Darboux transformations for quasi-birth-and-death processes*

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VII Iberoamerican Workshop in Orthogonal Polynomials and Applications (EIBPOA2018)
Leganés, July 3rd, 2018

*Joint work with F. Alberto Grünbaum

OUTLINE

1. Stochastic Darboux transformations for random walks

2. Stochastic Darboux transformations for quasi-birth-and-death processes (QBD)

3. The (2x2) Jacobi type example

1. Stochastic Darboux transformations

for random walks

UL (LU) STOCHASTIC FACTORIZATION

Let $\{X_n : n = 0, 1, \ldots\}$ be an irreducible random walk with space state $\mathbb{Z}_{\geq 0}$ and P its one-step transition probability matrix. We would like to perform a UL decomposition of the matrix P in the following way

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with the condition that P_U and P_L are also stochastic matrices, i.e. $x_n = 1 - y_n, s_0 = 1, r_n = 1 - s_n$, and nonnegative entries.

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Comparing the coefficients, the only relevant equations are

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UL and LU decompositions of stochastic matrices have been considered earlier in the literature (W.K. Grassmann, D.P. Heyman, V. Vigon, etc.) in a different context related with censored Markov chains and Wiener-Hopf factorizations.

WHEN IS IT POSSIBLE?

From the relations

$$a_n = (1 - y_n)s_{n+1}, \quad c_{n+1} = y_{n+1}(1 - s_{n+1}), \quad n \geq 0$$

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We will need that the following continued fraction

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}} \doteq 1 - \frac{a_0 |}{| 1} - \frac{c_1 |}{| 1} - \frac{a_1 |}{| 1} - \frac{c_2 |}{| \cdots}$$

is convergent and $0 < H < 1$.

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In Grünbaum-MdI (2017) we proved the following result

Theorem. *Let H the continued fraction given before and the corresponding convergents $h_n = A_n/B_n$. Assume that*

$$0 < A_n < B_n, \quad n \geq 1$$

Then H is convergent. Moreover, if $P = P_U P_L$, then both P_U and P_L are stochastic matrices if and only if we choose y_0 in the following range

$$0 \leq y_0 \leq 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots}$$

STOCHASTIC DARBOUX TRANSFORMATION

If $P = P_U P_L$, then by inverting the order of multiplication we obtain another tridiagonal matrix of the form

$$\widetilde{P} = P_L P_U = \begin{pmatrix} s_0 & 0 & & \\ r_1 & s_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y_0 & x_0 & & \\ 0 & y_1 & x_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_0 & & \\ \tilde{c}_1 & \tilde{b}_1 & \tilde{a}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

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This is called a discrete Darboux transformation. It appeared for the first time in Matveev-Salle in connection with Toda lattices. Later, many other authors (Grünbaum, Haine, Horozov, Iliev, etc.) have used this transformation in the description of some families of Krall polynomials.

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The matrix $\widetilde{P}$ is actually stochastic, since the multiplication of two stochastic matrices is again a stochastic matrix. Therefore it gives a *family* of new random walks with coefficients $(\tilde{a}_n)_n$, $(\tilde{b}_n)_n$ and $(\tilde{c}_n)_n$ and depending on a free parameter y_0.

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Probabilistic interpretation. In terms of a model driven by urn experiments the factorization $P = P_U P_L$ may be thought as two urn experiments, Experiment 1 and Experiment 2, respectively. We first perform the Experiment 1 and with the result we immediately perform the Experiment 2. The urn model for $\widetilde{P} = P_L P_U$ will proceed in the reversed order, first the Experiment 2 and with the result the Experiment 1. The same can be done for the LU decomposition.

SPECTRAL MEASURES

One important property of the Darboux transformation is how to transform the spectral measure associated P. It is very well known that for every tridiagonal stochastic matrix P (or Jacobi matrix) there exists an unique positive measure ω supported on the interval $-1 \leq x \leq 1$ (Spectral or Favard's Theorem).

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The Darboux transformation gives a family of random walks $\widetilde{P}$ which is also a tridiagonal stochastic matrix. If the moment $\mu_{-1} = \int_{-1}^{1} d\omega(x)/x$ is well defined, then a candidate for the family of spectral measures is then

$$\widetilde{\omega}(x) = y_0 \frac{\omega(x)}{x} + M\delta_0(x), \quad M = 1 - y_0\mu_{-1}$$

where $\delta_0(x)$ is the Dirac delta located at $x = 0$ and y_0 is the free parameter from the UL factorization. This transformation of the spectral measure ω is also known as a Geronimus transformation.

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Similarly, for the LU decomposition, the corresponding Darboux transformation $\widehat{P}$ gives rise to a tridiagonal stochastic matrix and a spectral measure $\widehat{\omega}$. In this case, it is possible to see that this new spectral measure is given by

$$\widehat{\omega}(x) = x\omega(x)$$

or, in other words, a Christoffel transformation of ω.

2. Stochastic Darboux transformations for quasi-birth-and-death processes (QBD)

QUASI-BIRTH-AND-DEATH PROCESSES

Let P be the one-step transition probability matrix of a discrete-time quasi-birth-and-death (QBD) process with state space $\mathbb{Z}_{\geq 0} \times \{1, 2, \ldots, d\}$, $d \geq 1$, given by

$$P = \begin{pmatrix} B_0 & A_0 & 0 & & \\ C_1 & B_1 & A_1 & & \\ 0 & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

By definition of the process, we must have that all entries of P are nonnegative and

$$(B_0 + A_0)\boldsymbol{e}_d = \boldsymbol{e}_d, \quad (C_n + B_n + A_n)\boldsymbol{e}_d = \boldsymbol{e}_d, \quad n \geq 1, \quad \boldsymbol{e}_d = (1, 1, \ldots, 1)^T$$

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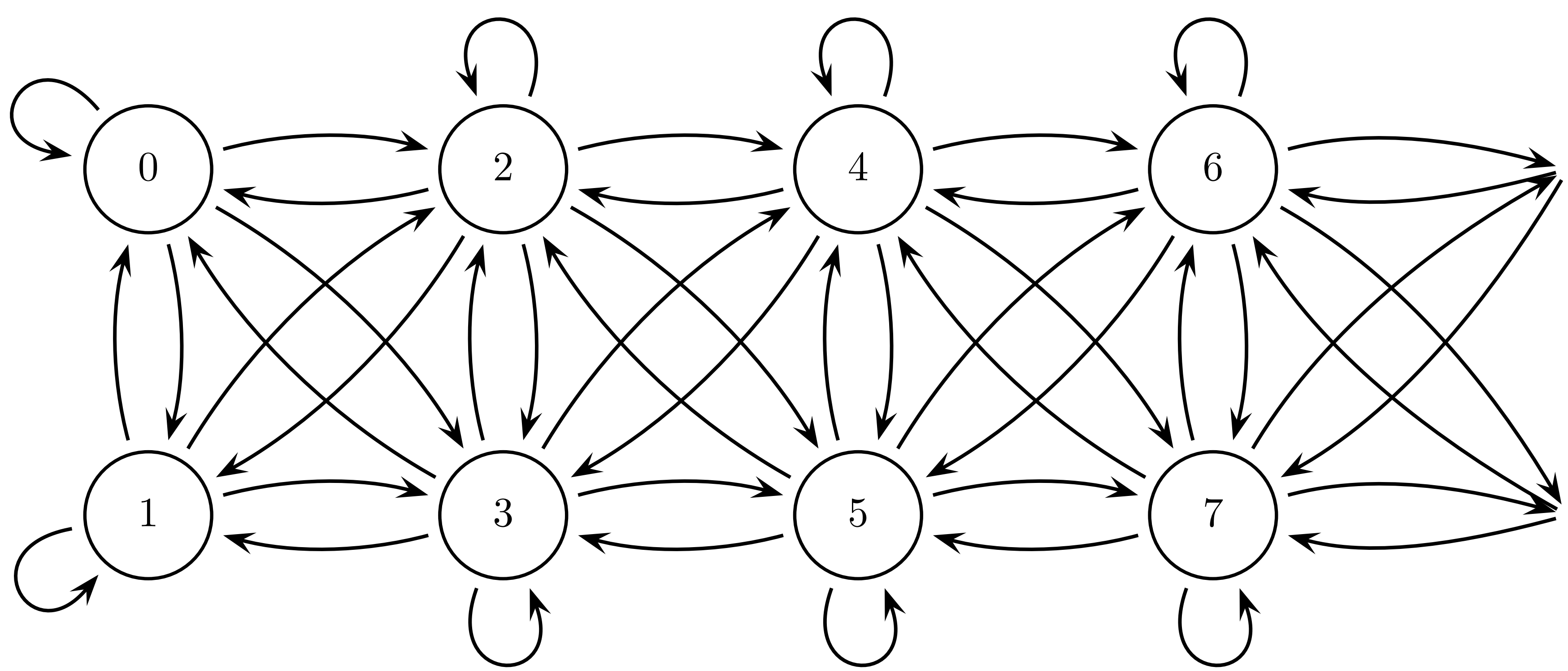
A diagram of the transitions between states looks as follows (for $d = 2$)



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with the condition that P_U and P_L are also stochastic matrices, i.e. all (scalar) entries are nonnegative and

$$(X_n + Y_n)\boldsymbol{e}_d = \boldsymbol{e}_d, \quad n \geq 0, \quad S_0\boldsymbol{e}_d = \boldsymbol{e}_d, \quad (R_n + S_n)\boldsymbol{e}_d = \boldsymbol{e}_d, \quad n \geq 1$$

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A direct computation shows that

$$\begin{aligned} A_n &= X_n S_{n+1}, \quad n \geq 0, \\ B_n &= X_n R_{n+1} + Y_n S_n, \quad n \geq 0, \\ C_n &= Y_n R_n \quad n \geq 1. \end{aligned}$$

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IMPORTANT DIFFERENCE: In the scalar situation the UL factorization has exactly one free parameter y_0, while in the LU factorization case the factorization is unique. This is not the case for the UL and LU block factorizations, where there may be many degrees of freedom. For instance, it is not possible to compute all entries of S_0 by having only the information that $S_0\boldsymbol{e}_d = \boldsymbol{e}_d$. The same is true for the rest of coefficients X_n, Y_n, R_n, S_n.

One way of computing X_n, Y_n, S_n, R_n is using what corresponds to the "monic" version of P, i.e. writing $P = LJL^{-1}$, where $L = \mathrm{diag}\{L_0, L_1, \cdots\}$, $L_n = (A_0 \cdots A_{n-1})^{-1}, n \geq 1, L_0 = I$

$$J = \begin{pmatrix} \widehat{B}_0 & I & & \\ \widehat{C}_1 & \widehat{B}_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

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Consider now the UL block factorization of the "monic" operator J in the following way

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With this decomposition, it is possible to compute all the coefficients in terms only of α_0.

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This is not enough to guarantee stochastic factors, so we introduce a new block diagonal matrix $T = \mathrm{diag}\{\tau_0, \tau_1, \cdots\}$ such that P can be written as

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Since both factor are assumed to be stochastic matrices, we get conditions on the sequence $(\tau_n)_n$, but not uniquely determined. Indeed they have to satisfy

$$L_n(\alpha_n \tau_n + \tau_{n+1})\boldsymbol{e}_d = \boldsymbol{e}_d, \quad n \geq 0$$
$$(\beta_{n+1} L_n^{-1} + L_{n+1}^{-1})\boldsymbol{e}_d = \tau_{n+1}\boldsymbol{e}_d, \quad n \geq 0,$$
$$\tau_0^{-1}\boldsymbol{e}_d = \boldsymbol{e}_d,$$

and it is possible to see that the first one implies the second.

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where $\widehat{B}_n = L_n^{-1} B_n L_n$ and $\widehat{C}_n = L_n^{-1} C_n L_{n-1}$.

Consider now the UL block factorization of the "monic" operator J in the following way

$$J = \begin{pmatrix} \widehat{B}_0 & I & & \\ \widehat{C}_1 & \widehat{B}_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \alpha_0 & I & & \\ 0 & \alpha_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I & 0 & & \\ \beta_1 & I & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \boldsymbol{\alpha}\boldsymbol{\beta}$$

With this decomposition, it is possible to compute all the coefficients in terms only of α_0.

This is not enough to guarantee stochastic factors, so we introduce a new block diagonal matrix $T = \mathrm{diag}\{\tau_0, \tau_1, \cdots\}$ such that P can be written as

$$P = [L\boldsymbol{\alpha}T]\left[T^{-1}\boldsymbol{\beta}L^{-1}\right] = P_U P_L$$

Since both factor are assumed to be stochastic matrices, we get conditions on the sequence $(\tau_n)_n$, but not uniquely determined. Indeed they have to satisfy

$$\begin{aligned} L_n(\alpha_n \tau_n + \tau_{n+1})\boldsymbol{e}_d &= \boldsymbol{e}_d, \quad n \geq 0 \\ (\beta_{n+1} L_n^{-1} + L_{n+1}^{-1})\boldsymbol{e}_d &= \tau_{n+1}\boldsymbol{e}_d, \quad n \geq 0, \\ \tau_0^{-1}\boldsymbol{e}_d &= \boldsymbol{e}_d, \end{aligned}$$

and it is possible to see that the first one implies the second.

Therefore, if we are able to propose a good candidate for $(\tau_n)_n$, then we can compute all block entries X_n, Y_n, R_n, S_n in terms only of $Y_0 = \alpha_0 \tau_0$.

BLOCK STOCHASTIC DARBOUX TRANSFORMATION

If $P = P_U P_L$ then by reversing the order of multiplication we obtain another block tridiagonal matrix of the form

$$\widetilde{P} = P_L P_U = \begin{pmatrix} S_0 & 0 & & \\ R_1 & S_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} Y_0 & X_0 & & \\ 0 & Y_1 & X_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \widetilde{B}_0 & \widetilde{A}_0 & & \\ \widetilde{C}_1 & \widetilde{B}_1 & \widetilde{A}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

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The matrix $\widetilde{P}$ is again stochastic. Therefore it is a new QBD process depending on many free parameters. The same probabilistic interpretation applies here.

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As in the scalar case, if the moment $\mu_{-1} = \int_{-1}^{1} dW(x)/x$ is well defined, then a candidate for the family of matrix-valued spectral measures associated with the Darboux transformation $\widetilde{P}$ is again a Geronimus transformation of W, i.e.

$$\widetilde{W}(x) = \frac{W(x)}{x} + \boldsymbol{M}\delta_0(x), \quad \boldsymbol{M} = \alpha_0^{-1}\mu_0 - \mu_{-1}$$

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The free parameters of $\widetilde{W}$ only depend on α_0 and not on the sequence $(\tau_n)_n$, which will only interfere in the normalization of the corresponding matrix-valued polynomials. In the case where α_0 is a singular matrix, we will have a degenerate matrix-valued spectral measure. Also we observe that $\widetilde{W}$ in general is neither symmetric nor positive semidefinite.

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Similarly, for the LU factorization, the corresponding Darboux transformation gives rise to matrix-valued spectral measure $\widehat{W}$, which is a Christoffel transformation of W, i.e. $\widehat{W}(x) = xW(x)$. In this case the weight matrix $\widehat{W}$ is unique and positive semidefinite.

3. The (2x2) Jacobi type example

JACOBI TYPE EXAMPLE

This example comes from group representation theory and was introduced for the first time in Grünbaum-Pacharoni-Tirao (2002). In Grünbaum-MdI (2008) we studied the probabilistic aspects of this example and gave an explicit expression of the block entries of P. The most general situation is considered in Grünbaum-Pacharoni-Tirao (2013), where the authors also give two stochastic models in terms of urns and Young diagrams.

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For $\alpha, \beta > -1$ and $0 < k < \beta + 1$, the coefficients A_n, B_n, C_n of P are given by (2×2)

$$A_n = \begin{pmatrix} \frac{(\beta+n+2)(k+n)(\alpha+\beta+n+2)}{(k+n+1)(\alpha+\beta+2n+2)(\alpha+\beta+2n+3)} & 0 \\ \frac{k(\beta+n+2)}{(\alpha+\beta-k+n+3)(\alpha+\beta+2n+3)(k+n+1)} & \frac{(\beta+n+2)(\alpha+\beta+n+3)(\alpha+\beta-k+n+2)}{(\alpha+\beta+2n+3)(\alpha+\beta+2n+4)(\alpha+\beta-k+n+3)} \end{pmatrix},$$

$$B_n = \begin{pmatrix} B_n^{11} & \frac{(\beta-k+1)(\alpha+\beta+n+2)}{(k+n+1)(\alpha+\beta+2n+2)(\alpha+\beta-k+n+2)} \\ \frac{(\alpha+n+1)k}{(k+n)(\alpha+\beta-k+n+2)(\alpha+\beta+2n+3)} & B_n^{22} \end{pmatrix},$$

$$C_n = \begin{pmatrix} \frac{n(\alpha+n)(\alpha+\beta-k+n+2)}{(\alpha+\beta-k+n+1)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)} & \frac{n(\beta-k+1)}{(\alpha+\beta-k+n+1)(\alpha+\beta+2n+2)(k+n)} \\ 0 & \frac{n(\alpha+n+1)(k+n+1)}{(k+n)(\alpha+\beta+2n+2)(\alpha+\beta+2n+3)} \end{pmatrix},$$

where

$$B_n^{11} = \frac{(n+k)(n+\beta+2)(n+1)}{(\alpha+\beta+2n+2)(n+k+1)(\alpha+\beta+2n+3)} + \frac{(n+\alpha)(\alpha+\beta-k+n+2)(n+\alpha+\beta+1)}{(\alpha+\beta+2n+2)(\alpha+1+n-k+\beta)(\alpha+\beta+2n+1)}$$
$$+ \frac{k(\beta-k+1)}{(\alpha+1+n-k+\beta)(n+k+1)(\alpha+\beta-k+n+2)(n+k)},$$

$$B_n^{22} = \frac{(n+\beta+2)(n+1)(n+k+2)}{(\alpha+\beta+2n+3)(\alpha+\beta+2n+4)(n+k+1)} + \frac{(\alpha+n+1)(\alpha+\beta+n+2)(\alpha+1+n-k+\beta)}{(\alpha+\beta+2n+3)(\alpha+\beta+2n+2)(\alpha+\beta-k+n+2)}.$$

UL DECOMPOSITION (SPECIAL CASE)

If we choose the free matrix parameter α_0 as the following matrix (Grünbaum-Pacharoni-Tirao, 2013)

$$\alpha_0 = \begin{pmatrix} \frac{\beta-k+1}{(1+\alpha+\beta-k)(1+k)(2+\alpha+\beta-k)} + \frac{\alpha(2+\alpha+\beta-k)}{(2+\alpha+\beta)(1+\alpha+\beta-k)} & \frac{\beta-k+1}{(1+k)(2+\alpha+\beta-k)} \\ \frac{1+\alpha}{(3+\alpha+\beta)(2+\alpha+\beta-k)} & \frac{(1+\alpha)(1+\alpha+\beta-k)}{(3+\alpha+\beta)(2+\alpha+\beta-k)} \end{pmatrix}$$

then the explicit expression for τ_n in the (monic) UL decomposition is given by

$$\tau_0^{-1}\tau_n = \begin{pmatrix} \frac{k(\beta+2)_n}{(n+k)(\alpha+\beta+n+1)_n} & 0 \\ \frac{n(\beta+2)_n}{(n+k)(\alpha+\beta+n-k+1)(\alpha+\beta+n+1)_n} & \frac{(\alpha+\beta-k+1)(\beta+2)_n}{(\alpha+\beta+n-k+1)(\alpha+\beta+n+2)_n} \end{pmatrix}$$

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The block entries of the stochastic matrices P_U and P_L are given by

$$X_n = \begin{pmatrix} \frac{(n+k)(n+\beta+2)}{(2n+\alpha+\beta+2)(n+k+1)} & 0 \\ 0 & \frac{n+\beta+2}{2n+\alpha+\beta+3} \end{pmatrix}, \quad Y_n = \begin{pmatrix} \frac{(n+\alpha)(n+\alpha+\beta-k+2)}{(2n+\alpha+\beta+2)(n+\alpha+1-k+\beta)} & \frac{\beta-k+1}{(n+\alpha+1-k+\beta)(n+k+1)} \\ 0 & \frac{n+\alpha+1}{2n+\alpha+\beta+3} \end{pmatrix},$$

$$S_n = \begin{pmatrix} \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} & 0 \\ \frac{k}{(n+\alpha+\beta-k+2)(n+k)} & \frac{(n+\alpha+\beta+2)(n+\alpha+1-k+\beta)}{(2n+\alpha+\beta+2)(n+\alpha+\beta-k+2)} \end{pmatrix}, \quad R_n = \begin{pmatrix} \frac{n}{2n+\alpha+\beta+1} & 0 \\ 0 & \frac{n(n+k+1)}{(2n+\alpha+\beta+2)(n+k)} \end{pmatrix}.$$

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The weight matrix $\widetilde{W}$ of the Darboux transformation is given by the Geronimus transformation $\widetilde{W}(x) = W(x)/x$ (there is no mass at 0!).

GENERAL α_0

Let us now study different situations with

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In the case of this example we have that

$$\mu_0 = \frac{\Gamma(\alpha+1)\Gamma(\beta+2)(\alpha+\beta-k+2)}{\Gamma(\alpha+\beta+3)} \begin{pmatrix} 1 & 0 \\ 0 & \frac{(\alpha+1)(k+1)}{(\alpha+\beta+3)(\beta-k+1)} \end{pmatrix}$$

and (assuming $\alpha > 0, \beta > -1$)

$$\mu_{-1} = \frac{\Gamma(\alpha)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \begin{pmatrix} \alpha+\beta-k+1 & -1 \\ -1 & \frac{(\alpha+1)(k+1)(\alpha+\beta-k+2)-k(\beta-k+1)}{(\alpha+\beta+2)(\beta-k+1)} \end{pmatrix}$$

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With these data we have that $\boldsymbol{M}$ is symmetric if and only if one of the entries of α_0 is chosen according to the following relation

$$s_{12} = \frac{(\beta-k+1)(\alpha+\beta+3)}{(\alpha+1)(k+1)} s_{21}$$

GENERAL α_0

Let us study the particular case of (two free parameters s_{11} and s_{21})

$$\alpha_0 = \begin{pmatrix} \frac{(\alpha+\beta+3)(\beta-k+1)}{(k+1)(\alpha+1)(\alpha+\beta-k+1)} s_{11} & \frac{(\alpha+\beta+3)(\beta-k+1)}{(k+1)(\alpha+1)} s_{21} \\ s_{21} & (\alpha+\beta-k+1) s_{21} \end{pmatrix}$$

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For this α_0 we have that $\boldsymbol{M}$ can be written as

$$\boldsymbol{M} = \tau_0 \begin{pmatrix} \frac{(\alpha)_2(k+1)(\alpha+\beta-k+2)}{(\beta-k+1)(\alpha+\beta+2)_2(s_{11}-s_{21})} - 1 & 0 \\ 0 & \frac{\alpha+1}{(\alpha+\beta+2)(\alpha+\beta-k+2)s_{21}} - 1 \end{pmatrix} \mu_0\big|_{\alpha=\alpha-1} \tau_0^*$$

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In this case we are able to find a sequence of matrices $(\tau_n)_n$ such that X_n and S_n are lower triangular, while Y_n and R_n are upper triangular (satisfying that $(X_n + Y_n)\boldsymbol{e}_d = \boldsymbol{e}_d$, $S_0\boldsymbol{e}_d = \boldsymbol{e}_d$ and $(R_n + S_n)\boldsymbol{e}_d = \boldsymbol{e}_d$).

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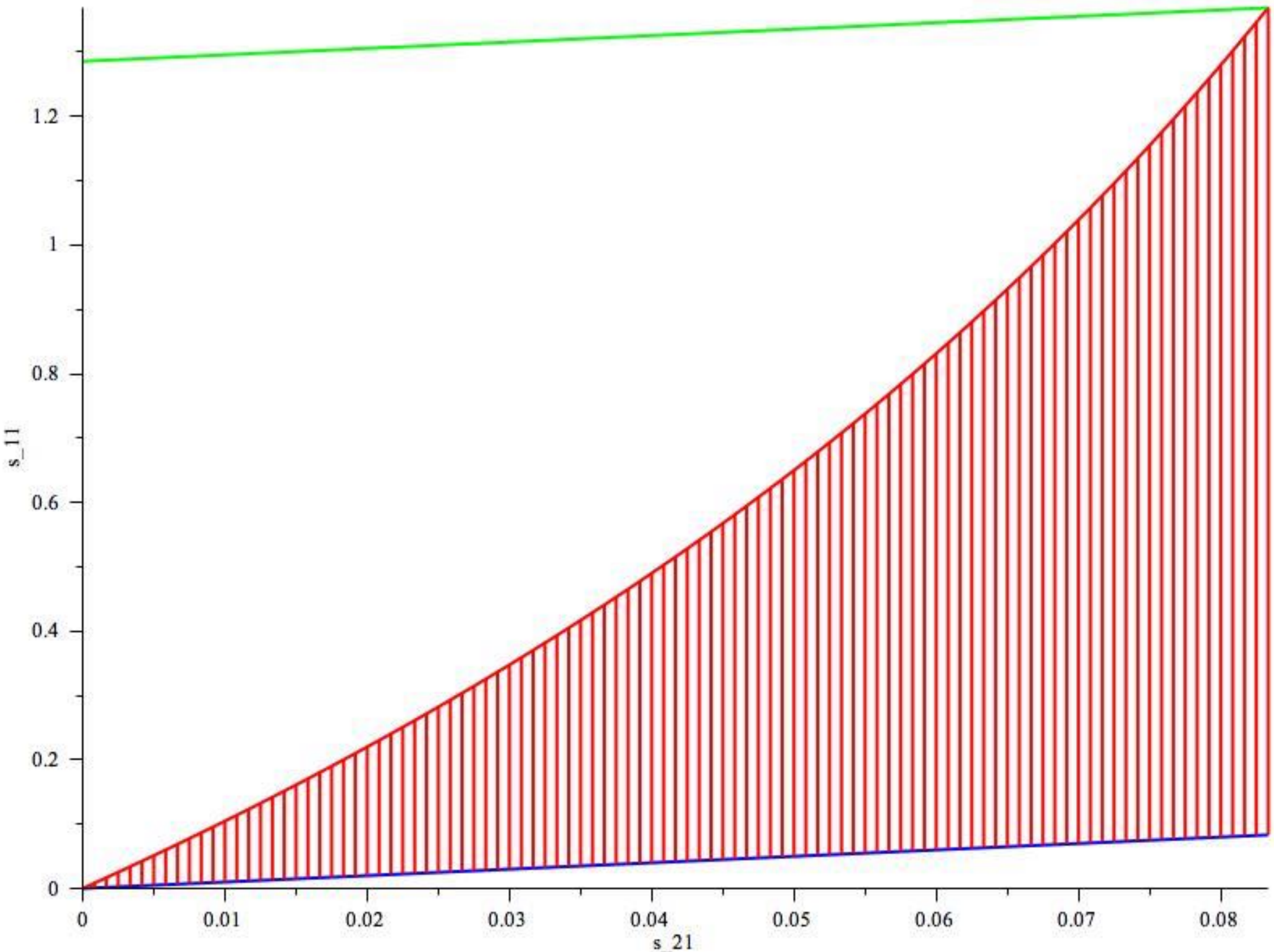
But we also need that all entries of X_n, Y_n, S_n, R_n to be nonnegative. After extensive symbolic computations we find that this holds (and therefore P_U and P_L are stochastic matrices) if the parameters s_{11} and s_{21} are chosen in the following range

$$0 < s_{21} \leq \frac{\alpha+1}{(\alpha+\beta+3)(\alpha+\beta-k+2)},$$

$$s_{21} < s_{11} \leq \frac{s_{21}\left(s_{21} - \frac{(\alpha+1)^2(k+1)}{k(\beta-k+1)(\alpha+\beta+3)}\right)}{s_{21} - \frac{(\alpha+1)(k+1)}{k(\alpha+\beta-k+1)(\alpha+\beta+3)}}$$

GENERAL α_0



The region with red stripes (shaded area) gives all possible values of s_{21} and s_{11} for which all entries of X_n, Y_n, S_n, R_n are nonnegative for the values of $\alpha = 3, \beta = 2, k = 1$. The green line is the upper bound for which M is positive semidefinite.

FINAL COMMENTS

Remark 1. We have been able to find a nice urn model for the case where

$$X_n = \begin{pmatrix} \frac{(n+k)(n+\beta+2)}{(2n+\alpha+\beta+2)(n+k+1)} & 0 \\ 0 & \frac{n+\beta+2}{2n+\alpha+\beta+3} \end{pmatrix}, \quad Y_n = \begin{pmatrix} \frac{(n+\alpha)(n+\alpha+\beta-k+2)}{(2n+\alpha+\beta+2)(n+\alpha+1-k+\beta)} & \frac{\beta-k+1}{(n+\alpha+1-k+\beta)(n+k+1)} \\ 0 & \frac{n+\alpha+1}{2n+\alpha+\beta+3} \end{pmatrix},$$

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$$X_n = \begin{pmatrix} \frac{(n+k)(n+\beta+2)}{(2n+\alpha+\beta+2)(n+k+1)} & 0 \\ 0 & \frac{n+\beta+2}{2n+\alpha+\beta+3} \end{pmatrix}, \quad Y_n = \begin{pmatrix} \frac{(n+\alpha)(n+\alpha+\beta-k+2)}{(2n+\alpha+\beta+2)(n+\alpha+1-k+\beta)} & \frac{\beta-k+1}{(n+\alpha+1-k+\beta)(n+k+1)} \\ 0 & \frac{n+\alpha+1}{2n+\alpha+\beta+3} \end{pmatrix},$$

$$S_n = \begin{pmatrix} \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} & 0 \\ \frac{k}{(n+\alpha+\beta-k+2)(n+k)} & \frac{(n+\alpha+\beta+2)(n+\alpha+1-k+\beta)}{(2n+\alpha+\beta+2)(n+\alpha+\beta-k+2)} \end{pmatrix}, \quad R_n = \begin{pmatrix} \frac{n}{2n+\alpha+\beta+1} & 0 \\ 0 & \frac{n(n+k+1)}{(2n+\alpha+\beta+2)(n+k)} \end{pmatrix}.$$

For α, β and k nonnegative integers with $1 \leq k \leq \beta$, the discrete-time QBD process on $\mathbb{Z}_{\geq 0} \times \{1, 2\}$ generated by the coefficients A_n, B_n, C_n can be decompose into two easier urn experiments (Experiment 1 and Experiment 2).

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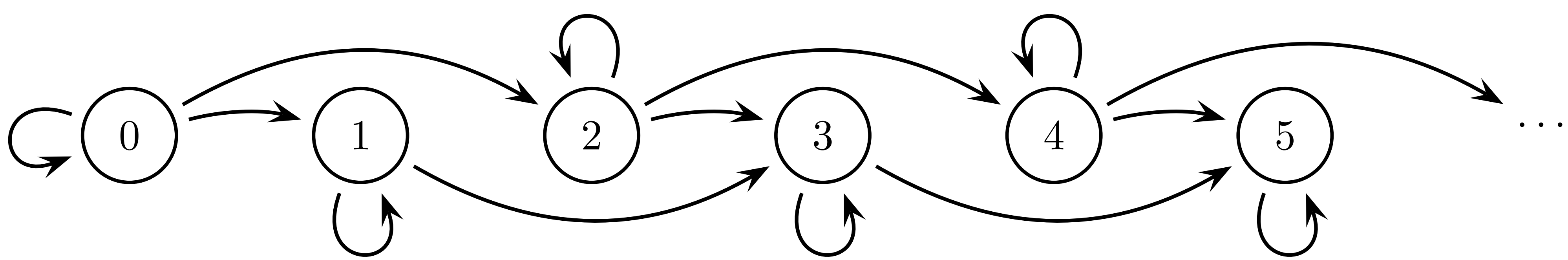
The Experiment 1 is can be interpreted as a pure-birth discrete-time Markov chain on $\mathbb{Z}_{\geq 0}$ with transitions between not only adjacent states but second adjacent ones too, and with diagram



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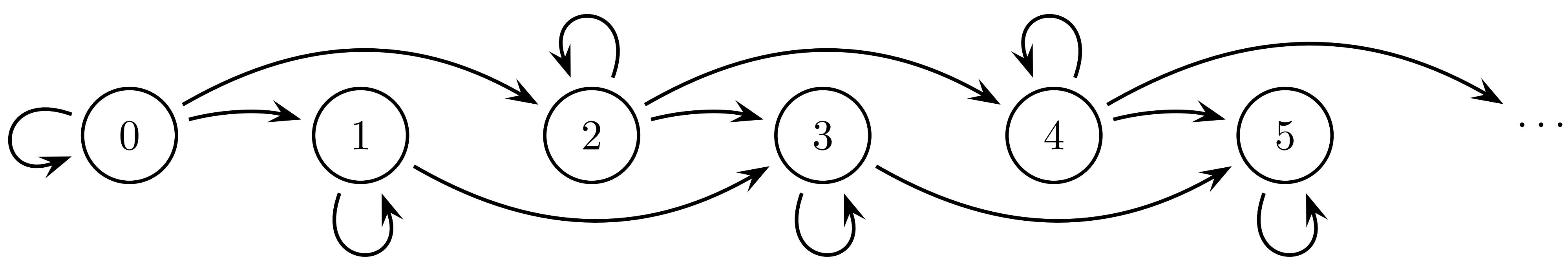

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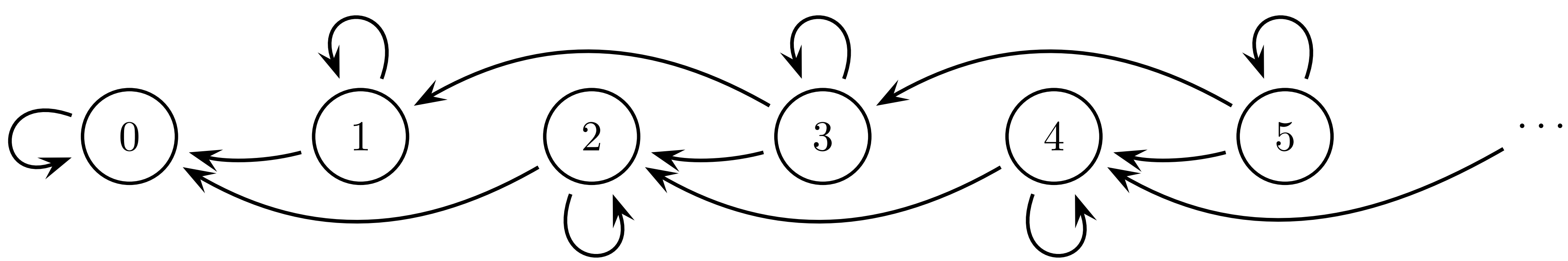

Remark 2. It is well known that the original matrix-valued orthogonal polynomials P_n satisfy a second-order differential equation of the form

$$P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0 = \Lambda_n P_n(x)$$

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Indeed, in this situation (and only in this situation) the matrix-valued polynomials $\widetilde{P}_n$ obtained by performing the Darboux transformation also satisfy a second-order differential equation of the form with coefficients $\widetilde{F}_2, \widetilde{F}_1, \widetilde{F}_0$ given by

$$\widetilde{F}_2(x) = x\left[\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} \frac{\beta-k+1}{\alpha+\beta-k+2} & -\frac{\beta-k+1}{\alpha+\beta-k+2} \\ -\frac{\alpha+1}{\alpha+\beta-k+2} & \frac{\alpha+1}{\alpha+\beta-k+2} \end{pmatrix}\right],$$

$$\widetilde{F}_1(x) = x\begin{pmatrix} 0 & 0 \\ k+1 & -(\alpha+\beta+3) \end{pmatrix} + \begin{pmatrix} -\frac{\beta-k+1}{\alpha+\beta-k+2} & -\frac{(\beta-k+1)(\alpha+\beta-k+1)}{\alpha+\beta-k+2} \\ \frac{\alpha+1}{\alpha+\beta-k+2} & \frac{(\alpha+1)(\alpha+\beta-k+1)}{\alpha+\beta-k+2} \end{pmatrix},$$

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Typically, in the scalar case, and for some special values of the parameters involved, the order of the differential equation satisfied by the Darboux polynomials is higher than 2. In the matrix case we have a family of matrix-valued orthogonal polynomials $\widetilde{P}_n$ (depending on one free parameter s_{11}) satisfying again a second-order differential equation with coefficients independent of s_{11}. This phenomenon is not new and appeared for the first time in Durán-MdI (2008) using a different method.


AUDITORIO


THANK YOU