Sobre algunas extensiones de polinomios ortogonales y sus aplicaciones

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Outline

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1. Classical orthogonal polynomials
2. Krall orthogonal polynomials
3. Orthogonal matrix polynomials

The space $L^2_\omega(\mathcal{S})$

Let ω be a positive measure on $\mathcal{S} \subset \mathbb{R}$ and consider the space of functions $L^2_\omega(\mathcal{S})$ with the inner product

$$\langle f, g\rangle_\omega = \int_{\mathcal{S}} f(x)g(x)d\omega(x)$$

We say that $f \in L^2_\omega(\mathcal{S})$ if $\langle f, f\rangle_\omega = \|f\|^2_\omega < \infty$.

$\mathcal{S}$ can be a continuous interval, a discrete set of points or a combination of both. The discrete component of the measure is usually written as

$$\omega_d(x) = \sum_{x=0}^{N} a_x \delta_{t_x}, \quad t_{x_0}, \ldots, t_{x_N} \in \mathbb{R}$$

In that case the inner product can be thought of as

$$\langle f, g\rangle_{\omega_d} = \sum_{x=0}^{N} a_x \int f(x)g(x)\delta_{t_x} = \sum_{x=0}^{N} a_x f(t_x)g(t_x)$$

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Orthogonal polynomials

A system of polynomials $(p_n)_n = \{p_0(x), p_1(x), \ldots\}$ with $\deg(p_n) = n$ is orthogonal in $L^2_\omega(\mathcal{S})$ if (Gramm-Schmidt)

$$\langle p_n, p_m \rangle_\omega = \int_{\mathcal{S}} p_n(x) p_m(x) d\omega(x) = \|p_n\|^2_\omega \delta_{nm}, \quad n, m \geq 0$$

Every family of OP's $(p_n)_n$ satisfy a three-term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \geq 1$$

where $a_n, c_n \neq 0$, $b_n \in \mathbb{R}$ and $p_0(x) = 1, p_{-1}(x) = 0$.
Jacobi operator (tridiagonal):

$$Jp = \begin{pmatrix} b_0 & a_1 & & & \\ c_1 & b_1 & a_2 & & \\ & c_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = xp, \quad x \in \mathcal{S}$$

The converse result is also true (Favard's or spectral theorem)

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Classical families (continuous case)

Bochner (1929)

$$\sigma(x)\frac{d^2}{dx^2}p_n(x)+\tau(x)\frac{d}{dx}p_n(x)+\lambda_n p_n(x)=0,\quad x\in\mathcal{S}\subset\mathbb{R}$$

$\deg\sigma\leq 2,\quad \deg\tau=1$

- Hermite (Normal, Gaussian): $\omega(x)=e^{-x^2}$, $x\in\mathbb{R}$

$$H_n(x)''-2xH_n(x)'=-2nH_n(x)$$

- Laguerre (Gamma, Exponential): $\omega(x)=x^\alpha e^{-x}$, $x>0$, $\alpha>-1$

$$xL_n^\alpha(x)''+(\alpha+1-x)L_n^\alpha(x)'=-nL_n^\alpha(x)$$

- Jacobi (Beta, Uniform): $\omega(x)=x^\alpha(1-x)^\beta$, $x\in(0,1)$, $\alpha,\beta>-1$

$$x(1-x)P_n^{(\alpha,\beta)}(x)''+(\alpha+1-(\alpha+\beta+2)x)P_n^{(\alpha,\beta)}(x)'=$$
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Classical families (discrete case)

If we set

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1)$$

the classification problem is to find discrete OP's $(p_n)_n$

Lancaster (1941)

$$\sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0, \quad x \in \mathcal{S} \subset \mathbb{N}$$
$$\deg \sigma \leq 2, \quad \deg \tau = 1$$

In other words, if we call the shift operator

$$\mathfrak{S}_j f(x) = f(x+j)$$

the difference equation reads

$$[\sigma(x) + \tau(x)]\mathfrak{S}_1 p_n(x) - [2\sigma(x) + \tau(x)]\mathfrak{S}_0 p_n(x)$$
$$+ \sigma(x)\mathfrak{S}_{-1} p_n(x) + \lambda_n p_n(x) = 0, \quad x \in \mathcal{S} \subset \mathbb{N}$$

Classical families

- Charlier (Poisson): $\mathcal{S} = \{0, 1, 2, \dots\}$.

$$\omega_a(x) = \sum_{x=0}^{\infty} \frac{a^x}{x!}\delta_x, \quad a > 0$$

$$ac_n^a(x+1) - (x+a)c_n^a(x) + xc_n^a(x-1) = -nc_n^a(x)$$

- Meixner (Pascal, Geometric): $\mathcal{S} = \{0, 1, 2, \dots\}$.

$$\omega_{a,c}(x) = \Gamma(c)(1-a)^c \sum_{x=0}^{\infty} \frac{(c)_x a^x}{x!}\delta_x, \quad 0 < a < 1, \quad c > 0$$

$(i)_j = i(i+1)\cdots(i+j-1)$ is the Pochhammer symbol

$$a(x+c)m_n^{a,c}(x+1) - (x + a(x+c))m_n^{a,c}(x) + xm_n^{a,c}(x-1) = n(a-1)m_n^{a,c}(x)$$

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Classical families

- Krawtchuok (Binomial, Bernoulli): $\mathcal{S} = \{0, 1, 2, \dots N-1\}$.

$$\omega_{a,N}(x) = \frac{1}{(1+a)^{N-1}} \sum_{x=0}^{N-1} \binom{N-1}{x} a^x \delta_x, \quad a > 0$$

$$a(N-x-1)k_n^{a,N}(x+1) - [x + a(N-x-1)]k_n^{a,N}(x) + xk_n^{a,N}(x-1) = -n(1+a)k_n^{a,N}(x)$$

- Hahn (Hypergeometric): $\mathcal{S} = \{0, 1, 2, \dots N\}$

$$\omega_{a,b,N}(x) = N! \sum_{x=0}^{N} \binom{a+x}{x} \binom{b+N-x}{N-x} \delta_x, \quad a, b > -1, \quad a, b < -N$$

$$B(x)h_n^{a,b,N}(x+1) - [B(x) + D(x)]h_n^{a,b,N}(x) + D(x)h_n^{a,b,N}(x-1) = n(n+a+b+1)h_n^{a,b,N}(x)$$

where $B(x) = (x+a+1)(x-N)$ and $D(x) = x(x-b-N-1)$.

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EXTENSIONS

- q-polynomials.
- OP's on the unit circle (or any other curves).
- Krall orthogonal polynomials.
- Sobolev type orthogonal polynomials.
- Matrix-valued orthogonal polynomials.
- Multiple orthogonal polynomials.
- Multivariate orthogonal polynomials.
- $\cdots$

Outline

Krall polynomials (continuous case)

GOAL (H.L. Krall, 1939): find families of OP's $(q_n)_n$ which are also eigenfunctions of a higher-order differential operator of the form

$$D_c = \sum_{j=0}^{2m} h_j(x) \frac{d^j}{dx^j}, \quad \deg(h_j) \leq j \quad \Rightarrow \quad D_c(q_n) = \lambda_n q_n$$

A.M. Krall, Littlejohn, Koornwinder, Koekoek's, Lesky, Grünbaum, Heine, Iliev, Horozov, Zhedanov, etc (80's, 90's, 00's).

$(q_n)_n$ are typically orthogonal with respect to the measure

$$\omega(x) + \sum_{j=0}^{m-1} a_j \delta_{x_0}^{(j)}, \quad a_j \in \mathbb{R}$$

where ω is a (modified) classical weight and x_0 is an endpoint of the support of orthogonality of ω.

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Krall polynomials (discrete case)

The same question arise in the discrete setting, i.e. find families of OP's $(q_n)_n$ which are also eigenfunctions of a higher order difference operator

$$D_d = \sum_{j=r}^{s} h_j(x)\mathfrak{S}_j, \quad h_s, h_r \neq 0, \quad \Rightarrow \quad D_d(q_n) = \lambda_n q_n$$

Bavinck-van Haeringen-Koekoek, 1994: adding deltas at the endpoints of the support does not work (infinite order difference operator).

Surprisingly, it has not been until very recently (Durán, 2012) when the first examples appeared. Also $s - r = 2m$.

$(q_n)_n$ are typically orthogonal with respect to the measure

$$\omega^F(x) = \prod_{f \in F}(x - f)\, \omega(x)$$

where ω is a discrete classical weight and F is a finite set of numbers. This is also called a Christoffel transform of ω.

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Conjectures (Durán, 2012)

For a finite set F consider $r_F = \sum_{f\in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$.

Conjecture A: Let ω_a be the Charlier weight and consider (F finite)

$$\omega_a^F = \prod_{f\in F}(x-f)\omega_a$$

The OP's $(q_n)_n$ with respect to ω_a^F are eigenfunctions of a higher-order difference operator with $-s = r = r_F$.

Conjecture B: Let $\omega_{a,c}$ be the Meixner weight and consider (F_1, F_2 finite)

$$\omega_{a,c}^{F_1,F_2} = \prod_{f\in F_1}(x+c+f)\prod_{f\in F_2}(x-f)\omega_{a,c}$$

The OP's $(q_n)_n$ with respect to $\omega_a^{F_1,F_2}$ are eigenfunctions of a higher-order difference operator with $-s = r = r_{F_1} + r_{F_2} - 1$.

Conjecture C: Let $\omega_{a,N}$ be the Krawtchouk weight and consider (F_1, F_2 finite)

$$\omega_{a,N}^{F_1,F_2} = \prod_{f\in F_1}(x-f)\prod_{f\in F_2}(N-1-f-x)\omega_{a,N}$$

The OP's $(q_n)_n$ with respect to $\omega_a^{F_1,F_2}$ are eigenfunctions of a higher-order difference operator with $-s = r = r_{F_1} + r_{F_2} - 1$.

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For a finite set F consider $r_F = \sum_{f\in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$.

Conjecture A: Let ω_a be the Charlier weight and consider (F finite)

$$\omega_a^F = \prod_{f\in F}(x-f)\omega_a$$

The OP's $(q_n)_n$ with respect to ω_a^F are eigenfunctions of a higher-order difference operator with $-s = r = r_F$.

Conjecture B: Let $\omega_{a,c}$ be the Meixner weight and consider (F_1, F_2 finite)

$$\omega_{a,c}^{F_1,F_2} = \prod_{f\in F_1}(x+c+f)\prod_{f\in F_2}(x-f)\omega_{a,c}$$

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Conjecture C: Let $\omega_{a,N}$ be the Krawtchouk weight and consider (F_1, F_2 finite)

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$\mathcal{D}$-OPERATORS

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $\mathcal{D}$ is an $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

- Laguerre: $\varepsilon_n = -1 \Rightarrow \mathcal{D} = \dfrac{d}{dx}$.
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- Meixner:
  $$\varepsilon_n^1 = \frac{a}{1-a} \Rightarrow \mathcal{D}_1 = \frac{a}{1-a}\Delta, \quad \varepsilon_n^2 = \frac{1}{1-a} \Rightarrow \mathcal{D}_2 = \frac{1}{1-a}\nabla.$$
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$\mathcal{D}$-OPERATORS

THEOREM (DURÁN, 2013)

Let $\mathcal{A}$, $(p_n)_n$, $D_p(p_n) = np_n$, $(\varepsilon_n)_n$ and $\mathcal{D}$.
For an arbitrary polynomial R such that $R(n) \neq 0$, $n \geq 0$, we define a new polynomial P by

$$P(x) - P(x-1) = R(x)$$

and a sequence of polynomials $(q_n)_n$ by $q_0 = 1$ and

$$q_n = p_n + \beta_n p_{n-1}, \quad n \geq 1$$

where the numbers β_n, $n \geq 0$, are given by

$$\beta_n = \varepsilon_n \frac{R(n)}{R(n-1)}, \quad n \geq 1$$

Then there exist $D_q \in \mathcal{A}$ sucht that $D_q(q_n) = P(n)q_n$ where

$$D_q = P(D_p) + \mathcal{D}R(D_p)$$

$\mathcal{D}$-OPERATORS

GOAL: Extend the previous Theorem for the case that we consider a linear combination of $m+1$ consecutive p_n's:

$$q_n = p_n + \beta_{n,1}p_{n-1} + \beta_{n,2}p_{n-2} + \cdots + \beta_{n,m}p_{n-m}$$

Let $R_1, R_2, \ldots, R_m$ be m arbitrary polynomials and m $\mathcal{D}$-operators $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ defined by the sequences $(\varepsilon_n^h)_n$, $h = 1, \ldots, m$.

Define the auxiliary functions $\xi_{n,i}^h$ by

$$\xi_{n,i}^h = \varepsilon_n^h \varepsilon_{n-1}^h \cdots \varepsilon_{n-i+1}^h$$

and assume that the following Casorati determinant never vanish ($n \geq 0$)

$$\Omega(n) = \begin{vmatrix} \xi_{n-1,m-1}^1 R_1(n-1) & \xi_{n-2,m-2}^1 R_1(n-2) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n-1,m-1}^m R_m(n-1) & \xi_{n-2,m-2}^m R_m(n-2) & \cdots & R_m(n-m) \end{vmatrix} \neq 0$$

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Now consider the sequence of polynomials $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi^1_{n,m} R_1(n) & \xi^1_{n-1,m-1} R_1(n-1) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^m_{n,m} R_m(n) & \xi^m_{n-1,m-1} R_m(n-1) & \cdots & R_m(n-m) \end{vmatrix}$$

Observation: q_n is a linear combination of of $m+1$ consecutive p_n's.

Define for $h = 1, \ldots, m$, the following functions

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi^h_{x,m-j} \det\left(\xi^l_{x+j-r,m-r} R_l(x+j-r)\right)_{\left\{\begin{array}{c} l \neq h \\ r \neq j \end{array}\right\}}$$

Observation: M_h are linear combinations of adjoint determinants of $\Omega(x)$.

If we assume that $\Omega(x)$ and $M_h(x)$ are polynomials in x, then $\exists\, D_q \in \mathcal{A}$ with $D_q(q_n) = P(n) q_n$ and $P(x) - P(x-1) = \Omega(x)$, where

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ORTHOGONALITY (CHOICE OF $R_1, R_2, \ldots, R_m$)

GOAL: Make $(q_n)_n$ bispectral (we already have $D_q(q_n) = \lambda_n q_n$).

For that we have to make an appropriate choice of the arbitrary polynomials $R_1, R_2, \ldots, R_m$. This choice is based on the following recurrence formula ($h = 1, \ldots, m$):

$$\varepsilon^h_{n+1} a_{n+1} R^h_j(n+1) - b_n R^h_j(n) + \frac{c_n}{\varepsilon^h_n} R^h_j(n-1) = (\eta_h j + \kappa_h) R^h_j(n), \quad n \in \mathbb{Z}$$

where η_h and κ_h are real numbers independent of n and j, $(a_n)_{n\in\mathbb{Z}}$, $(b_n)_{n\in\mathbb{Z}}$, $(c_n)_{n\in\mathbb{Z}}$ are the coefficients in the TTRR for the OP's $(p_n)_n$, and $(\varepsilon^h_n)_n$ defines a $\mathcal{D}$-operator for $(p_n)_n$.

Classical discrete family	$\mathcal{D}$-operators	$R_j(x)$
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Identifying the measure

Given a set G of m positive integers, $G = \{g_1, \dots, g_m\}$ we then define the sequence of polynomials $(q_n^G)_n$ by

$$q_n^G(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi_{n,m}^1 R_{g_1}^1(n) & \xi_{n-1,m-1}^1 R_{g_1}^1(n-1) & \cdots & R_{g_1}^1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m R_{g_m}^m(n) & \xi_{n-1,m-1}^m R_{g_m}^m(n-1) & \cdots & R_{g_m}^m(n-m) \end{vmatrix}$$

$(q_n^G)_n$ will be orthogonal w.r.t a Christoffel transform of ω (or several)

$$\omega^F(x) = \prod_{f \in F} (x-f)\, \omega(x)$$

How is the set G related with the set F?: G will be identified by one of the following sets:

$$I(F) = \{1, 2, \dots, f_k\} \setminus \{f_k - f, f \in F\},$$
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Charlier polynomials

Let $F \subset \mathbb{N}$ be finite and consider $G = I(F) = \{g_1, \ldots, g_m\}$.
Let ω_a be the Charlier measure and $(c_n^a)_n$ its sequence of OP's. Assume that $\Omega_G(n) = \det\left(c_{g_l}^{-a}(-n-j-1)\right)_{l,j=1}^{m} \neq 0$.

If we define $(q_n)_n$ by

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then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\tilde{\omega}_a^F = \prod_{f \in F}(x + f_k + 1 - f)\omega_a(x + f_k + 1)$$

and they are eigenfunctions of a higher order difference operator D_q with $-s = r = \sum_{f \in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$ and $f_k = \max F$.

Proof of Conjecture A: $\omega_a^F = a^{f_k+1}\tilde{\omega}_a^F(x - f_k - 1)$.

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Charlier polynomials: explicit example

Let $a = 1$, $F = \{1, 3\}$, $G = I(F) = \{1, 3\}$.

$\dfrac{q_n^G}{\Omega(n)} = c_n^1 + \beta_{n,1} c_{n-1}^1 + \beta_{n,2} c_{n-2}^1$ are orthogonal w.r.t

$$\tilde{\omega}_1^F = (x+3)(x+1)\omega_1(x+4)$$

The difference operator (of order 8) satisfying $D_q(q_n^G) = P(n) q_n^G$ is

$$D_q = P(D_1) + M_1(D_1)\nabla R_1(D_1) + M_2(D_1)\nabla R_2(D_1)$$

where

$$D_1 = -x\mathfrak{S}_{-1} + (x+1)\mathfrak{S}_0 - \mathfrak{S}_1, \quad D_1(c_n^1) = nc_n^1, \quad n \geq 0$$

$$R_1(x) = -x, \qquad R_2(x) = -\frac{1}{6}(x^3 + 3x^2 + 5x + 2)$$

$$M_1(x) = x^2 + 2x + 2, \qquad M_2(x) = -2$$

$$P(x) = -\frac{x}{12}(x^3 - 2x^2 - x - 2)$$

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Outline

Orthogonal matrix polynomials

Matrix polynomials on the real line:

$$\mathbf{E}_n x^n + \cdots + \mathbf{E}_1 x + \mathbf{E}_0, \quad \mathbf{E}_i \in \mathbb{C}^{N\times N}$$

Krein (1949): Orthogonal matrix polynomials (OMP)

Orthogonality: weight matrix $\mathbf{W}$ supported on $\mathcal{S} \subset \mathbb{R}$ (positive definite with finite moments) and a matrix-valued inner product ($L^2_{\mathbf{W}}(\mathcal{S};\mathbb{C}^{N\times N})$):

$$\langle \mathbf{P}, \mathbf{Q}\rangle_{\mathbf{W}} = \int_{\mathcal{S}} \mathbf{P}(x)\mathbf{W}(x)\mathbf{Q}^*(x)\,dx$$

A sequence of OMP $(\mathbf{Q}_n)_n$ ($\langle \mathbf{Q}_n, \mathbf{Q}_m\rangle_{\mathbf{W}} = \|\mathbf{Q}_n\|^2_{\mathbf{W}}\delta_{nm}$) satisfies a three-term recurrence relation ($\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I}$)

$$x\mathbf{Q}_n(x) = \mathbf{A}_n\mathbf{Q}_{n+1}(x) + \mathbf{B}_n\mathbf{Q}_n(x) + \mathbf{C}_n\mathbf{Q}_{n-1}(x), \quad \det(\mathbf{A}_n), \det(\mathbf{C}_n) \neq 0$$

Jacobi operator (block tridiagonal)

$$\mathbf{J}\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & \\ \mathbf{C}_1 & \mathbf{B}_1 & \mathbf{A}_1 & & \\ & \mathbf{C}_2 & \mathbf{B}_2 & \mathbf{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{Q}_0(x) \\ \mathbf{Q}_1(x) \\ \mathbf{Q}_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} \mathbf{Q}_0(x) \\ \mathbf{Q}_1(x) \\ \mathbf{Q}_2(x) \\ \vdots \end{pmatrix} = x\mathbf{Q}, \quad x \in \mathcal{S}$$

The converse is also true (Favard's or spectral theorem**)

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Differential properties

Durán (1997): characterize families of OMP $(\mathbf{Q}_n)_n$ satisfying

$$\mathbf{Q}_n(x)\mathcal{D} \equiv \mathbf{Q}_n''(x)\mathbf{F}_2(x) + \mathbf{Q}_n'(x)\mathbf{F}_1(x) + \mathbf{Q}_n(x)\mathbf{F}_0(x) = \mathbf{\Gamma}_n\mathbf{Q}_n(x)$$

where

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and $\mathbf{\Gamma}_n$ is a Hermitian matrix.

Equivalent to the symmetry of the second-order differential operator

$$\mathcal{D} = \partial^2\mathbf{F}_2(x) + \partial^1\mathbf{F}_1(x) + \partial^0\mathbf{F}_0(x), \quad \partial = \frac{d}{dx}$$

$$\text{with} \quad \mathbf{Q}_n\mathcal{D} = \mathbf{\Gamma}_n\mathbf{Q}_n$$

$\mathcal{D}$ is symmetric with respect to $\mathbf{W}$ if $\langle \mathbf{P}\mathcal{D}, \mathbf{Q}\rangle_{\mathbf{W}} = \langle \mathbf{P}, \mathbf{Q}\mathcal{D}\rangle_{\mathbf{W}}$

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How to generate examples

- Group representation theory: matrix-valued spherical functions associated with different groups (Grünbaum, Pacharoni, Tirao, Román, Zurrián, Koelink).
- Moment equations: solving moment equations from the symmetry equations (Durán, Grünbaum, MdI). Used in Durán-MdI (2008) to generate examples of OMP orthogonal with respect to a weight matrix plus a Dirac delta at some point.
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- Group representation theory: matrix-valued spherical functions associated with different groups (Grünbaum, Pacharoni, Tirao, Román, Zurrián, Koelink).
- Moment equations: solving moment equations from the symmetry equations (Durán, Grünbaum, MdI). Used in Durán-MdI (2008) to generate examples of OMP orthogonal with respect to a weight matrix plus a Dirac delta at some point.
- Matrix bispectral problem: solving the so-called *ad-conditions* ($\mathrm{ad}_{\mathbf{J}}^{k+1}(\mathbf{\Gamma}) = \mathbf{0}$) where k is the order of the differential operator (Castro, Grünbaum, Tirao). Used to generate examples of OMP satisfying differential equations of order $k = 1$ in Castro-Grünbaum (2005, 2008).

How to generate examples

- Symmetry equations: Durán-Grünbaum (2004):

$$\begin{aligned}
\mathbf{W}(x)\mathbf{F}_2^*(x) &= \mathbf{F}_2(x)\mathbf{W}(x)\\
\mathbf{W}(x)\mathbf{F}_1^*(x) &= 2(\mathbf{F}_2(x)\mathbf{W}(x))' - \mathbf{F}_1(x)\mathbf{W}(x)\\
\mathbf{W}(x)\mathbf{F}_0^*(x) &= (\mathbf{F}_2(x)\mathbf{W}(x))'' - (\mathbf{F}_1(x)\mathbf{W}(x))' + \mathbf{F}_0(x)\mathbf{W}(x)
\end{aligned}$$

$$\lim_{x\to t} \mathbf{F}_2(x)\mathbf{W}(x) = \mathbf{0} = \lim_{x\to t}(\mathbf{F}_1(x)\mathbf{W}(x) - \mathbf{W}(x)\mathbf{F}_1^*(x)), \quad t = a, b$$

General method: Assume $\mathbf{F}_2(x) = f_2(x)\mathbf{I}$. Factorize

$$\mathbf{W}(x) = \omega(x)\mathbf{T}(x)\mathbf{T}^*(x),$$

where ω is an scalar weight (Hermite, Laguerre or Jacobi) and $\mathbf{T}$ is a matrix function, solution of the first order differential equation

$$\mathbf{T}'(x) = \mathbf{G}(x)\mathbf{T}(x), \quad \mathbf{T}(c) = \mathbf{I}, \quad c \in (a, b)$$

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1. The first symmetry equation is trivial.
2. Defining
$$\mathbf{F}_1(x) = 2f_2(x)\mathbf{G}(x) + \frac{(f_2(x)\omega(x))'}{\omega(x)}\mathbf{I}$$
the second of the symmetry equations also holds.
3. Finally, the third of the symmetry equations is equivalent to
$$(\mathbf{F}_1(x)\mathbf{W}(x) - \mathbf{W}(x)\mathbf{F}_1^*(x))' = \mathbf{F}_0\mathbf{W}(x) - \mathbf{W}(x)\mathbf{F}_0^*$$
Therefore, it is enough to find $\mathbf{F}_0$ such that the matrix function
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Examples

Let $\mathbf{F}_2 = \mathbf{I}$ and $\omega = e^{-x^2}$. Then $\mathbf{G}(x) = \mathbf{A} + 2\mathbf{B}x$

$$\begin{cases} \text{If } \mathbf{B} = \mathbf{0} \Rightarrow \mathbf{W}(x) = e^{-x^2} e^{\mathbf{A}x} e^{\mathbf{A}^* x} \\ \text{If } \mathbf{A} = \mathbf{0} \Rightarrow \mathbf{W}(x) = e^{-x^2} e^{\mathbf{B}x^2} e^{\mathbf{B}^* x^2} \end{cases}$$

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$$\mathbf{A} = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

For the second case, a solution such that

$$\chi(x) = 2\mathbf{B} + (4\mathbf{B}^2 - 4\mathbf{B})x^2 - e^{-\mathbf{B}x^2}\mathbf{F}_0 e^{\mathbf{B}x^2}$$

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New phenomena

- For a fixed family of OMP there exist several differential operators having them as eigenfunctions (Castro, Durán, MdI, Grünbaum, Pacharoni, Tirao, Román).
- For a fixed second-order differential operator, there may be infinitely many linearly independent families of OMP having them as eigenfunctions (Durán, MdI).
- There exist families of OMP satisfying odd order differential equations (Castro, Grünbaum, Durán, MdI).
- There exists a family of ladder operators (upper and lower) for some examples of OMP, some of them of order 0 (Grünbaum, MdI, Martínez-Finkelshtein).

Applications

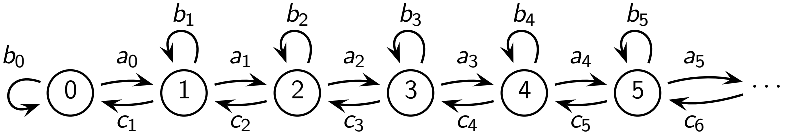
- Bivariate Markov processes: The first component is a regular Markov process (*level*) while the second is a finite Markov chain (discrete or continuous time) representing *N phases* (Dette, Reuther, Zygmunt, Grünbaum, Pacharoni, Tirao, MdI).
- Matrix-valued harmonic analysis: Matrix functions that are eigenfunctions at the same time of a second-order differential operator of Schrödinger type and an integral operator of Fourier type (MdI).
- Other applications: scattering theory (Geronimo), Lanczos method for block matrices (Golub, Underwood), asymptotic results (Durán, López-Rodríguez), quadrature formuale (Durán, Polo, van Assche, Sinap), Sobolev OP's (Durán, van Assche), doubly infinite Jacobi matrices (Berezanskii, Nikishin, van Assche), Dirac equation (Durán-Grünbaum), time-and-band limiting problems (Durán, Grünbaum, Pacharoni, Zurrián), noncommutative integrable systems (Cafasso, MdI), Riemann-Hilbert problems (Grünbaum, MdI, Martínez-Finkelshtein, Delvaux), etc.

Random walks

The state space is $\mathcal{S} = \{0, 1, 2, \ldots\}$, while time is $\mathcal{T} = \{0, 1, 2, \ldots\}$

$$P = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_i \geq 0, a_i, c_i > 0, \quad a_i + b_i + c_i = 1$$

$P(n) = P^n$ is the *n-step transition probability matrix.*



Spectral theorem (Karlin-MacGregor, 1959): there exists a measure ω associated with P which OP's $(q_n)_n$ satisfy $(q_{-1} = 0, q_0 = 1)$

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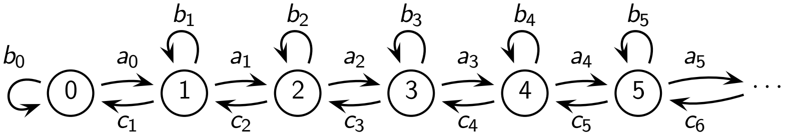

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We get spectral representations of the transition probabilities:

$$P_{ij}^n = \Pr(X_n = j | X_0 = i) = \frac{1}{\|q_i\|_\omega^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

and the invariant measure

Non-null vector $\pi = (\pi_0, \pi_1, \dots) \geq 0$ such that

$$\pi P = \pi \quad \Rightarrow \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\|q_i\|_\omega^2}$$

Other probabilistic models related with OP's: birth-and-death processes (linear growth models, queuing theory, etc) and diffusion processes (Orstein-Uhlenbeck process, Wright-Fisher models, etc)

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BIVARIATE MARKOV PROCESSES

Quasi-birth-and-death processes: Now we have a bivariate or 2-component Markov process of the form

$$\{(X_t, Y_t) : t \in \mathcal{T}\}$$

indexed by time $\mathcal{T} = \{0, 1, 2, \dots\}$ and with state space $\{0, 1, 2, \dots\} \times \{1, 2, \dots, N\}$.
The first component is the level while the second component is the phase.
The allowed transitions satisfy

$$(\mathbf{P}_{ij})_{i'j'} = \Pr(X_{n+1} = j, Y_{n+1} = j' | X_n = i, Y_n = i') = 0 \quad \text{for} \quad |i - j| > 1.$$

i.e. an stochastic block transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & \\ \mathbf{C}_1 & \mathbf{B}_1 & \mathbf{A}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

The invariant measure ($n \to \infty$) is now

$$\pi = (\pi_0; \pi_1; \cdots) \geq 0, \quad \pi_i \in \mathbb{R}^N$$

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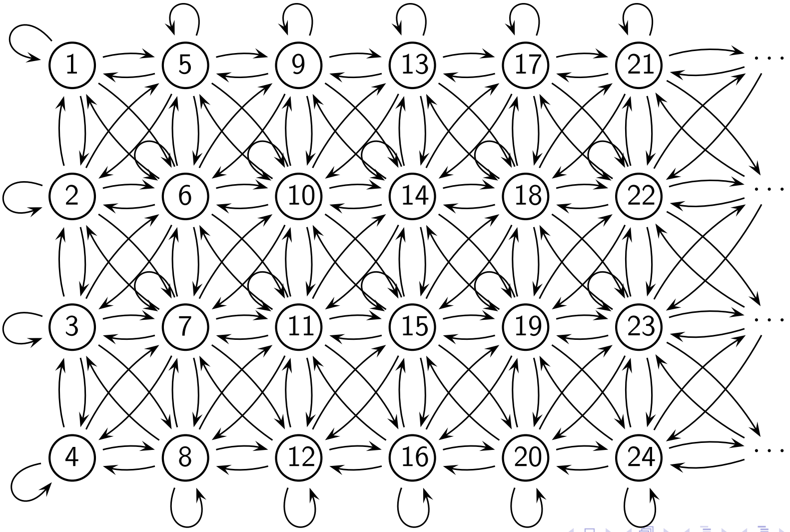
Stochastic representation ($N = 4$ phases)

Special case of $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ tridiagonal:



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Spectral theorem (Grünbaum, Dette et al., 2006): $\exists^*$ $\mathbf{W}$ on $[-1,1]$ associated with $\mathbf{P}$ with OMP $(\mathbf{Q}_n)_n$ satisfying $\mathbf{PQ} = x\mathbf{Q}$ $(\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I})$

$$\mathbf{P}_{ij}^n = \left(\int_{-1}^{1} x^n \mathbf{Q}_i(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)\left(\int_{-1}^{1} \mathbf{Q}_j(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)^{-1}$$

Invariant measure (MdI, 2011)

$\pi = (\pi_0; \pi_1; \cdots) \equiv (\Pi_0\mathbf{e}_N; \Pi_1\mathbf{e}_N; \cdots)$ such that $\pi\mathbf{P} = \pi$

$$\Pi_n = (\mathbf{C}_1^T \cdots \mathbf{C}_n^T)^{-1}\Pi_0(\mathbf{A}_0 \cdots \mathbf{A}_{n-1}) = \left(\int_{-1}^{1} \mathbf{Q}_n(x)\mathbf{W}(x)\mathbf{Q}_n^*(x)dx\right)^{-1}$$

and $\mathbf{e}_N = (1, 1, \ldots, 1)^T$.

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Spectral theorem (Grünbaum, Dette et al., 2006): $\exists^*$ $\mathbf{W}$ on $[-1,1]$ associated with $\mathbf{P}$ with OMP $(\mathbf{Q}_n)_n$ satisfying $\mathbf{PQ} = x\mathbf{Q}$ $(\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I})$

$$\mathbf{P}_{ij}^n = \left(\int_{-1}^{1} x^n \mathbf{Q}_i(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)\left(\int_{-1}^{1} \mathbf{Q}_j(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)^{-1}$$

Invariant measure (MdI, 2011)

$\boldsymbol{\pi} = (\boldsymbol{\pi}_0; \boldsymbol{\pi}_1; \cdots) \equiv (\boldsymbol{\Pi}_0\mathbf{e}_N; \boldsymbol{\Pi}_1\mathbf{e}_N; \cdots)$ such that $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$

$$\boldsymbol{\Pi}_n = (\mathbf{C}_1^T \cdots \mathbf{C}_n^T)^{-1}\boldsymbol{\Pi}_0(\mathbf{A}_0 \cdots \mathbf{A}_{n-1}) = \left(\int_{-1}^{1} \mathbf{Q}_n(x)\mathbf{W}(x)\mathbf{Q}_n^*(x)dx\right)^{-1}$$

and $\mathbf{e}_N = (1, 1, \ldots, 1)^T$.

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Hermite functions

Hermite or wave functions

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x)$$

where $H_n(x) = (-1)^n e^{x^2} (e^{-x^2})^{(n)}$, are the Hermite polynomials

$(\psi_n)_n$ is a complete orthonormal set in $L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} \psi_m^*(x) \psi_n(x) dx = \delta_{nm}, \quad n, m \geq 0$$

$(\psi_n)_n$ are eigenfunctions of the Schrödinger operator

$$\psi_n''(x) - x^2 \psi_n(x) = -(2n+1)\psi_n(x), \quad x \in \mathbb{R}$$

and at the same time of the Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_n(t) e^{ixt} dt = (i)^n \psi_n(x), \quad x \in \mathbb{R}$$

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Matrix-valued Hermite functions

Matrix case: Let $\mathbf{W}(x) = e^{-x^2} e^{\mathbf{A}x} e^{\mathbf{A}^* x}$ ($\mathbf{A}$ nilpotent)

We know from Durán-Grünbaum (2004) that

$$\mathbf{Q}_n''(x) - 2\mathbf{Q}_n'(x)(x\mathbf{I} - \mathbf{A}) + \mathbf{Q}_n(x)(\mathbf{A}^2 - 2\mathbf{J}) = (-2n\mathbf{I} - 2\mathbf{J})\mathbf{Q}_n(x)$$

The family $\Phi_n(x) = e^{-x^2/2}\mathbf{Q}_n(x)e^{\mathbf{A}x}$ is orthogonal in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$

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Case $N = 2$

The *normalized* family is

$$\mathbf{\Phi}_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

That means $\int_{-\infty}^{\infty} \mathbf{\Phi}_n(x)\mathbf{\Phi}_m^*(x)dx = \delta_{nm}\mathbf{I}$, so the diagonal entries are probability distributions on $\mathbb{R}$. The plots for the first values of n and $\nu_1 = 1$ are given by

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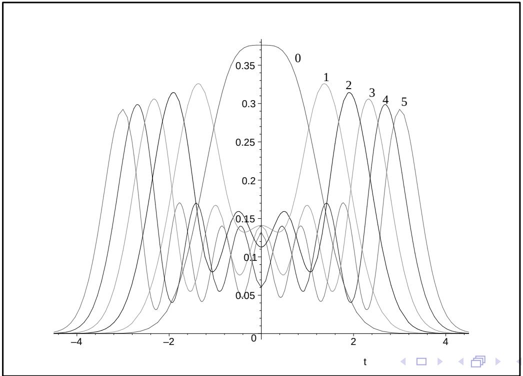

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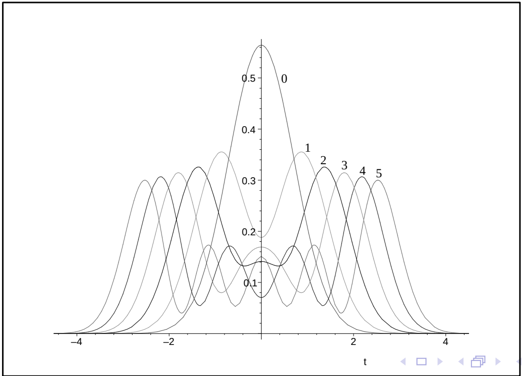

We also have

$$(x\mathbf{I})_{nm} \doteq \int_{-\infty}^{\infty} x\Phi_n(x)\Phi_m^*(x)dx = \begin{pmatrix} \sqrt{\frac{n\gamma_{n+1}}{2\gamma_n}} & 0 \\ 0 & \sqrt{\frac{n\gamma_{n-1}}{2\gamma_n}} \end{pmatrix} \delta_{m,n-1}$$

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Therefore $(x\mathbf{I})$ is the matrix of the homomorphism $\mathbf{F} \mapsto x\mathbf{F}$ in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$ with respect to the basis $(\Phi_n)_n$

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