# Matrix valued orthogonal polynomials satisfying differential equations

Manuel Domínguez de la Iglesia

Departamento de Análisis Matemático
Universidad de Sevilla

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# Outline

1. Scalar case

2. Matrix case
   - How to get examples?
   - New phenomena

3. Applications

# Scalar case

- Equivalence between $(p_n)_n$, orthonormal w. r. t. $\rho > 0$

$$(p_n, p_m)_\rho = \int_{\mathbb{R}} p_n(t) p_m(t) d\rho(t) = \delta_{nm}, \quad n, m \geq 0$$

and a Jacobi (tridiagonal) operator

$$\mathcal{L} = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

- Three term recurrence relation

$$tp_n(t) = a_{n+1}p_{n+1}(t) + b_n p_n(t) + a_n p_{n-1}(t), \quad n \geq 0$$
$$a_{n+1} \neq 0 \quad b_n \in \mathbb{R}$$

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- (Bochner, 1929) Characterize $(p_n)_n$ satisfying

$$\sigma(t)p_n''(t) + \tau(t)p_n'(t) + \lambda_n p_n(t) = 0$$
$$\deg\sigma \leq 2, \quad \deg\tau = 1$$

- Equivalent to the symmetry (i.e. $(dp, q) = (p, dq)$) of

$$d = \sigma(t)\partial^2 + \tau(t)\partial^1, \quad \partial = \frac{d}{dt}$$

- Also equivalent to the Pearson equation:

$$(\sigma\rho)' = \tau\rho$$

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Classical families of

- Hermite: $\rho(t) = e^{-t^2}, \; t \in \mathbb{R}$

$$H_n(t)'' - 2tH_n(t)' = -2nH_n(t)$$

- Laguerre: $\rho(t) = t^\alpha e^{-t}, \; t > 0, \; \alpha > -1$

$$tL_n^\alpha(t)'' + (\alpha + 1 - t)L_n^\alpha(t)' = -nL_n^\alpha(t)$$

- Jacobi: $\rho(t) = t^\alpha(1-t)^\beta, \; t \in (0,1), \; \alpha, \beta > -1$

$$t(1-t)P_n^{(\alpha,\beta)}(t)'' + (\alpha + 1 - (\alpha+\beta+2)t)P_n^{(\alpha,\beta)}(t)' = -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(t)$$

# Matrix case

- The theory of orthogonal matrix polynomials (OMP) was introduced by Kreĭn in 1949.
- Equivalence between $(P_n)_n$, orthonormal w. r. t. $W$

$$(P_n, P_m)_W = \int_{\mathbb{R}} P_n(t)\, W(t)\, P_m^*(t)\, dt = \delta_{nm} I, \quad n, m \geq 0$$

and a Jacobi (block tridiagonal) operator

$$\mathcal{L} = \begin{pmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & A_2^* & B_2 & A_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0$$
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- (Durán, 1997) Characterize all families of OMP $(P_n)_n$ satisfying

$$P_n D \equiv P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0 = \Lambda_n P_n(t), \quad n \geq 0$$
$$\deg F_i \leq i$$

- The side where the coefficients are multiplied allows us to find "interesting" examples.
- Equivalent to the symmetry (i.e. $(PD, Q)_W = (P, QD)_W$) of

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^1 F_0, \quad \partial = \frac{d}{dt}$$

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# Generating examples

- Symmetry equations

$$F_2 W = W F_2^*$$
$$2(F_2 W)' = F_1 W + W F_1^*,$$
$$(F_2 W)'' - (F_1 W)' + F_0 W = W F_0^*$$

$\Rightarrow W(t) = \rho(t) T(t) T^*(t)$

- Matrix spherical functions associated with $P_N(\mathbb{C}) = \mathrm{SU}(N+1)/\mathrm{U}(N)$
- Bispectral problem $\Rightarrow$ ad-conditions

$$\begin{cases} \mathcal{L}P = tP \\ PD = \Lambda P \end{cases} \Leftrightarrow \mathrm{ad}_{\mathcal{L}}^{k+1}(\Lambda) = 0,$$

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# New phenomena I

Example:

$$W(t) = t^{\alpha} e^{-t} \begin{pmatrix} t(1+a^2 t) & at \\ at & 1 \end{pmatrix}, \quad t > 0, \quad \alpha > -1, \quad a \in \mathbb{R} \setminus \{0\}$$

Several linearly independent second-order differential operators for a fixed family of OMP.

$$D_1 = \partial^2 tI + \partial^1 \begin{pmatrix} \alpha + 2 - t & at \\ 0 & \alpha + 1 - t \end{pmatrix} + \partial^0 \begin{pmatrix} -\frac{1+a^2}{a^2} & (1+\alpha)a \\ 0 & -\frac{1}{a^2} \end{pmatrix}$$

$$D_2 = \partial^2 \begin{pmatrix} t & -2at^2 \\ 0 & -t \end{pmatrix} + \partial^1 \begin{pmatrix} \alpha + 2 + t & -\frac{(2+a^2(2\alpha+5))t}{a} \\ \frac{2}{a} & -t - \alpha - 1 \end{pmatrix} + \partial^0 \begin{pmatrix} \frac{1+a^2}{a^2} & -\frac{(1+\alpha)(2+a^2)}{a} \\ 0 & -\frac{1}{a^2} \end{pmatrix}$$

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# New phenomena II

OMP satisfy odd order differential equations.

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\begin{aligned}
D = {} & \partial^3 \begin{pmatrix} -a^2t^2 & at^2(1+a^2t) \\ -at & a^2t^2 \end{pmatrix} + \\
& \partial^2 \begin{pmatrix} -t(2+a^2(\alpha+5)) & at(2\alpha+4+t(1+a^2(\alpha+5))) \\ -a(\alpha+2) & t(2+a^2(\alpha+2)) \end{pmatrix} + \\
& \partial^1 \begin{pmatrix} t-2(\alpha+2)(1+a^2) & \frac{a^2(\alpha+1)(\alpha+2)+t(1+2a^2(1+a^2(\alpha+2)))}{a} \\ -\frac{1}{a} & 2\alpha+2-t \end{pmatrix} + \\
& \partial^0 \begin{pmatrix} 1+\alpha & -\frac{1}{a}(1+\alpha)(a^2\alpha-1) \\ \frac{1}{a} & -(1+\alpha) \end{pmatrix}
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# New phenomena III

For a fixed second-order differential operator $D$, there can be more than one family of linearly independent (monic) OMP $(P_{n,\gamma})_n$, $\gamma > 0$, having them as eigenfunctions

$$W_\gamma = t^\alpha e^{-t} \begin{pmatrix} t(1+a^2t) & at \\ at & 1 \end{pmatrix} + \gamma \begin{pmatrix} a^2(\alpha+1)^2 & a(\alpha+1) \\ a(\alpha+1) & 1 \end{pmatrix} \delta(t)$$

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# Applications

## Quantum mechanics

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## Time-and-band limiting

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