Riemann-Hilbert techniques in the theory of orthogonal matrix polynomials [1]

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[1] joint work with F. A. Grünbaum and A. Martínez Finkelshtein

Outline

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1. What is a Riemann-Hilbert problem?

2. The Riemann-Hilbert problem for orthogonal matrix polynomials

Hilbert's 21st problem

Riemann: The problem that an analytic function could be completely defined by its singularities and monodromy properties.

Hilbert: A $N \times N$ linear system of differential equations

$$\frac{d\Psi(x)}{dx} = A(x)\Psi(x)$$

is called *Fuchsian* if $A(x)$ is a rational function with simple poles.

If S is the punctured (at the poles) Riemann sphere $\mathbb{C} \cup \{\infty\}$ then

$$\Psi : \Pi_1(S) \to GL(N, \mathbb{C})$$

gives a representation, called *monodromy group*.

Question: Does always exist a Fuchsian system with given poles and a given monodromy group?

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Some history

J. Plemelj (1909) published a solution relating the problem into the context of analytic factorization of matrix-valued functions and methods of singular integral equations.

A. Bolibruch (1989) gave a counterexample of size $N = 3$. For $N = 2$ Plemelj's claim is true (Dekkers, 1978).

There is now a modern version (D-module and derived category), the Riemann-Hilbert correspondence.

The Riemann-Hilbert method consists of reconstructing an analytic function from jump conditions or the analytic factorization of a given matrix-valued function defined on a curve. This is what we will call *Riemann-Hilbert problems* (RHP).

Applications: Integrable systems, Orthogonal polynomials and Random matrices, Combinatorial probability.

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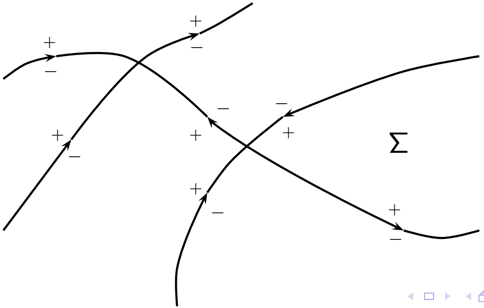
Riemann-Hilbert factorization problem

Let $G : \Sigma \to GL(N, \mathbb{C})$ be a matrix-valued function. The RHP determined by (Σ, G) consists of finding an $N \times N$ matrix-valued function $Y(z)$ s.t.

1. $Y(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
2. $Y_+(z) = Y_-(z)G(z)$ when $z \in \Sigma$
   $Y_\pm(z) = \lim_{z' \to z, \pm\text{side}} Y(z)$
3. $Y(z) = I$ as $z \to \infty$



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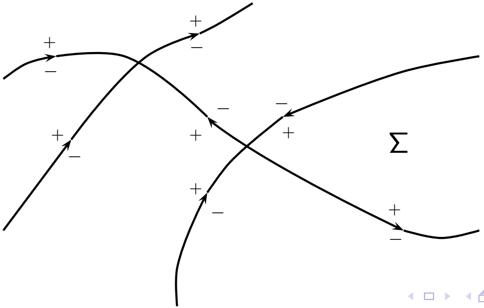

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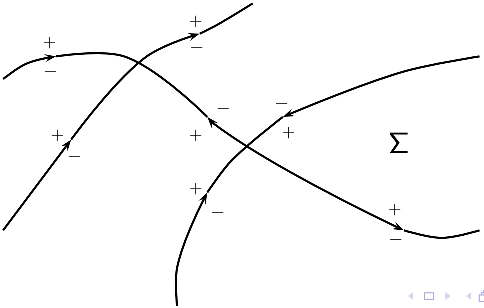

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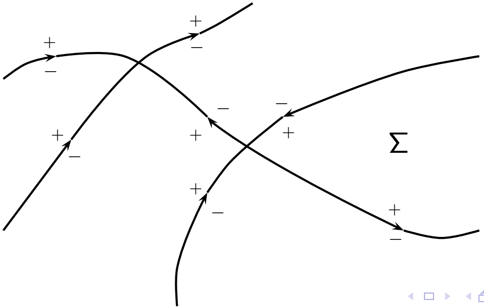

Scalar and additive RHP ($N = 1$)

Let $\omega : \mathbb{R} \to \mathbb{R}$ be a $L^1(\mathbb{R})$ and Hölder continous function. The Riemann-Hilbert problem determined by $(\mathbb{R}, \omega)$ consists of finding a function $f : \mathbb{C} \to \mathbb{C}$ such that

1. $f(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $f_+(x) = f_-(x) + \omega(x)$ when $x \in \mathbb{R}$
3. $f(z) = \mathcal{O}(1/z)$ as $z \to \infty$

The unique solution of the RH problem above is the Stieltjes or Cauchy transform of ω, i.e.

$$f(z) = C(\omega)(z) \doteq \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(t)}{t - z}$$

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RHP for orthogonal polynomials ($N = 2$)

Let $d\mu$ be a positive Borel measure.
We will assume $d\mu(x) = \omega(x)dx$, $\omega \geq 0$ and $x^i\omega, x^j\omega' \in L^1(\mathbb{R})$.
We can then construct a family of orthonormal polynomials $(p_n)_n$ s.t.

$$(p_n, p_m)_\omega = \int_{\mathbb{R}} p_n(x)p_m(x)\omega(x)dx = \delta_{n,m}, \quad n, m \geq 0$$

$$p_n(x) = \gamma_n(x^n + a_{n,n-1}x^{n-1} + \cdots) = \gamma_n\widehat{p}_n(x)$$

We try to find a 2×2 matrix-valued function $Y^n : \mathbb{C} \to \mathbb{C}^{2\times 2}$ such that

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Solution of the RHP for orthogonal polynomials

Fokas-Its-Kitaev (1990): For $n \geq 1$ the unique solution of the RHP above is given by

$$Y^n(z) = \begin{pmatrix} \widehat{p}_n(z) & C(\widehat{p}_n\omega)(z) \\ c_n\widehat{p}_{n-1}(z) & c_n C(\widehat{p}_{n-1}\omega)(z) \end{pmatrix}$$

where $c_n = -2\pi i \gamma_{n-1}^2$.

The existence and unicity is a consequence of the Morera's theorem, Liouville's theorem, the additive RHP and $\det Y^n(z) = 1$.

The Liouville-Ostrogradski formula

$$q_n(z)p_{n-1}(z) - p_n(z)q_{n-1}(z) = \gamma_n/\gamma_{n-1}$$

where $q_n(z) = \int_{\mathbb{R}} \frac{p_n(t)-p_n(z)}{t-z}\omega(t)dt$ is a consequence of $Y^n(Y^n)^{-1} = I$.

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Example: Hermite polynomials

The solution $Y^n(z)$ of the RHP

1. Y^n is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $Y_+^n(x) = Y_-^n(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$
3. $Y^n(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

is given by

$$Y^n(z) = \begin{pmatrix} h_n(z) & C(h_n e^{-t^2})(z) \\ c_n h_{n-1}(z) & c_n C(h_{n-1} e^{-t^2})(z) \end{pmatrix}$$

where $(h_n)_n$ is the family of monic Hermite polynomials.

A recurrence relation

If we call $R = Y^{n+1}(Y^n)^{-1}$ and denoting

$$Y^n(z) = \left(I + \frac{1}{z}\begin{pmatrix} d_n & e_n \\ f_n & g_n \end{pmatrix} + \mathcal{O}_n(1/z^2)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty$$

we have that

1. R is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $R_+(x) = R_-(x)$ for all $x \in \mathbb{R}$
3. $R(z) = \begin{pmatrix} z + d_{n+1} - d_n & -e_n \\ f_{n+1} & 0 \end{pmatrix}$ as $z \to \infty$

Therefore

$$Y^{n+1}(z) = \begin{pmatrix} z + d_{n+1} - d_n & -e_n \\ f_{n+1} & 0 \end{pmatrix} Y^n(z)$$

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$$Y^{n+1}(z) = \begin{pmatrix} z + d_{n+1} - d_n & -e_n \\ f_{n+1} & 0 \end{pmatrix} Y^n(z)$$

A recurrence relation

If we call $R = Y^{n+1}(Y^n)^{-1}$ and denoting

$$Y^n(z) = \left(I + \frac{1}{z}\begin{pmatrix} d_n & e_n \\ f_n & g_n \end{pmatrix} + \mathcal{O}_n(1/z^2) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty$$

we have that

1. R is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $R_+(x) = R_-(x)$ for all $x \in \mathbb{R}$
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A recurrence relation II

If we put

$$\alpha_n = d_n - d_{n+1}, \quad \text{and} \quad \beta_n = e_n f_n$$

entry $(1,1)$ gives the three-term recurrence formula

$$zh_n(z) = h_{n+1}(z) + \alpha_n h_n(z) + \beta_n h_{n-1}(z)$$

The entry $(2,1)$ gives

$$f_n = c_n = -2\pi i \gamma_{n-1}^2$$

From the explicit expression of α_n and β_n we have

$$d_n = a_{n,n-1}, \quad \text{and} \quad c_{n+1}^{-1} = e_n$$

Also we have the following

$$f_{n+1}e_n = 1 \quad d_n + g_n = 0$$

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A differential equation

Consider the transformation

$$X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2} & 0 \\ 0 & e^{z^2/2} \end{pmatrix}$$

We observe that X^n is invertible and that

$$X^n_+(x) = Y^n_+(x) \begin{pmatrix} e^{-x^2/2} & 0 \\ 0 & e^{x^2/2} \end{pmatrix} = Y^n_-(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-x^2/2} & 0 \\ 0 & e^{x^2/2} \end{pmatrix}$$
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That means that X^n has a constant jump.

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A differential equation II

Consider $R = \frac{d}{dz}X^n(X^n)^{-1}$. We have that

1. R is analytic in $\mathbb{C} \setminus \mathbb{R}$
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   as $z \longrightarrow \infty$

Therefore

$$\frac{d}{dz}X^n(z) = \begin{pmatrix} -z & 2c_{n+1}^{-1} \\ -2c_n & z \end{pmatrix} X^n(z)$$

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The Lax pair and compatibility conditions

X^n satisfies the following Lax pair

$$X^{n+1}(z) = \underbrace{\begin{pmatrix} z-\alpha_n & -c_{n+1}^{-1} \\ c_{n+1} & 0 \end{pmatrix}}_{E_n(z)} X^n(z), \quad \frac{d}{dz}X^n(z) = \underbrace{\begin{pmatrix} -z & 2c_{n+1}^{-1} \\ -2c_n & z \end{pmatrix}}_{F_n(z)} X^n(z)$$

Combining them we get

$$E_n'(z) + E_n(z)F_n(z) = F_{n+1}(z)E_n(z)$$

In this case we get (using $\beta_n = c_n/c_{n+1}$)

$$\alpha_n = 0, \quad \beta_{n+1} - \beta_n = \frac{1}{2} \Rightarrow \beta_n = \frac{n}{2}$$

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The ladder operators and second order differential equation

From entries $(1,1)$ and $(2,1)$ of $\frac{d}{dz}X^n(z) = \begin{pmatrix} -z & 2c_{n+1}^{-1} \\ -2c_n & z \end{pmatrix} X^n(z)$ we get the ladder operators

$$h_n'(z) = nh_{n-1}(z), \quad h_{n-1}'(z) - 2zh_{n-1}(z) = -2h_n(z)$$

Combining them we get the second order differential equation

$$h_n''(z) - 2zh_n'(z) + 2nh_n(z) = 0$$

Conclusion: advantages

1. Algebraic properties: three term recurrence relation, ladder operators, second order differential equation
2. Uniform asymptotics: *steepest descent analysis for RHP* (Deift-Zhou,1993). The idea is to transform Y^n into another RHP R^n where $R^n(z) = I + \mathcal{O}(1/z)$ as $z \to \infty$ and $R_+(z) = R_-(z)(I + \mathcal{O}(1/n))$. Therefore $R^n(z) = I + \mathcal{O}(1/n)$ as $n \to \infty$ uniformly for all $z \in \mathbb{C}$.

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Outline

Orthogonal matrix polynomials

The theory of orthogonal matrix polynomials (OMP) was introduced by Krein in 1949.

A $N \times N$ matrix polynomial on the real line is

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0, \quad x \in \mathbb{R} \quad A_i \in \mathbb{C}^{N \times N}$$

Let W be a $N \times N$ a matrix of measures or weight matrix.
We will assume $dW(x) = W(x)dx$ and $\int x^n W(x)dx$ $\int x^m W'(x)dx$ exist.
We can construct a family of OMP with respect to the inner product

$$(P, Q)_W = \int_{\mathbb{R}} P(x)W(x)Q^*(x)dx \in \mathbb{C}^{N \times N}$$

such that

$$(P_n, P_m)_W = \int_{\mathbb{R}} P_n(x)W(x)P_m^*(x)dx = \delta_{n,m} I, \quad n, m \geq 0$$

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Solution of the RHP for OMP

$Y^n : \mathbb{C} \to \mathbb{C}^{2N\times 2N}$ such that

1. Y^n is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $Y^n_+(x) = Y^n_-(x) \begin{pmatrix} I & W(x) \\ 0 & I \end{pmatrix}$ when $x \in \mathbb{R}$
3. $Y^n(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n I & 0 \\ 0 & z^{-n} I \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RH problem above is given by

$$Y^n(z) = \begin{pmatrix} \widehat{P}_n(z) & C(\widehat{P}_n W)(z) \\ c_n \widehat{P}_{n-1}(z) & c_n C(\widehat{P}_{n-1} W)(z) \end{pmatrix}$$

where $C(F)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(t)}{t-z} dt$ and $c_n = -2\pi i \gamma^*_{n-1} \gamma_{n-1}$.

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Again, we have that $\det Y^n(z) = 1$, so we find a solution of the inverse

$$(Y^n)^{-1} = \begin{pmatrix} C(W\widehat{P}^*_{n-1})(z)c_n & -C(W\widehat{P}^*_n)(z) \\ -\widehat{P}^*_{n-1}(\bar{z})c_n & \widehat{P}^*_n(\bar{z}) \end{pmatrix}$$

- The Liouville-Ostrogradski formula (Durán, 1996)

$$Q_n(z)P^*_{n-1}(\bar{z}) - P_n(z)Q^*_{n-1}(\bar{z}) = \gamma_n\gamma_{n-1}^{-1}$$

- The Hermitian property (Durán, 1996)

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$$(Y^n)^{-1} = \begin{pmatrix} C(W\widehat{P}^*_{n-1})(z)c_n & -C(W\widehat{P}^*_n)(z) \\ -\widehat{P}^*_{n-1}(\bar{z})c_n & \widehat{P}^*_n(\bar{z}) \end{pmatrix}$$

- The Liouville-Ostrogradski formula (Durán, 1996)

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A special case

Let us consider

$$W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R}$$

for any $A \in \mathbb{C}^{N\times N}$.

The solution $Y^n(z)$ of the RHP

1. Y^n is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $Y^n_+(x) = Y^n_-(x) \begin{pmatrix} I & e^{-x^2} e^{Ax} e^{A^* x} \\ 0 & I \end{pmatrix}$ when $x \in \mathbb{R}$
3. $Y^n(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n I & 0 \\ 0 & z^{-n} I \end{pmatrix}$ as $z \to \infty$

is given by

$$Y^n(z) = \begin{pmatrix} \widehat{P}_n(z) & C(\widehat{P}_n e^{-t^2} e^{At} e^{A^* t})(z) \\ c_n \widehat{P}_{n-1}(z) & c_n C(\widehat{P}_{n-1} e^{-t^2} e^{At} e^{A^* t})(z) \end{pmatrix}$$

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A recurrence relation

If we call $R = Y^{n+1}(Y^n)^{-1}$ and denoting

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If we put

$$\alpha_n = \left(Y_1^n\right)_{11} - \left(Y_1^{n+1}\right)_{11}, \quad \beta_n = \left(Y_1^n\right)_{12}\left(Y_1^n\right)_{21}$$

block entry $(1,1)$ gives the three-term recurrence formula

$$z\widehat{P}_n(z) = \widehat{P}_{n+1}(z) + \alpha_n\widehat{P}_n(z) + \beta_n\widehat{P}_{n-1}(z)$$

The block entry $(2,1)$ gives

$$c_n = \left(Y_1^n\right)_{21}$$

From the explicit expression of α_n and β_n we have

$$\left(Y_1^{n+1}\right)_{11} = a_{n,n-1}, \quad \text{and} \quad c_{n+1}^{-1} = \left(Y_1^{n-1}\right)_{12}$$

Also we have the following

$$\left(Y_1^{n+1}\right)_{21}\left(Y_1^n\right)_{12} = \left(Y_1^n\right)_{12}\left(Y_1^{n+1}\right)_{21} = I, \quad \left(Y_1^n\right)_{11} + \left(Y_1^n\right)_{22}^{*} = 0$$

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A differential equation

Consider the transformation

$$X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2} e^{Az} & 0 \\ 0 & e^{z^2/2} e^{-A^* z} \end{pmatrix}$$

We observe that X^n is invertible and that

$$X_+^n(x) = X_-^n(x) \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

That means that X^n has a constant jump.

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   $$= \begin{pmatrix} -zI + A & 2\left(Y_1^n\right)_{12} \\ -2\left(Y_1^{n+1}\right)_{21} & zI - A^* \end{pmatrix}$$
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The Lax pair and compatibility conditions

X^n satisfies the following Lax pair

$$X^{n+1}(z) = \underbrace{\begin{pmatrix} zI - \alpha_n & -c_{n+1}^{-1} \\ c_{n+1} & 0 \end{pmatrix}}_{E_n(z)} X^n(z), \quad \frac{d}{dz}X^n(z) = \underbrace{\begin{pmatrix} -zI + A & 2c_{n+1}^{-1} \\ -2c_n & zI - A^* \end{pmatrix}}_{F_n(z)} X^n(z)$$

Combining them we get

$$E_n'(z) + E_n(z)F_n(z) = F_{n+1}(z)E_n(z)$$

In this case we get (using $\beta_n = c_n c_{n+1}^{-1}$)

Compatibility conditions

$$2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I$$

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The ladder operators and second order differential equation

From block entries $(1,1)$ and $(2,1)$ of $\frac{d}{dz}X^n(z) = \begin{pmatrix} -zI + A & 2c_{n+1}^{-1} \\ -2c_n & zI - A^* \end{pmatrix} X^n(z)$ we get the ladder operators

Ladder operators

$$\widehat{P}_n'(z) + \widehat{P}_n(z)A - A\widehat{P}_n(z) = 2\beta_n \widehat{P}_{n-1}(z)$$

$$-\widehat{P}_n'(z) + 2(z-\alpha_n)\widehat{P}_n(z) + A\widehat{P}_n(z) - \widehat{P}_n(z)A = 2\widehat{P}_{n+1}(z)$$

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Second order differential equation

$$\widehat{P}_n''(z) + 2\widehat{P}_n'(z)(A - zI) + \widehat{P}_n(z)A^2 - A^2\widehat{P}_n(z) + 4\beta_n\widehat{P}_n(z) = -2z(\widehat{P}_n(z)A - A\widehat{P}_n(z)) + 2(\alpha_n - A)(\widehat{P}_n'(z) + \widehat{P}_n(z)A - A\widehat{P}_n(z))$$

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