# A variant of the Wright-Fisher diffusion model coming from the theory of matrix-valued spherical functions*,†

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## The Wright-Fisher model

Consider a gene population of constant size $M$ composed of two types $A$ and $B$ with mutations allowed (only) $A \xrightarrow{a} B$, $B \xrightarrow{b} A, a, b > 0$. Call $\#A = i$. The next generation is determined by $M$ independent binomial trials where each trial results in $A$ or $B$ with probabilities

$$p_i = \frac{i}{M}(1-a) + \left(1 - \frac{i}{M}\right) b, \quad q_i = 1 - p_i.$$

Therefore we generate a discrete-time Markov chain $\{X(n)\}$ where $X(n) = \{\#A \text{ in the } n\text{-th generation}\}$ with state space $\mathcal{S} = \{0, 1, \ldots, M\}$ and transition probability matrix $P = (P_{i,j})$ with

$$P_{i,j} = \Pr\{X(n+1) = j | X(n) = i\} = \binom{M}{j} p_i^j q_i^{M-j}.$$

Let us consider a huge population size ($M \to \infty$) and define the associated rescaled process

$$Y_t = \lim_{M\to\infty} Y_M(t) = \lim_{M\to\infty} \frac{X([Mt])}{M}, \quad t \in [0, +\infty).$$

If we call $h = 1/M$ and $x = i/M$, we have that

$$\tau(x) = \lim_{h\to 0^+} \frac{1}{h} E\left[Y_M(t+h) - Y_M(t) | Y_M(t) = x\right] = -\gamma_1 x + (1-x)\gamma_2,$$

$$\sigma^2(x) = \lim_{h\to 0^+} \frac{1}{h} E\left[(Y_M(t+h) - Y_M(t))^2 | Y_M(t) = x\right] = x(1-x),$$

where $\gamma_1 = aM$ and $\gamma_2 = bM$ are the intensities of mutation. Therefore $Y_t$ is a continuous-time diffusion process with state space $\mathcal{S} = [0,1]$, *drift* $\tau(x)$ and *diffusion* coefficient $\sigma^2(x)$. $Y_t$ evolves according to the stochastic differential equation

$$dY_t = \tau(Y_t) + \sigma(Y_t) dB_t.$$

Write $\gamma_1 = \frac{1+\beta}{2}$ and $\gamma_2 = \frac{1+\alpha}{2}$. The infinitesimal operator $\mathcal{A}$ of the process $Y_t$ is

$$\mathcal{A} = x(1-x)\frac{d^2}{dx^2} + (1 + \alpha - x(\alpha+\beta+2))\frac{d}{dx}, \quad \alpha, \beta > -1.$$

The orthonormal Jacobi polynomials $P_n^{\alpha,\beta}(x)$ (orthogonal with respect to $\omega(x) = x^\alpha(1-x)^\beta$) are eigenfunctions of the infinitesimal operator $\mathcal{A}$, i.e.,

$$\mathcal{A}P_n^{\alpha,\beta}(x) = \lambda_n P_n^{\alpha,\beta}(x), \quad \lambda_n = -n(n+\alpha+\beta+1).$$

As a consequence, we get a spectral representation of the probability density

$$p(t;x,y) = \sum_{n=0}^{\infty} e^{\lambda_n t} P_n^{\alpha,\beta}(x) P_n^{\alpha,\beta}(y) y^\alpha (1-y)^\beta,$$

and the invariant distribution of the process ($\alpha, \beta \geqslant 0$)

$$\psi(y) = \lim_{t\to\infty} p(t;x,y) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} y^\alpha (1-y)^\beta.$$

We know from [2] that the boundaries $0, 1$ are absorbing if $-1 < \alpha, \beta < 0$ and reflecting if $\alpha, \beta \geqslant 0$.

## A variant of the Wright-Fisher model

We consider now a *hybrid process* of the form $\{(Y_t, R_t) : t \in [0, +\infty)\}$ where $Y_t \in [0,1]$ is a Wright-Fisher type diffusion process and $R_t \in \{1, 2, \ldots, N\}$ is a continuous-time Markov chain representing $N$ different *phases* for which the coefficients of the process $Y_t$ may change. These processes are also known as diffusions with Markovian switching. Our process evolves according to the stochastic differential equation

$$dY_t = \tau_{R_t}(Y_t) + \sigma_{R_t}(Y_t) dB_t,$$

$$\tau_i(x) = \alpha + 1 + N - i - x(\alpha + \beta + 2 + N - i), \quad \sigma_i^2(x) = 2x(1-x).$$

Now the intensities of mutations depend on the phase so

$$A \xrightarrow{\frac{\beta+1}{2}} B \quad \text{and} \quad B \xrightarrow{\frac{\alpha+N-i+1}{2}} A, \quad i = 1, 2, \ldots, N.$$

Observe that at phase $N$ we recover the original Wright-Fisher model, but the intensity of mutation $B \to A$ grows as we get closer to the first phases.

The continuous-time process $R_t$ (depending also on the position $Y_t$) evolves according to a birth-and-death process whose infinitesimal operator is given by an $N \times N$ tridiagonal matrix $\boldsymbol{Q}(x)$:

$$\boldsymbol{Q}_{i,i-1}(x) = \frac{1}{1-x}(N-i)(i+\beta-k), \quad \boldsymbol{Q}_{i,i+1}(x) = \frac{x}{1-x}(i-1)(N-i+k), \tag{1}$$

$$\boldsymbol{Q}_{i,i}(x) = -(\boldsymbol{Q}_{i,i-1}(x) + \boldsymbol{Q}_{i,i+1}(x)), \quad 0 < k < \beta + 1. \tag{2}$$

$\boldsymbol{Q}(x)$ only depends on $\beta$ and a NEW parameter $k$. We have that the boundary $0$ (or $1$) is reflecting if $\alpha \geqslant 0$ (or $\beta \geqslant 0$) and absorbing if $-1 < \alpha < 0$ AND the process is on phase $N$ (or $-1 < \beta < 0$).

Let us now study a couple of aspects from the discrete component $R_t$. First we will study the waiting times at each phase depending on the parameters and secondly we will study the tendency of moving forward or backward in phases. All these facts depend on the position of $X_t$.

For the waiting times we have to take a look to the diagonal entries of $\boldsymbol{Q}(x)$ (see (2)). We remark that if $x \to 1^-$ then all phases are instantaneous. If $x \to 0^+$ or $k \to 0^+$ then the phase $N$ is absorbing (see first plot in Figure 1). Finally, if $k \to \beta + 1$ then the phase 1 is absorbing (see second plot in Figure 1).

For the tendency, we observe from (1), that if $k \to \beta + 1$ then we have a backward tendency, meaning that the parameter $k$ helps the population of $A$'s to survive against the population of $B$'s. If $k \to 0^+$ then we have a forward tendency, meaning that both populations $A$ and $B$ fight in the same conditions (see Figure 1). Finally, for middle values of $k$ we may have forward and backward tendency, depending on the position of the process.

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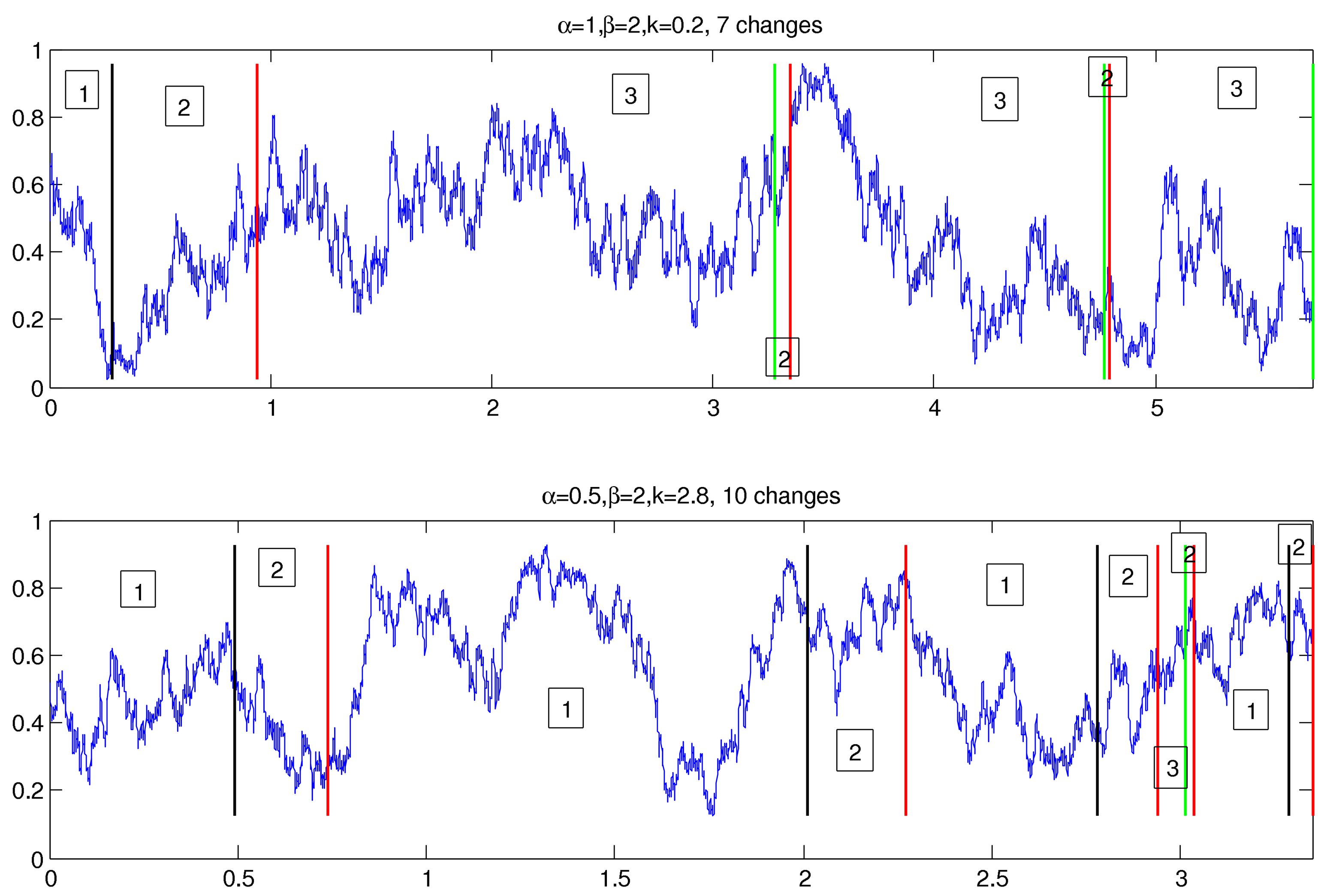


Figure 1: In the first plot $k$ is close to 0, so the process tends to spend more time in the last absorbing phase 3. In the second plot, $k$ is close to $\beta+1$, so the phase 1 is absorbing.

The infinitesimal operator $\mathcal{A}$ of the process $(Y_t, R_t)$ is now *matrix-valued*

$$\mathcal{A} = \frac{1}{2}\boldsymbol{A}(x)\frac{d^2}{dx^2} + \boldsymbol{B}(x)\frac{d}{dx} + \boldsymbol{Q}(x)\frac{d^0}{dx^0}, \quad \boldsymbol{A}(x) = 2x(1-x)\boldsymbol{I}, \quad \boldsymbol{B}_{ii}(x) = \tau_i(x).$$

We already know (see [1]) a family of matrix-valued orthonormal eigenfunctions $\boldsymbol{\Phi}_n(x)$ of $\mathcal{A}$, i.e,

$$\mathcal{A}\boldsymbol{\Phi}_n(x) = \boldsymbol{\Phi}_n(x)\boldsymbol{\Gamma}_n, \quad \boldsymbol{\Gamma}_n \quad \text{diagonal.}$$

They are called the matrix-valued spherical functions associated with the complex projective space $P_n(\mathbb{C}) = SU(n+1)/U(n)$. The corresponding weight matrix $\boldsymbol{W}(x)$ is diagonal with entries

$$\boldsymbol{W}_{ii}(x) = x^\alpha(1-x)^\beta \binom{\beta-k+i-1}{i-1}\binom{N+k-i-1}{N-i} x^{N-i}.$$

As a consequence, we get a spectral representation of the *matrix-valued* probability density

$$\boldsymbol{P}(t;x,y) = \sum_{n=0}^{\infty} \boldsymbol{\Phi}_n(x) e^{\boldsymbol{\Gamma}_n t} \boldsymbol{\Phi}_n^*(y) \boldsymbol{W}(y).$$

The invariant distribution ($\alpha, \beta \geqslant 0$) is now a row vector $\boldsymbol{\psi}(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y))$ with $0 \leqslant \psi_j(y) \leqslant 1$ and $\sum_{j=1}^N \int_0^1 \psi_j(y) dy = 1$. In our case, we have

$$\boldsymbol{\psi}(y) = \left(\int_0^1 e_N^T \boldsymbol{W}(x) e_N dx\right)^{-1} e_N^T \boldsymbol{W}(y).$$

where $e^T = (1, 1, \ldots, 1)$. In particular, for our example (see Figure 2),

$$\psi_j(y) = y^{\alpha+N-j}(1-y)^\beta \binom{N-1}{j-1}\binom{\alpha+\beta+N}{\alpha}\frac{(\beta+N)(k)_{N-j}(\beta-k+1)_{j-1}}{(\alpha+\beta-k+2)_{N-1}}.$$

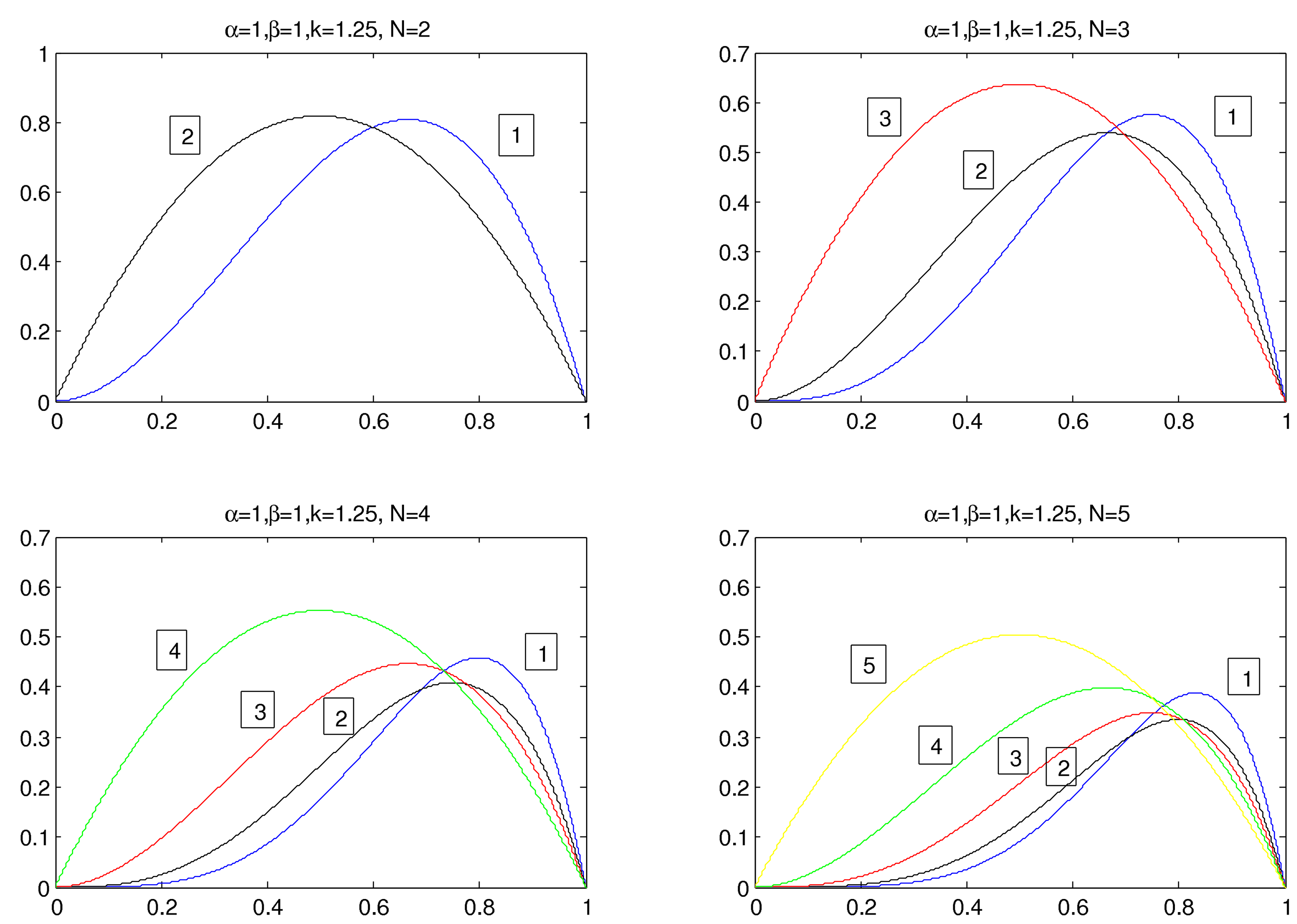


Figure 2: The components $\psi_j(y), j = 1, \ldots, N$, for $N = 2, 3, 4, 5$, and $\alpha, \beta = 1, k = 5/4$.

## References

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