{-x} + M\delta_0$ and $(1-x)^{\alpha}\chi_{[0,1]} + M\delta_0$
- $k \geq 6$: A. M. Krall, L. Littlejohn, Koekoek's, Zhedanov, Kwon, Lee, Grünbaum-Haine (Darboux transformation), Iliev, etc.

Typically all examples are orthogonal with respect to a measure of the form

$$\omega(x) + \sum_{j=0}^{m-1} a_j \delta_{x_0}^{(j)}, \quad a_j \in \mathbb{R}$$

where ω is a classical weight and x_0 is an endpoint of the $\sup(\omega)$.

Über Sturm-Liouvillesche Polynomsysteme.

Von

S. Bochner in München.

Wir betrachten irgendeine Differentialgleichung der Form

(1) $p_0(x)y'' + p_1(x)y' + p_2(x)y = \lambda y.$

Die Koeffizienten $p_0(x), p_1(x), p_2(x)$ sind irgendwelche reell- oder komplex-wertige Funktionen der Variablen x; von denen wir in erster Linie nur anzunehmen brauchen, daß sie in einem gemeinsamen Intervall J der x-Achse definiert sind; und λ bedeutet einen Parameter, der aller komplexen Werte fähig ist.

DISCRETE CASE

Families of discrete OPs $(p_n)_n$ satisfying difference equations

$$D(p_n(x)) = \lambda_n p_n(x), \quad \text{where} \quad D = \sum_{l=s}^{r} f_l(x)\mathfrak{S}_l$$

where $f_l, l = 1, \dots, k$, are polynomials independent of n and $\mathfrak{S}_l(p) = p(x+l)$.
Here $r, s \in \mathbb{Z}, r \geq s$, and the order is defined by $r - s \geq 0$.

- $r = -s = 1$: Chebychev (1858).

  Charlier, Meixner, Krawtchouk and Hahn (20th century).

  O. E. Lancaster (1941): complete classification.

- $r = -s \geq 2$: The trick of adding deltas at the endpoints of the support does not work here (Bavinck-van Haeringen-Koekoek, 1994). However it works if one considers q-difference equations (Grünbaum-Haine, 1996 and Vinet-Zhedanov, 2001).

  It was not until very recently (Durán, 2012) where the first examples appeared (he also proved that $s = -r$. Then the order must be even).
  Typically now the measures are of the form

$$\omega^F(x) = \prod_{f \in F}(x - f)\omega(x)$$

where ω is a classical discrete weight and F is a finite set of real numbers.
This is also called a Christoffel transform of ω (related to the Geronimus transform).

ORTHOGONAL POLYNOMIALS DEFINED BY DIFFERENCE EQUATIONS.*†

By Otis E. Lancaster.

1. Introduction. Many analogous properties of differential and difference equations have been studied. Here these analogies are extended to include some ideas relative to orthogonal solutions of difference equations.

Although some general theorems are given, the main study is confined to polynomial solutions of difference equations of the form

(1) $(ax^2 + bx + c)\underset{h}{\Delta^2} y(x) + (dx + f)\underset{h}{\Delta} y(x) + \lambda y(x+h) = 0$

where $h > 0$ is the interval of difference, a, b, c, d and f are constants and λ is a parameter which is determined so as to insure polynomial solutions.[1]

FIRST EXAMPLES: DURÁN (2012)

Let $\omega_a(x) = \sum_{x\in\mathbb{N}} \frac{a^x}{x!}\delta_x, a > 0$ be the Charlier weight

- $F = \{1\}$: $\omega^F(x) = (x-1)\omega_a(x) = -\delta_0 + \sum_{x=2}^{\infty} \frac{a^x}{x(x-2)!}\delta_x, \quad a > 0$

Then, the corresponding OPs are eigenfunctions of a fourth-order difference operator

$$D = x(x-3)\mathfrak{S}_{-2} - 2x(x-2)\mathfrak{S}_{-1} + x(x+2a-1)\mathfrak{S}_0 - 2xa\mathfrak{S}_1 + a^2\mathfrak{S}_2, \quad a \neq 1, 2, \ldots$$

- $F = \{1, 2\}$: $\omega^F(x) = (x-1)(x-2)\omega_a(x) = 2\delta_0 + \sum_{x=3}^{\infty} \frac{a^x}{x(x-3)!}\delta_x, \quad a > 0$

Then, the corresponding OPs are eigenfunctions of a sixth-order difference operator

$$\begin{aligned} D =& 2x\binom{x-4}{2}\mathfrak{S}_{-3} - 6x\binom{x-3}{2}\mathfrak{S}_{-2} + 3x(x-3)(x+a-2)\mathfrak{S}_{-1} \\ & - x(x-2)(x+6a-1)\mathfrak{S}_0 + 3ax(x+a-1)\mathfrak{S}_1 - 3a^2x\mathfrak{S}_2 + a^3\mathfrak{S}_3 \end{aligned}$$

The same can be done with Meixner, Krawtchouk and Hahn families.

FIRST EXAMPLES

If we call $\mathfrak{p}_F(x) = \prod_{f\in F}(x-f)$, then it is possible to generate examples of OPs with respect to $\omega^F = \mathfrak{p}_F\omega$ satisfying higher-order difference equations in the following situations:

$$\mathfrak{p}_F(x), \quad \mathfrak{p}_{-c-F}(x), \quad \breve{\mathfrak{p}}_{N-1-F}(x), \quad \breve{\mathfrak{p}}_{\alpha+F}(x)$$

where $\breve{p}(x) = p(-x)$, $c, \alpha, N \in \mathbb{R}$ and F is a finite set of positive integers.

Positivity: If $F = \bigcup F_i$ where F_i contains consecutive nonnegative integers, then ω^F is positive if and only if the cardinal of each F_i is even for all i.

In all the examples gave by Durán in 2012, the sets F contains consecutive positive integers. A number of conjectures were proposed stating that this can always be done for any set F (not necessarilly consecutive). Moreover, the order of the higher-order difference operator can be explicitly computed.

Some of these conjectures (for the Charlier, Meixner and Krawtchouk families) were recently proved by Durán-MdI in 2014. In these cases the corresponding eigenvalue of the second-order difference equation is linear in n.

More recently, in 2015, Durán-MdI worked out the Hahn case, where now the difficulty comes in the fact that the corresponding eigenvalue of the second-order difference equation is nonlinear in n (quadratic).

Krall-Hahn polynomials. An example

HAHN POLYNOMIALS

For $a, a+b+1, a+b+N+1 \neq -1, -2, \ldots$ we consider the Hahn polynomials defined by

$$h_n^{a,b,N}(x) = \sum_{j=0}^{n} \frac{(-x)_j (N-n+1)_{n-j} (a+b+1)_{j+n}}{(2+a+b+N)_n (a+1)_j (n-j)! j!}, \quad n \geq 0$$

Hahn polynomials are eigenfunctions of the second-order difference operator

$$D_{a,b,N} = x(x-b-N-1)\mathfrak{s}_{-1} - \big[(x+a+1)(x-N) + x(x-b-N-1)\big]\mathfrak{s}_0 + (x+a+1)(x-N)\mathfrak{s}_1$$

That is

$$D_{a,b,N}(h_n^{a,b,N}) = \theta_n h_n^{a,b,N}, \quad \theta_n = n(n+a+b+1), \quad n \geq 0$$

They satisfy the following three-term recurrence formula (TTRR) ($h_{-1}^{a,b,N} = 0$)

$$x h_n^{a,b,N} = a_{n+1} h_{n+1}^{a,b,N} + b_n h_n^{a,b,N} + c_n h_{n-1}^{a,b,N}, \quad n \geq 0,$$

where

$$a_n = -\frac{n(n+a)(n+a+b+N+1)}{(2n+a+b-1)(2n+a+b)},$$

$$b_n = \frac{N(a+1)(a+b) + n(2N+b-a)(n+a+b+1)}{(2n+a+b)(2n+a+b+2)},$$

$$c_n = -\frac{(n+a+b)(n+b)(N-n+1)}{(2n+a+b)(2n+a+b+1)}$$

HAHN POLYNOMIALS

When $N+1$ is a positive integer, $a, b \neq -1, -2, \ldots, -N$, and $a+b \neq -1, -2, \ldots, -2N-1$, the first $N+1$ Hahn polynomials are orthogonal with respect to the Hahn measure

$$\rho_{a,b,N} = N! \sum_{x=0}^{N} \frac{\Gamma(a+x+1)\Gamma(N-x+b+1)}{x!(N-x)!} \delta_x$$

The discrete measure $\rho_{a,b,N}$ is positive only when $a, b > -1$ or $a, b < -N$.

We also need the so-called dual Hahn polynomials, $a \neq -1, -2, \ldots$

$$R_n^{a,b,N}(x) = \sum_{j=0}^{n} \frac{(-1)^j (-n)_j (-N+j)_{n-j}}{(a+1)_j j!} \prod_{i=0}^{j-1} [x - i(i+a+b+1)], \quad n \geq 0$$

Observe that $(-1)^j \prod_{i=0}^{j-1} (x(x+a+b+1) - i(a+b+1+i)) = (-x)_j (x+a+b+1)_j$, therefore we have the duality

$$R_x^{a,b,N}(n(n+a+b+1)) = \frac{(-1)^n n! (N+a+b+2)_n (-N)_x}{(a+b+1)_n (-N)_n} h_n^{a,b,N}(x), \quad x, n \geq 0$$

AN EXPLICIT EXAMPLE

Let $F=\{1,3\}$ and consider the discrete weight

$$\rho^F_{a,b,N}(x)=(x-1)(x-3)\rho_{a,b,N}(x)$$

Now consider the change of variables

$$a\to a-4,\quad b\to b,\quad N\to N+4,\quad x\to x+4$$

The new weight is given by

$$\tilde\rho^F_{a,b,N}(x)=(x+3)(x+1)\rho_{a-4,b,N+4}(x+4),\quad x=-4,-2,0,1,\ldots,N$$

For this new weight we try to write its OPs $(q_n)_n$ as the following Casorati determinant

$$q_n(x)=\begin{vmatrix} h_n^{a,b,N}(x) & h_{n-1}^{a,b,N}(x) & h_{n-2}^{a,b,N}(x)\\ \xi_{n,2}Y_1(\theta_n) & \xi_{n-1,1}Y_1(\theta_{n-1}) & Y_1(\theta_{n-2})\\ \xi_{n,2}Y_2(\theta_n) & \xi_{n-1,1}Y_2(\theta_{n-1}) & Y_2(\theta_{n-2})\end{vmatrix}$$

where $\xi_{x,j}=(-1)^j\frac{(x-j+1)_j}{(x-j+a+1)_j}$ and Y_1,Y_2 are certain polynomials.

The key to prove that $\langle\tilde\rho^F,x^jq_n\rangle=0$ for $0\le j\le n-1$ and $\langle\tilde\rho^F,x^nq_n\rangle\neq 0$ is

$$\langle\tilde\rho^F,h_n^{a,b,N}\rangle=A_1(n,F)Y_1(\theta_n)+A_2(n,F)Y_2(\theta_n),\quad n\ge 0,$$
$$0=B_1(F)Y_1(\theta_{-1})+B_2(F)Y_1(\theta_{-1}),$$
$$0=C_1(F)Y_1(\theta_{-2})+C_2(F)Y_2(\theta_{-2}),$$

for certain sequences $A_i(n,F)$ and constants $B_i(F),C_i(F)$, $i=1,2$.

This happens when Y_1,Y_2 satisfy the following recurrence relations (as a function of n)

$$\varepsilon_{n+1}a_nY_j(n+1)-b_nY_j(n)+\frac{c_n}{\varepsilon_n}Y_j(n-1)=(j+1)Y_j(n)$$

where $\varepsilon_n=-\frac{n+b}{n+a}$ and a_n,b_n,c_n are the coefficients of the TTRR for the Hahn family.

The solutions are given by the dual Hahn polynomials $R_j^{-a,-b,-2-N}(n+a+b)$.

AN EXPLICIT EXAMPLE

The right choice Y_1, Y_2 comes from the following involution of the the set $F = \{1, 3\}$:

$$I(F) = \{1, 2, \dots, \max F\} \setminus \{\max F - f, f \in F\}$$

In this case $I(F) = \{1, 3\}$. That means that

$$Y_1(x) = R_1^{-a,-b,-2-N}(x + a + b), \quad Y_2(x) = R_3^{-a,-b,-2-N}(x + a + b)$$

The OPs $(q_n)_n$ with respect to $\tilde{\rho}^F$ are then given by

$$q_n = \begin{vmatrix} h_n^{a,b,N}(x) & -h_{n-1}^{a,b,N}(x) & h_{n-2}^{a,b,N}(x) \\ \xi_{n,2} R_1^{-a,-b,-2-N}(\theta_n + a + b) & \xi_{n-1,1} R_1^{-a,-b,-2-N}(\theta_{n-1} + a + b) & R_1^{-a,-b,-2-N}(\theta_{n-2} + a + b) \\ \xi_{n,2} R_3^{-a,-b,-2-N}(\theta_n + a + b) & \xi_{n-1,1} R_3^{-a,-b,-2-N}(\theta_{n-1} + a + b) & R_3^{-a,-b,-2-N}(\theta_{n-2} + a + b) \end{vmatrix}$$

Now that we have the idea of constructing the polynomials $(q_n)_n$ let us compute the explicit expression of the higher-order difference operator for which the polynomials $(q_n)_n$ are eigenfunctions and the corresponding order.

AN EXPLICIT EXAMPLE

The higher-order difference operator can be written as

$$D_q = \frac{1}{2}P(D_{a,b,N}) + \tilde{M}_1(D_{a,b,N})\mathcal{D}R_1^{-a,-b,-2-N}(D_{a,b,N}) + \tilde{M}_2(D_{a,b,N})\mathcal{D}R_3^{-a,-b,-2-N}(D_{a,b,N})$$

Here P, $\tilde{M}_1$ and $\tilde{M}_2$ are certain polynomials in θ_x (we will explain later how to calculate them) and $\mathcal{D}$ is what is called a $\mathcal{D}$-operator for the Hahn family:

$$\mathcal{D} = \frac{a+b+1}{2}I + (x-b-N-1)\nabla$$

where $\nabla(f) = (\mathfrak{S}_0 - \mathfrak{S}_{-1})f = f(x) - f(x-1)$.

For instance, if we fix the parameters $a = 1/2, b = -1/2$ and $N = 9$, then we have

$$P(x) = \frac{2}{70785}x(2x^3 + 192x^2 + 3925x + 34656)$$

$$\tilde{M}_1(x) = -\frac{4}{2145}x^2 - \frac{84}{715}x - \frac{12}{11}$$

$$\tilde{M}_2(x) = \frac{4}{11}$$

order(D_q)=8

and the OPs $(q_n)_n$ are eigenfunctions of D_q, i.e. $D_q(q_n) = \lambda_n q_n$, where

$$\lambda_n = \frac{2}{70785}n(n+1)(2n^6 + 6n^5 + 198n^4 + 386n^3 + 4117n^2 + 3925n + 34656)$$

Krall-Hahn polynomials. General case

GENERAL CASE

In order to study the Krall-Hahn OPs $(q_n)_n$ we will follow the following guideline:

1. Computation of the higher-order difference operator

This result will be valid for any family of polynomials $(p_n)_n$ (not necessarily orthogonal) eigenfunctions of certain operator, i.e. $D(p_n) = \theta_n p_n$. It is based on the abstract concept of $\mathcal{D}$-operator and there will be $Y_1, \ldots, Y_m$ arbitrary polynomials.

2. Orthogonality

Only for a convenient choice of the polynomials $Y_1, \ldots, Y_m$, the polynomials $(q_n)_n$ are also orthogonal with respect to a measure. We already have a clue on how to construct this measure in terms of Christoffel transforms of the Hahn weight. The right choice of the polynomials $Y_1, \ldots, Y_m$ will be certain families of dual Hahn polynomials.

D-OPERATORS

Let $\mathcal{A}$ be an algebra of difference operators acting in the linear space of polynomials $\mathbb{P}$

$$\mathcal{A} = \left\{ \sum_{l=s}^{r} h_l \mathfrak{S}_l : h_l \in \mathbb{P}, l = s, \dots, r, s \leq r \right\}$$

We will work with the Hahn polynomials $(h_n^{a,b,N})_n$ for which we know that there exists a second-order difference operator $D_{a,b,N} \in \mathcal{A}$ such that

$$D_{a,b,N}(h_n^{a,b,N}) = \theta_n h_n^{a,b,N}, \quad \theta_n = n(n+a+b+1)$$

Given two sequences of numbers $(\varepsilon_n)_n$ and $(\sigma_n)_n$, a $\mathcal{D}$-operator (of type 2) associated to the algebra $\mathcal{A}$ and the Hahn polynomials is defined as

$$\mathcal{D}(h_n^{a,b,N}) = -\frac{1}{2}\sigma_{n+1} h_n^{a,b,N} + \sum_{j=1}^{n} (-1)^{j+1} \sigma_{n-j+1} \varepsilon_n \cdots \varepsilon_{n-j+1} h_{n-j}^{a,b,N}, \quad n \geq 0$$

We then say that $\mathcal{D}$ is a $\mathcal{D}$-operator if $\mathcal{D} \in \mathcal{A}$.

D-OPERATORS

There are four different $\mathcal{D}$-operators for the Hahn polynomials. They are defined by the sequences $(\varepsilon_{n,h})_n$ and $(\sigma_n)_n$, $h = 1, 2, 3, 4$, given by

$$\varepsilon_{n,1} = -\frac{n-N+1}{n+a+b+N+1}, \qquad \sigma_n = -(2n+a+b-1),$$

$$\varepsilon_{n,2} = \frac{(n+b)(n-N+1)}{(n+a)(n+a+b+N+1)}, \qquad \sigma_n = -(2n+a+b-1),$$

$$\varepsilon_{n,3} = 1, \qquad \sigma_n = -(2n+a+b-1),$$

$$\varepsilon_{n,4} = -\frac{n+b}{n+a}, \qquad \sigma_n = -(2n+a+b-1).$$

These sequences define four $\mathcal{D}$-operators:

$$\mathcal{D}_1 = \frac{a+b+1}{2}I + x\nabla, \qquad \mathcal{D}_2 = \frac{a+b+1}{2}I + (x-N)\Delta,$$

$$\mathcal{D}_3 = \frac{a+b+1}{2}I + (x+a+1)\Delta, \qquad \mathcal{D}_4 = \frac{a+b+1}{2}I + (x-b-N-1)\nabla,$$

where $\Delta(f) = f(x+1) - f(x), \nabla(f) = f(x) - f(x-1)$.

From now on we will restrict our attention ONLY to the fourth $\mathcal{D}$-operator, in which case we will use the notation ε_n, σ_n and $\mathcal{D}$. All the results can be generalize to include all the $\mathcal{D}$-operators for the Hahn polynomials.

THEOREM (DURÁN-MDI, 2015)

Let $\xi_{x,j} = \prod_{h=0}^{j-1} \varepsilon_{x-h} = (-1)^j \frac{(x-j+b+1)_j}{(x-j+a+1)_j}$ and $Y_1, \ldots, Y_m$ arbitrary polynomials such that

$$\boxed{\Omega(n) = \det\left(\xi_{n-j,m-j} Y_l(\theta_{n-j})\right)_{l,j=1}^{m} \neq 0}, \quad n \geq 0$$

Consider the sequence of polynomials $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} h_n^{a,b,N}(x) & -h_{n-1}^{a,b,N}(x) & \cdots & (-1)^m h_{n-m}^{a,b,N}(x) \\ \xi_{n,m} Y_1(\theta_n) & \xi_{n-1,m-1} Y_1(\theta_{n-1}) & \cdots & Y_1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m} Y_m(\theta_n) & \xi_{n-1,m-1} Y_m(\theta_{n-1}) & \cdots & Y_m(\theta_{n-m}) \end{vmatrix}$$

REMARK: $q_n(x)$ is a linear combination of $m+1$ consecutive Hahn polynomials.
Define the rational function $S(x)$ given by

$$S(x) = -\sigma_{x-\frac{m-1}{2}} \frac{\left[(x-m+a+1)_{m-1}\right]^m}{q(x) \displaystyle\prod_{i=1}^{m-1} (x-m+b+1)_{m-i}(x-m+i+a)_{m-i}}$$

where $q(x) = (-1)^{\frac{m(m-1)}{2}} \displaystyle\prod_{p=1}^{m-1} \left(\prod_{s=1}^{p} \sigma_{x-m+\frac{s+p+1}{2}} \right).$

THEOREM (DURÁN-MDI, 2015)

With this choice of $S(x)$ for the Hahn polynomials we get the following properties

1. $S(x)\Omega(x)$ is a polynomial in x

Then, define the polynomial (eigenvalue) λ_x by

$$\lambda_x - \lambda_{x-1} = S(x)\Omega(x)$$

2. There exist $\tilde{M}_1, \dots, \tilde{M}_m$, polynomials in x such that $M_h(x) = \sigma_{x+1}\tilde{M}_h(\theta_x)$, where

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi_{x,m-j} S(x+j) \det \left(\xi_{x+j-r,m-r} Y_l(\theta_{x+j-r}) \right)_{l \in \mathbb{I}_h; r \in \mathbb{I}_j}$$

where $\mathbb{I}_h = \{1, 2, \dots, m\} \setminus \{h\}$.

3. There exists a polynomial P_S such that $P_S(\theta_x) = 2\lambda_x + \sum_{h=1}^{m} Y_h(\theta_x) M_h(x)$

P_S also satisfies $P_S(\theta_x) - P_S(\theta_{x-1}) = S(x)\Omega(x) + S(x+m)\Omega(x+m)$.

Then there exists an operator $D_{q,S} \in \mathcal{A}$ such that $D_{q,S}(q_n) = \lambda_n q_n, \quad n \geq 0$, where the operator $D_{q,S}$ is defined by

$$D_{q,S} = \frac{1}{2} P_S(D_{a,b,N}) + \sum_{h=1}^{m} \tilde{M}_h(D_{a,b,N}) \mathcal{D} Y_h(D_{a,b,N})$$

ORTHOGONALITY

Consider the m-tuple G of m positive integers, $G = (g_1, \dots, g_m)$ and define

$$q_n^G(x) = \begin{vmatrix} h_n^{a,b,N}(x) & -h_{n-1}^{a,b,N}(x) & \cdots & (-1)^m h_{n-m}^{a,b,N}(x) \\ \xi_{n,m} Z_{g_1}(\theta_n) & \xi_{n-1,m-1} Z_{g_1}(\theta_{n-1}) & \cdots & Z_{g_1}(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m} Z_{g_m}(\theta_n) & \xi_{n-1,m-1} Z_{g_m}(\theta_{n-1}) & \cdots & Z_{g_m}(\theta_{n-m}) \end{vmatrix}$$

The polynomials $(q_n^G)_n$ are orthogonal with respect to certain measure $\tilde{\rho}$ if

$$\begin{aligned} \langle \tilde{\rho}, h_n^{a,b,N} \rangle &= \sum_{i=1}^m A_n(G) Z_{g_i}^i(\theta_n), \quad n \geq 0, \\ 0 &= \sum_{i=1}^m B_n(G) Z_{g_i}^i(\theta_n), \quad 1 - m \leq n < 0, \\ 0 &\neq \sum_{i=1}^m C(G) Z_{g_i}^i(\theta_{-m}), \end{aligned}$$

This will always be possible if $(Z_j(n))_{n \in \mathbb{Z}}$ satisfies

$$\varepsilon_{n+1} a_{n+1} Z_j(n+1) - b_n Z_j(n) + \frac{c_n}{\varepsilon_n} Z_j(n-1) = (j+1) Z_j(n), \quad n \in \mathbb{Z}$$

where $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ are the coefficients of the TTRR for the Hahn polynomials.
In this case we have that this recurrence relation is satisfied by the dual Hahn polynomial

$$R_j^{-a,-b,-2-N}(x + a + b), \quad j \geq 0$$

ORTHOGONALITY

For any set F of positive integers, call $\max F = f_M$, and consider

$$G = \{1, 2, \dots, f_M\} \setminus \{f_M - f, f \in F\} = (g_1, \dots, g_m)$$

and the measure

$$\tilde{\rho}^F_{a,b,N} = \prod_{f\in F} (x + f_M + 1 - f)\rho_{a-f_M-1,b,N+f_M+1}(x + f_M + 1)$$

Then, for $0 \leq n \leq N + m$, the polynomials

$$q_n(x) = \left| \begin{array}{l} (-1)^{j-1} h^{a,b,N}_{n+1-j}(x) \qquad\qquad {}_{j=1,\dots,m+1} \\ \begin{bmatrix} \xi_{n-j,m-j} R_g^{-a,-b,-2-N}(\theta_{n-j} + a + b) \\ g \in G \end{bmatrix} \end{array} \right|$$

are orthogonal with respect to $\tilde{\rho}^F_{a,b,N}$ and they are eigenfunctions of a higher-order difference operator $D_{q,S}$ of order

$$\operatorname{ord}(D_{q,S}) = 2\left(\sum_{f\in F} f - \binom{\#F}{2} + 1\right)$$

REMARK: After a proper change of variables the measure $\tilde{\rho}^F_{a,b,N}$ can be transformed into the measure $\rho^F_{a,b,N} = \prod_{f\in F}(x - f)\rho_{a,b,N}$ and the conjecture of A. Durán is proved.

COMBINATION OF ALL D-OPERATORS

It is possible to combine the four $\mathcal{D}$-operators and get the same result:

$$q_n(x) = \left| \begin{array}{l} (-1)^{j-1} h^{a,b,N}_{n+1-j}(x) \qquad\qquad\qquad {}_{j=1,\dots,m+1} \\ \left[\begin{array}{l} \xi^1_{n-j,m-j} R_g^{-b,-a,a+b+N}(\theta_{n-j}+a+b) \\ g \in G_1 \end{array}\right] \\ \left[\begin{array}{l} \xi^2_{n-j,m-j} R_g^{-a,-b,a+b+N}(\theta_{n-j}+a+b) \\ g \in G_2 \end{array}\right] \\ \left[\begin{array}{l} \xi^3_{n-j,m-j} R_g^{-b,-a,-2-N}(\theta_{n-j}+a+b) \\ g \in G_3 \end{array}\right] \\ \left[\begin{array}{l} \xi^4_{n-j,m-j} R_g^{-a,-b,-2-N}(\theta_{n-j}+a+b) \\ g \in G_4 \end{array}\right] \end{array} \right|$$

where $m = m_1 + m_2 + m_3 + m_4$, and m_i denotes de cardinal of G_i.

Then, the associate measure is, for certain sets of positive integers $F_i, i = 1, 2, 3, 4$

$$\begin{aligned} \tilde{\rho}_{a,b,N} = & \prod_{f \in F_1} (b + N + 1 - f - x) \prod_{f \in F_2} (x + a + 1 - f) \\ & \times \prod_{f \in F_3} (N + f - x) \prod_{f \in F_4} (x + f_{4,M} + 1 - f) \rho_{\tilde{a},\tilde{b},\tilde{N}}(x + f_{4,M} + 1) \end{aligned}$$

where $f_{i,M} = \max F_i$ and

$$\tilde{a} = a - f_{2,M} - f_{4,M} - 2, \quad \tilde{b} = b - f_{1,M} - f_{3,M} - 2, \quad \tilde{N} = N + f_{3,M} + f_{4,M} + 2$$


THANK YOU