Differential equations for discrete Laguerre-Sobolev orthogonal polynomials*

Manuel Domínguez de la Iglesia

Instituto de Matemáticas, C.U., UNAM

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*Joint work with Antonio J. Durán

Goal of the talk

Let $\mu_\alpha(x)$ be the Laguerre weight

$$\mu_\alpha(x) = x^\alpha e^{-x}, \quad x > 0, \quad \alpha > -1$$

For $m \geq 1$, let $\mathbf{M} = (M_{ij})_{i,j=0}^{m-1}$ be any $m \times m$ matrix.
We will consider the discrete Laguerre-Sobolev bilinear form

$$\langle p, q \rangle = \int_0^\infty p(x)q(x)\mu_{\alpha-m}(x)dx + \mathbb{P}(0)\,\mathbf{M}\,\mathbb{Q}(0)^T$$

$\mathbb{P}(0) = \big(p(0), p'(0), \ldots, p^{(m-1)}(0)\big)$ and $\alpha \neq m-1, m-2, \ldots$

- Give an expression of the (left) OP $(q_n)_n$ as a linear combination of $m+1$ consecutive Laguerre polynomials $(L_n^\alpha)_n$.
- Find a higher-order differential operator D_q such that $D_q(q_n) = \lambda_n q_n$, where D_q is a polynomial combination of D_α.

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Some history

- $m = 0 \Rightarrow \mathbf{M} = 0 \Rightarrow \mu_\alpha(x)$.

  Laguerre OP's $(L_n^\alpha)_n$ (Laguerre, 1879). They are eigenfunctions of the second-order differential operator

$$D_\alpha = -x\left(\frac{d}{dx}\right)^2 - (\alpha+1-x)\frac{d}{dx}, \quad D_\alpha(L_n^\alpha) = nL_n^\alpha, \quad n \geq 0$$

- $m = 1 \Rightarrow \mathbf{M} = M_0 \in \mathbb{R} \Rightarrow \mu_{\alpha-1}(x) + M_0\delta_0$.

  Krall-Laguerre OP's.

  - H. L. Krall, 1939 ($\alpha = 1$). Fourth-order differential operator.
  - L. L. Littlejohn, 1984 ($\alpha = 2, 3$). Sixth and eighth-order differential operator.
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- A natural extension of the Krall-Laguerre OP's is to consider the moment functional (Grünbaum-Haine-Horozov, 1999)

$$\mu_{\alpha-m} + \sum_{i=0}^{m-1} M_i \delta_0^{(i)}$$

This case represents our bilinear form with $\mathbf{M}$ the symmetric matrix ($M_{m-1} \neq 0$)

$$\mathbf{M} = \begin{pmatrix} M_0 & M_1 & \cdots & M_{m-2} & M_{m-1} \\ M_1 & M_2 & \cdots & M_{m-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ M_{m-2} & M_{m-1} & \cdots & 0 & 0 \\ M_{m-1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

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- $m = 2$ and $\mathbf{M}$ is the 2×2 diagonal matrix

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These are the Laguerre-Sobolev OP's.

R. Koekoek (1991) and later Koekoek's-Bavinck (1998) proved that if $\alpha \geq 2$ is a nonnegative integer

$$\text{order diff. op.} = \begin{cases} 2\alpha + 8, & \text{if } M_0 = 0, M_1 > 0, \\ 4\alpha + 10, & \text{if } M_0, M_1 > 0; \end{cases}$$

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Orthogonality

Theorem (Durán-MdI, 2014)

Let $\mathbf{M} = (M_{ij})_{i,j=0}^{m-1}$ be any $m \times m$ matrix and $\alpha \neq m-1, m-2, \dots$ Consider

$$\langle p, q \rangle = \int_0^\infty p(x)q(x)\mu_{\alpha-m}(x)dx + \mathbb{P}(0)\,\mathbf{M}\,\mathbb{Q}(0) \qquad (1)$$

and the functions $(j = 1, 2, \dots m)$

$$\mathcal{R}_j(x) = \frac{\Gamma(\alpha - m + j)}{(m-j)!}(x+1)_{m-j} + (j-1)!\frac{\Gamma(\alpha+1+x)}{\Gamma(1+x)}\sum_{i=0}^{m-1}\frac{(-1)^i M_{j-1,i}}{\Gamma(\alpha+i+1)}(x-i+1)_i$$

Then the family $(q_n)_n$ defined by the Casorati determinant

$$q_n(x) = \begin{vmatrix} L_n^\alpha(x) & L_{n-1}^\alpha(x) & \cdots & L_{n-m}^\alpha(x) \\ \mathcal{R}_1(n) & \mathcal{R}_1(n-1) & \cdots & \mathcal{R}_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}_m(n) & \mathcal{R}_m(n-1) & \cdots & \mathcal{R}_m(n-m) \end{vmatrix}, \quad n \geq 0$$

is (left) orthogonal with respect to (1) if and only if

$$\Omega(n) = \det(\mathcal{R}_i(n-j))_{i,j=1}^m \neq 0, \quad n \geq 0$$

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Differential operator

Theorem (Durán-MdI, 2014)

Let $Y_1, Y_2, \dots, Y_m$ be m arbitrary polynomials and assume that

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Then there exists a differential operator D_q of the form

$$D_q = P(D_\alpha) + \sum_{h=1}^m M_h(D_\alpha)\frac{d}{dx}Y_h(D_\alpha)$$

such that $D_q(q_n) = P(n)q_n$ where $P(x) - P(x-1) = \Omega(x)$ and the polynomials $M_h(x), h = 1, \dots, m$ are defined by

$$M_h(x) = \sum_{j=1}^m (-1)^{h+j} \det\left(Y_l(x+j-r)\right)_{\left\{\substack{l \neq h \\ r \neq j}\right\}}$$

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In other to combine the previous two results we have to force that

$$Y_j(x) = \mathcal{R}_j(x) = \frac{\Gamma(\alpha - m + j)}{(m-j)!}(x+1)_{m-j} + (j-1)!\frac{\Gamma(\alpha+1+x)}{\Gamma(1+x)}\sum_{i=0}^{m-1}\frac{(-1)^i M_{j-1,i}}{\Gamma(\alpha+i+1)}(x-i+1)_i$$

are polynomials in x for $j = 1, 2, \ldots, m$.

This holds only when α is a nonnegative integer with $\alpha \geq m$.
In that case the functions $\mathcal{R}_l(x)$ are polynomials given by

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Order of the differential operator

The order of the differential operator D_q is given by twice the degree of the polynomial P defined by $P(x) - P(x-1) = \Omega(x)$ where

$$\Omega(x) = \det(\mathcal{R}_i(x-j))_{i,j=1}^m$$

Therefore this order should be $2(\deg \Omega(x) + 1)$.

This degree depends on the polynomials $\mathcal{R}_i(x)$ and therefore on the matrix $\mathbf{M}$. There can be two cases:

- If $\deg \mathcal{R}_i \neq \deg \mathcal{R}_j$ for $i \neq j$. Then (Durán-MdI, 2014)

$$\deg \Omega(x) = \sum_{j=1}^{m} \deg \mathcal{R}_j(x) - \frac{m(m-1)}{2}$$

- Otherwise (Durán-MdI, 2014)

$$\deg \Omega(x) = \alpha\text{-wr}(\mathbf{M}) = \sum_{j=1}^{m} n_j + \sum_{j=1}^{m-1} m_j - \frac{m(m-1)}{2}$$

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How to obtain α-wr(**M**)?

Write **M** by columns $\mathbf{M} = [c_1|c_2|\cdots|c_m]$.

Define the numbers $n_1, n_2, \ldots, n_m$ by

$$n_1 = \begin{cases} \alpha + m - 1, & \text{if } c_m \neq 0, \\ 0, & \text{if } c_m = 0; \end{cases}$$

and for $j = 2, \ldots, m$,

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Consider now $\tilde{\mathbf{M}}$ the matrix whose columns are c_i, $i \in \{j : n_{m-j+1} \neq 0\}$ and write $f_1, \ldots, f_m$, for the rows of $\tilde{\mathbf{M}}$.

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$$n_1 = \begin{cases} \alpha + m - 1, & \text{if } c_m \neq 0, \\ 0, & \text{if } c_m = 0; \end{cases}$$

and for $j = 2, \ldots, m$,

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Consider now $\tilde{\mathbf{M}}$ the matrix whose columns are c_i, $i \in \{j : n_{m-j+1} \neq 0\}$ and write $f_1, \ldots, f_m$, for the rows of $\tilde{\mathbf{M}}$.

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How to obtain α-wr($\mathbf{M}$)?

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Let $\mathbf{M}$ be the symmetric matrix $(M_{m-1} \neq 0)$

$$\mathbf{M} = \begin{pmatrix} M_0 & M_1 & \cdots & M_{m-2} & M_{m-1} \\ M_1 & M_2 & \cdots & M_{m-1} & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ M_{m-2} & M_{m-1} & \cdots & 0 & 0 \\ M_{m-1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$(q_n)_n$ are orthogonal with respect to $\mu_{\alpha-m} + \sum_{i=0}^{m-1} M_i \delta_0^{(i)}$.

The polynomials $\mathcal{R}_j(x)$ are in this case

$$\mathcal{R}_j(x) = \frac{(\alpha-m+j-1)!}{(m-j)!}(x+1)_{m-j} + (j-1)!(x+1)_\alpha \sum_{i=0}^{m-j} \frac{(-1)^i M_{j+i-1}}{(\alpha+i)!}(x-i+1)_i$$

Therefore $\deg \mathcal{R}_j = \alpha + m - j$ and $\deg \mathcal{R}_i \neq \deg \mathcal{R}_j$ for $i \neq j$, since $\alpha \geq m$.

$$\deg \Omega(x) = \alpha m \quad \Rightarrow \quad \operatorname{ord}(D_q) = 2(\alpha m + 1)$$

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In this case we have to use the definition of α-wr($\mathbf{M}$) to find $\deg \Omega(x)$.

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$$n_1 = \alpha + 2, n_2 = \alpha + 1, n_3 = 0, \quad m_1 = 2, m_2 = 0, \quad \Rightarrow \quad \alpha\text{-wr}(\mathbf{M}) = 2\alpha + 2$$

For $\alpha = 3$, $\mathcal{R}_1(x), \mathcal{R}_2(x), \mathcal{R}_3(x)$ are given by

$$\begin{aligned} \mathcal{R}_1(x) &= -\frac{(x+1)(x+2)(x^2 - x - 24)}{24} \\ \mathcal{R}_2(x) &= -\frac{(x+1)(x^3 + x^2 - 14x - 48)}{24} \\ \mathcal{R}_3(x) &= \frac{(x+4)(x^4 + x^3 + x^2 - 9x + 30)}{60} \end{aligned}$$

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