Differential equations for discrete Sobolev orthogonal polynomials[1]

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13th OPSFA
Gaithersburg, MD, June 2nd, 2015

[1]joint work with Antonio J. Durán
[2]Supported by PAPIIT-DGAPA-UNAM grant IA100515

Outline

1. Introduction
2. Discrete Laguerre-Sobolev
3. Discrete Jacobi-Sobolev

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Goal of the talk

For a given measure ν, a couple of real numbers $\lambda \neq \mu$, and $m_1, m_2 \geq 0$ nonnegative integers, consider the discrete Sobolev bilinear form

$$\langle p, q \rangle = \int p(x)q(x)d\nu(x) + \mathbb{T}_\lambda^{m_1}(p)\, \boldsymbol{M}\, \mathbb{T}_\lambda^{m_1}(q)^T + \mathbb{T}_\mu^{m_2}(p)\, \boldsymbol{N}\, \mathbb{T}_\mu^{m_2}(q)^T$$

where $\boldsymbol{M}$ and $\boldsymbol{N}$ are $m_1 \times m_1$ and $m_2 \times m_2$ matrices respectively, and $\mathbb{T}_\lambda^k(p) = \left(p(\lambda), p'(\lambda), \ldots, p^{(k-1)}(\lambda)\right)$. Call $m = m_1 + m_2$.

We will focus in two cases:

1. Laguerre-Sobolev: $m_1 = m, m_2 = 0, \lambda = 0$ and for $\alpha > m-1$

$$d\nu(x) = \mu_{\alpha-m}(x) = x^{\alpha-m}e^{-x}dx, \quad x > 0$$

2. Jacobi-Sobolev: $\lambda = -1, \mu = 1$ and for $\alpha > m_2 - 1, \beta > m_1 - 1$

$$d\nu(x) = \mu_{\alpha-m_2,\beta-m_1}(x) = (1-x)^{\alpha-m_2}(1+x)^{\beta-m_1}dx, \quad -1 < x < 1$$

- Give an expression of the (left) OP $(q_n)_n$ as a linear combination of $m+1$ consecutive Laguerre $(L_n^\alpha)_n$ or Jacobi $(P_n^{\alpha,\beta})_n$ polynomials.
- Find a higher-order differential operator D_q such that $D_q(q_n) = \lambda_n q_n$, where D_q is a polynomial combination of D_α or $D_{\alpha,\beta}$.

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$$\mathbf{M} = \begin{cases} M_{i+j}, & \text{if} \quad i + j \leq m - 1 \\ 0, & \text{if} \quad i + j > m - 1 \end{cases}$$

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Outline

1. Introduction
2. Discrete Laguerre-Sobolev
3. Discrete Jacobi-Sobolev

Orthogonality

Theorem (Durán-MdI, 2014)

Let $\boldsymbol{M} = (M_{ij})_{i,j=0}^{m-1}$ be any $m \times m$ matrix and $\alpha \neq m-1, m-2, \dots$ Consider

$$\langle p, q \rangle = \int_0^\infty p(x)q(x)\mu_{\alpha-m}(x)dx + \mathbb{T}_0^m(p)\,\boldsymbol{M}\,\mathbb{T}_0^m(q) \qquad (1)$$

and the functions ($j = 1, 2, \dots m$)

$$\mathcal{R}_j(x) = \frac{\Gamma(\alpha - m + j)}{(m-j)!}(x+1)_{m-j} + (j-1)!\frac{\Gamma(\alpha+1+x)}{\Gamma(1+x)}\sum_{i=0}^{m-1}\frac{(-1)^i M_{j-1,i}}{\Gamma(\alpha+i+1)}(x-i+1)_i$$

Then the family $(q_n)_n$ defined by the Casorati determinant

$$q_n(x) = \begin{vmatrix} L_n^\alpha(x) & L_{n-1}^\alpha(x) & \cdots & L_{n-m}^\alpha(x) \\ \mathcal{R}_1(n) & \mathcal{R}_1(n-1) & \cdots & \mathcal{R}_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_m(n) & \mathcal{R}_m(n-1) & \cdots & \mathcal{R}_m(n-m) \end{vmatrix}, \qquad n \geq 0$$

is (left) orthogonal with respect to (1) if and only if

$$\Omega(n) = \det(\mathcal{R}_i(n-j))_{i,j=1}^m \neq 0, \qquad n \geq 0$$

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Differential operator

Theorem (Durán-MdI, 2014)

Let $Y_1, Y_2, \dots, Y_m$ be m arbitrary polynomials and assume that

$$\Omega(n) = \det(Y_i(n-j))_{i,j=1}^m \neq 0, n \geq 0$$

Now consider the sequence of polynomials $(q_n)_n$ defined by

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Then there exists a differential operator D_q of the form

$$D_q = P(D_\alpha) + \sum_{h=1}^m M_h(D_\alpha) \frac{d}{dx} Y_h(D_\alpha)$$

such that $D_q(q_n) = P(n) q_n$ where $P(x) - P(x-1) = \Omega(x)$ and the polynomials $M_h(x), h = 1, \dots, m$ are defined by

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are polynomials in x for $j = 1, 2, \ldots, m$.

This holds only when α is a nonnegative integer with $\alpha \geq m$.
In that case the functions $\mathcal{R}_l(x)$ are polynomials given by

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Order of the differential operator

The order of the differential operator D_q is given by twice the degree of the polynomial P defined by $P(x) - P(x-1) = \Omega(x)$ where

$$\Omega(x) = \det(\mathcal{R}_i(x-j))_{i,j=1}^{m}$$

Therefore this order should be $2(\deg \Omega(x) + 1)$.

This degree depends on the polynomials $\mathcal{R}_i(x)$ and therefore on the matrix $\boldsymbol{M}$. There can be two cases:

- If $\deg \mathcal{R}_i \neq \deg \mathcal{R}_j$ for $i \neq j$. Then (Durán-MdI, 2014)

$$\deg \Omega(x) = \sum_{j=1}^{m} \deg \mathcal{R}_j(x) - \frac{m(m-1)}{2}$$

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How to obtain γ-wr($\boldsymbol{M}$)?

Write $\boldsymbol{M}$ by columns $\boldsymbol{M} = [c_1|c_2|\cdots|c_m]$.

Define the numbers $n_1, n_2, \ldots, n_m$ by

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$$m_j = \begin{cases} m - j, & \text{if } f_j \in < f_{j+1}, \ldots, f_m >, \\ 0, & \text{if } f_j \notin < f_{j+1}, \ldots, f_m >. \end{cases}$$

$$\gamma\text{-wr}(\boldsymbol{M}) = \sum_{j=1}^{m} n_j + \sum_{j=1}^{m-1} m_j - \frac{m(m-1)}{2}$$

How to obtain γ-wr($\boldsymbol{M}$)?

Write $\boldsymbol{M}$ by columns $\boldsymbol{M} = [c_1|c_2|\cdots|c_m]$.
Define the numbers $n_1, n_2, \ldots, n_m$ by

$$n_1 = \begin{cases} \gamma + m - 1, & \text{if } c_m \neq 0, \\ 0, & \text{if } c_m = 0; \end{cases}$$

and for $j = 2, \ldots, m$,

$$n_j = \begin{cases} \gamma + m - j, & \text{if } c_{m-j+1} \notin < c_{m-j+2}, \ldots, c_m >, \\ 0, & \text{if } c_{m-j+1} \in < c_{m-j+2}, \ldots, c_m >. \end{cases}$$

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Examples

- Grünbaum-Haine-Horozov: Let M be the matrix associated with the functional

$$\mu_{\alpha-m}(x)+\sum_{i=0}^{m-1} M_i\delta_0^{(i)}$$

Then $\deg \mathcal{R}_j=\alpha+m-j$ and $\deg \mathcal{R}_i\neq \deg \mathcal{R}_j$ for $i\neq j$. Therefore

$$\deg \Omega(x)=\alpha m \quad \Rightarrow \quad \mathrm{ord}(D_q)=2(\alpha m+1)$$

- Laguerre-Sobolev: Let M be the diagonal $M=\mathrm{diag}(M_0,\ldots,M_{m-1})$, $M_{m-1}\neq 0$. Therefore

$$\deg \mathcal{R}_j=\begin{cases}\alpha+j-1, & M_{j-1}\neq 0,\\ m-j, & M_{j-1}=0.\end{cases}$$

and $\deg \mathcal{R}_i\neq \deg \mathcal{R}_j$ for $i\neq j$. If $s=\#\{j:1\leq j\leq m, M_j\neq 0\}$

$$\deg \Omega(x)=s\alpha+(m-s)(m+1)-2\sum_{1\leq j\leq m, M_{j-1}=0} j$$

$$\Rightarrow \quad \mathrm{ord}(D_q)=2(\deg \Omega(x)+1)$$

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$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \deg \mathcal{R}_1 = \deg \mathcal{R}_2 = \alpha + 1, \deg \mathcal{R}_3 = \alpha + 2$$

In this case we have to use the definition of α-wr$(\boldsymbol{M})$ to find $\deg \Omega(x)$.

Therefore

$$n_1 = \alpha + 2, n_2 = \alpha + 1, n_3 = 0, \quad m_1 = 2, m_2 = 0, \quad \Rightarrow \quad \alpha\text{-wr}(\boldsymbol{M}) = 2\alpha + 2$$

For $\alpha = 3$, $\mathcal{R}_1(x), \mathcal{R}_2(x), \mathcal{R}_3(x)$ are given by

$$\begin{aligned} \mathcal{R}_1(x) &= -\frac{(x+1)(x+2)(x^2 - x - 24)}{24} \\ \mathcal{R}_2(x) &= -\frac{(x+1)(x^3 + x^2 - 14x - 48)}{24} \\ \mathcal{R}_3(x) &= \frac{(x+4)(x^4 + x^3 + x^2 - 9x + 30)}{60} \end{aligned}$$

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Outline

Orthogonality

Theorem (Durán-MdI, 2015)

Let $\boldsymbol{M} = (M_{ij})_{i,j=0}^{m_1-1}$ and $\boldsymbol{N} = (N_{ij})_{i,j=0}^{m_2-1}$ be any $m_1 \times m_1$ and $m_2 \times m_2$ matrices, respectively. For $\alpha \neq m_2-1, m_2-2, \dots$ and $\beta \neq m_1-1, m_1-2, \dots$, consider

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\mu_{\alpha-m_2,\beta-m_1}(x)dx + \mathbb{T}_{-1}^{m_1}(p)\,\boldsymbol{M}\,\mathbb{T}_{-1}^{m_1}(q) + \mathbb{T}_{1}^{m_2}(p)\,\boldsymbol{N}\,\mathbb{T}_{1}^{m_2}(q)$$

Let $m = m_1 + m_2 \geq 1$. Then the family $(q_n)_n$ defined by the Casorati determinant

$$q_n(x) = \begin{vmatrix} P_n^{\alpha,\beta}(x) & P_{n-1}^{\alpha,\beta}(x) & \cdots & P_{n-m}^{\alpha,\beta}(x) \\ \mathcal{R}_1(n) & \mathcal{R}_1(n-1) & \cdots & \mathcal{R}_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_m(n) & \mathcal{R}_m(n-1) & \cdots & \mathcal{R}_m(n-m) \end{vmatrix}, \quad n \geq 0$$

is (left) orthogonal with respect to (1) if and only if

$$\Omega(n) = \det(\mathcal{R}_i(n-j))_{i,j=1}^{m} \neq 0, \quad n \geq 0$$

Now the functions $\mathcal{R}_l$ depend on the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$, so there will be m_1 functions $\mathcal{R}_l$ associated with $\boldsymbol{M}$ and m_2 functions $\mathcal{R}_l$ associated with $\boldsymbol{N}$.

ORTHOGONALITY

In order to define the functions $\mathcal{R}_l$, let us introduce the notation

$$\mathcal{G}_{c,d}^{a,b}(x) = \frac{\Gamma(x+a+1)\Gamma(x+b+1)}{\Gamma(x+c+1)\Gamma(x+d+1)}, \quad a,b,c,d \in \mathbb{R}$$

Then, for $l = 1, \ldots, m_1$, we have

$$\mathcal{R}_l(n) = \frac{2^{\alpha+\beta-m_1+l}\Gamma(\beta-m_1+l)}{(-1)^n(m_1-l)!}\mathcal{G}_{0,\alpha+\beta-m_1+l}^{m_1-l,\alpha}(n) + \frac{2^{m_2}}{(-1)^n}\sum_{i=0}^{m_1-1}\left(\sum_{j=l}^{l+m_2\wedge m_1}\frac{(j-1)!\binom{m_2}{j-l}M_{i,j-1}}{(-2)^{i+j-l}}\right)\frac{\mathcal{G}_{-i,\alpha+\beta}^{\beta,\alpha+\beta+i}(n)}{\Gamma(\beta+i+1)}$$

and for $l = m_1+1, \ldots, m$, we have

$$\mathcal{R}_l(n) = \frac{2^{\alpha+\beta-m+l}\Gamma(\alpha-m+l)}{(m-l)!}\mathcal{G}_{0,\alpha+\beta-m+l}^{m-l,\beta}(n) + \sum_{i=0}^{m_2-1}\left(\sum_{j=l-m_1}^{l\wedge m_2}\frac{(j-1)!\binom{m_1}{l-j}N_{i,j-1}}{(-1)^{l-m_1-1}2^{i+j-l}}\right)\frac{\mathcal{G}_{-i,\alpha+\beta}^{\alpha,\alpha+\beta+i}(n)}{\Gamma(\alpha+i+1)}$$

where $\wedge$ is the minimum between two numbers.

Differential operator

Theorem (Durán-MdI, 2015)

Let $(P_n^{\alpha,\beta})_n$ de Jacobi polynomials and denote $\mathcal{D}_{\alpha,\beta}$ the Jacobi second-order differential operator satisfying

$$\mathcal{D}_{\alpha,\beta}\left(P_n^{\alpha,\beta}\right) = \theta_n P_n^{\alpha,\beta}, \quad \theta_n = n(n+\alpha+\beta+1)$$

Define the two $\mathcal{D}$-operators for the Jacobi polynomials

$$\mathcal{D}_1 = -\frac{\alpha+\beta+1}{2}I + (1-x)\frac{d}{dx}, \quad \text{and} \quad \mathcal{D}_2 = \frac{\alpha+\beta+1}{2}I + (1+x)\frac{d}{dx}$$

Define additionally the sequences

$$\xi_{n,i}^h = \frac{(n+\alpha-i+1)_i}{(n+\alpha+\beta-i+1)_i}, \quad h = 1,\dots,m_1$$

$$\xi_{n,i}^h = \frac{(-1)^i(n+\beta-i+1)_i}{(n+\alpha+\beta-i+1)_i}, \quad h = m_1+1,\dots,m$$

Let $Y_1, Y_2, \dots, Y_m$ be m arbitrary polynomials and assume that the $m \times m$ (quasi) Casorati determinant $\Omega(n)$ satisfies

$$\Omega(n) = \det\left(\xi_{n-j,m-j}^l Y_l(\theta_{n-j})\right)_{l,j=1}^m \neq 0$$

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Differential operator

Theorem (continuation)

Now consider the sequence of polynomials $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} P_n^{\alpha,\beta}(x) & -P_{n-1}^{\alpha,\beta}(x) & \cdots & (-1)^m p P_{n-m}^{\alpha,\beta}(x) \\ \xi_{n,m}^1 Y_1(\theta_n) & \xi_{n-1,m-1}^1 Y_1(\theta_{n-1}) & \cdots & Y_1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m Y_m(\theta_n) & \xi_{n-1,m-1}^m Y_m(\theta_{n-1}) & \cdots & Y_m(\theta_{n-m}) \end{vmatrix}.$$

Then there exists a differential operator $D_{q,S}$ of the form

$$D_{q,S} = \frac{1}{2} P_S(D_{\alpha,\beta}) + \sum_{h=1}^{m_1} \tilde{M}_h(D_{\alpha,\beta}) \mathcal{D}_1 Y_h(D_{\alpha,\beta}) + \sum_{h=m_1+1}^{m} \tilde{M}_h(D_{\alpha,\beta}) \mathcal{D}_2 Y_h(D_{\alpha,\beta})$$

such that $D_{q,S}(q_n) = \lambda_n q_n$ where $\tilde{M}_h$ are certain polynomials,

$$\lambda_n - \lambda_{n-1} = S(n)\Omega(n)$$

where $S(n)$ is certain rational function and

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Bispectrality

In other to combine the previous two results we have to choose α and β nonnegative integers with $\alpha \geq m_2$ and $\beta \geq m_1$. Then, the polynomials $Y_l(\theta_n)$ are given by

$$Y_l(\theta_n) = \frac{2^{\alpha+\beta-m_1+l}\Gamma(\beta-m_1+l)}{(m_1-l)!}\mathcal{U}^{\alpha}_{m_1-l}(\theta_n) + 2^{m_2}\sum_{i=0}^{m_1-1}\left(\sum_{j=l}^{l+m_2\wedge m_1}\frac{(j-1)!\binom{m_2}{j-l}M_{i,j-1}}{(-2)^{i+j-l}}\right)\frac{\mathcal{U}^{0}_{\beta+i}(\theta_n)}{\Gamma(\beta+i+1)}$$

$$Y_l(\theta_n) = \frac{2^{\alpha+\beta-m+l}\Gamma(\alpha-m+l)}{(m-l)!}\mathcal{U}^{\alpha}_{m-l}(\theta_n) + \sum_{i=0}^{m_2-1}\left(\sum_{j=l-m_1}^{l\wedge m_2}\frac{(j-1)!\binom{m_1}{l-j}N_{i,j-1}}{(-1)^{l-m_1-1}2^{i+j-l}}\right)\frac{\mathcal{U}^{\alpha-\beta}_{\alpha+i}(\theta_n)}{\Gamma(\alpha+i+1)}$$

where

$$\mathcal{U}^{\lambda}_{j}(\theta_n) = \prod_{i=1}^{j}\left[(\alpha-\lambda+i)(\beta+\lambda-i+1)-\theta_n\right]$$

If M and N *are not related* (in some sense) then the minimal order of the differential operator is given by

$$2(\beta\text{-wr}(M)+\alpha\text{-wr}(N)+1)$$

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If $\boldsymbol{M}$ and $\boldsymbol{N}$ *are not related* (in some sense) then the minimal order of the differential operator is given by

$$2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1)$$

Examples

- Grünbaum-Haine-Horozov: Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be the matrices associated with the functional

$$\mu_{\alpha-m_2,\beta-m_1}(x)+\sum_{i=0}^{m_1-1} M_i\delta_{-1}^{(i)}+\sum_{i=0}^{m_2-1} N_i\delta_{1}^{(i)}$$

with $M_{m_1-1}, N_{m_2-1}\neq 0$ and $M_i\neq N_j$. Then $\deg Y_l=\beta+m_1-l$ for $l=1,\dots,m_1$ and $\deg Y_l=\alpha+m-l$ for $l=m_1+1,\dots,m$. Therefore

$$\operatorname{ord}(D_q)=2(\alpha m_2+\beta m_1+1)$$

- Jacobi-Sobolev: Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be the diagonal matrices $\boldsymbol{M}=\operatorname{diag}(M_0,\dots,M_{m_1-1})$, $M_{m_1-1}\neq 0$, $\boldsymbol{N}=\operatorname{diag}(N_0,\dots,N_{m_2-1})$, $N_{m_2-1}\neq 0$ and $M_i\neq N_j$. If $s=\#\{j:1\leq j\leq m_1, M_j\neq 0\}$ and $t=\#\{j:1\leq j\leq m_2, N_j\neq 0\}$ then

$$\operatorname{ord}(D_q)=2\Big(t\alpha+s\beta+(m_1-s)(m_1+1)+(m_2-t)(m_2+1)-2\sum_{1\leq j\leq m_1, M_{j-1}=0} j-2\sum_{1\leq j\leq m_2, N_{j-1}=0} j+1\Big)$$

EXAMPLES

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with $M_{m_1-1}, N_{m_2-1}\neq 0$ and $M_i\neq N_j$. Then $\deg Y_l=\beta+m_1-l$ for $l=1,\ldots,m_1$ and $\deg Y_l=\alpha+m-l$ for $l=m_1+1,\ldots,m$. Therefore

$$\operatorname{ord}(D_q)=2(\alpha m_2+\beta m_1+1)$$

- Jacobi-Sobolev: Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be the diagonal matrices $\boldsymbol{M}=\operatorname{diag}(M_0,\ldots,M_{m_1-1})$, $M_{m_1-1}\neq 0$, $\boldsymbol{N}=\operatorname{diag}(N_0,\ldots,N_{m_2-1})$, $N_{m_2-1}\neq 0$ and $M_i\neq N_j$. If $s=\#\{j:1\leq j\leq m_1, M_j\neq 0\}$ and $t=\#\{j:1\leq j\leq m_2, N_j\neq 0\}$ then

$$\operatorname{ord}(D_q)=2\Bigg(t\alpha+s\beta+(m_1-s)(m_1+1)+(m_2-t)(m_2+1)-2\sum_{1\leq j\leq m_1, M_{j-1}=0} j-2\sum_{1\leq j\leq m_2, N_{j-1}=0} j+1\Bigg)$$

Another example

$m_1 = 2, m_2 = 1, \alpha = 1, \beta = 2$ and

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{N} = 2 \Rightarrow \quad \deg Y_1 = \deg Y_2 = 3, \deg Y_3 = 1$$

For $\boldsymbol{M}$ we have

$$n_1 = 3, n_2 = 0, \quad m_1 = 1, m_2 = 0, \quad \Rightarrow \quad \beta\text{-wr}(\boldsymbol{M}) = 3$$

while for $\boldsymbol{N}$ we have

$$n_1 = 1, \quad m_1 = 0, \quad \Rightarrow \quad \alpha\text{-wr}(\boldsymbol{N}) = 1$$

The polynomials $Y_1(x)$, $Y_2(x)$, $Y_3(x)$ are given by

$$\begin{aligned} Y_1(x) &= -\frac{(x-3)(x^2+2x-72)}{12} \\ Y_2(x) &= -\frac{x^3-x^2-30x+120}{6} \\ Y_3(x) &= 32-8x \end{aligned}$$

$$\text{ord}(D_q) = 2(\beta\text{-wr}(\boldsymbol{M}) + \alpha\text{-wr}(\boldsymbol{N}) + 1) = 10$$

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