SOME RECENT DEVELOPMENTS ABOUT BIRTH-AND-DEATH MODELS AND ORTHOGONAL POLYNOMIALS

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VI Iberoamerican Workshop on Orthogonal Polynomials and Applications
Uberaba, May 9-12, 2017

1 Joint work with Pablo Román
OUTLINE

1. Markov chains and OP
   - Markov chains
   - Bivariate Markov chains

2. The new example
   - Matrix-valued spherical functions
   - Two birth-and-death models
Markov chains and OP

1. Markov chains and OP
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2. The new example
   - Matrix-valued spherical functions
   - Two birth-and-death models
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. A (1-D) Markov chain with state space \(S \subset \mathbb{N}\) is a collection of \(S\)-valued random variables \(\{X_t : t \in \mathcal{T}\}\) indexed by a parameter set \(\mathcal{T}\) (time) such that they have the Markov property, i.e. the behavior of the future only depends on the present and not the past.

The main goal is to find a description of the transition probabilities

\[ P_{ij}(t) \equiv \mathbb{P}(X_t = j | X_0 = i), \quad i, j \in S \subset \mathbb{N} \]

- **Discrete time:** the process is characterized in terms of the one-step transition probability matrix \(P\), which entries are given by

\[ P_{ij} = \mathbb{P}(X_1 = j | X_0 = i) \quad \Rightarrow \quad P_{ij}(n) = (P^n)_{ij} \]

- **Continuous time:** in this case the process is characterized (among other properties) by the behavior near \(t \to 0^+\), i.e.

\[
\begin{align*}
P_{ij}(t) &= ta_{ij} + o(t), \quad t \to 0^+, \quad a_{ij} \geq 0, \\
P_{ii}(t) &= 1 - ta_i + o(t), \quad t \to 0^+, \quad a_i = \sum_{j \neq i} a_{ij}
\end{align*}
\]

If we denote by \(A = (a_{ij})\), then we have \(P'(t) = AP(t) = P(t)A\).
One dimensional Markov Chains

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A (1-D) Markov chain with state space $S \subset \mathbb{N}$ is a collection of $S$-valued random variables $\{X_t : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) such that they have the Markov property, i.e. the behavior of the future only depends on the present and not the past. The main goal is to find a description of the transition probabilities

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If we denote by $\mathcal{A} = (a_{ij})$, then we have $P'(t) = \mathcal{A}P(t) = P(t)\mathcal{A}$. 
**Examples related to OP**

1. **Random walks**: \( S = \{0, 1, 2, \ldots\} \), \( T = \{0, 1, 2, \ldots\} \). Transitions are only allowed between adjacent states, i.e.

\[
P_{ij} = \begin{cases} 
p_i, & \text{if } j = i + 1 \\
r_i, & \text{if } j = i \\
q_i, & \text{if } j = i - 1 \\
0, & \text{elsewhere} \end{cases}
\]

Therefore, \( P \) is a semi-infinite tridiagonal matrix (Jacobi matrix)

\[
P = \begin{pmatrix}
  r_0 & p_0 & 0 \\
  q_1 & r_1 & p_1 & 0 \\
  0 & q_2 & r_2 & p_2 \\
  \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}, \quad r_i \geq 0, p_i, q_i > 0, \quad p_i + r_i + q_i = 1
\]

The \( n \)-step transition probability matrix is then given by \( P^{(n)} = P^n \).

Some examples related to OP are the gambler’s ruin (Jacobi) and urn models like the Ehrenfest model (Krawtchouk) or the Laplace-Bernoulli model (Hahn).
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2. **Birth and death processes**: \( S = \{0, 1, 2, \ldots \} \), \( T = [0, \infty) \).

The infinitesimal transitions are only allowed between adjacent states, i.e.

\[
\lim_{t \to 0^+} P_{ij}(t) = \begin{cases} 
  t\lambda_i + o(t), & \text{if } j = i + 1 \\
  1 - (\lambda_i + \mu_i) t + o(t), & \text{if } j = i \\
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Therefore we have a semi-infinite tridiagonal matrix \( A \) (also a Jacobi matrix)

\[
A = \begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & \cdots \\
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The transition probability matrix \( P(t) \) satisfies the backward and forward Kolmogorov equations \( P'(t) = AP(t) = P(t)A, P(0) = I \).

Some examples of birth-and-death processes related to OP are the \( M/M/k \) queue (Chebychev) or linear birth-and-death processes (Charlier, Meixner, Krawtchouk, Laguerre).
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Spectral methods

Spectral or Favard’s Theorem: there exist a unique measure associated with $P$ or $A$. Therefore it is possible to find spectral representations of the transition probabilities (Karlin-McGregor formulas):

1. **Random walks**: the measure $\omega$ is supported on $[-1, 1]$. If we denote by $(q_i)$ the sequence of OP generated by $P$ then we have

   $$\mathbb{P}(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

2. **Birth-and-death processes**: now the measure $\omega$ is supported on $[0, \infty)$. If we denote by $(q_i)$ the sequence of OP generated by $A$ then we have

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**The spectral measure can either be discrete (finite or infinite) or continuous.**
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**The spectral measure can either be discrete (finite or infinite) or continuous.**
Bivariate Markov Chains

Now consider a bivariate or 2-component Markov chain of the form

$$\{(X_t, Y_t) : t \in \mathcal{T}\}, \quad X_t \in S \subset \mathbb{N}, \quad Y_t \in \{1, 2, \ldots, N\}$$

The first component is the level and the second component is the phase.

Now the transition probabilities can be written in terms of block matrix (each block of dimension $N \times N$)

$$(P_{ij}(t))_{i,j'} = \mathbb{P}(X_t = j, Y_t = j' | X_0 = i, Y_0 = i')$$

- Discrete time: as in one dimensional case the process is characterized by its one-step transition probability block matrix $P$.
- Continuous time: as in one dimensional case the process is characterized by the Kolmogorov equations $P'(t) = \mathcal{A}P(t) = P(t)\mathcal{A}$ but now $\mathcal{A}$ is a block matrix with same properties as before.

Ideas behind: *random evolutions* (Griego-Hersh-Papanicolaou-Pinsky-Kurtz...60’s and 70’s).
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(P_{ij}(t))_{i', j'} = \mathbb{P}(X_t = j, Y_t = j' | X_0 = i, Y_0 = i')
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Processes related to matrix-valued OP

Quasi-birth-and-death processes:
$P$ and $A$ are now block-tridiagonal matrices of the form

$$
\begin{pmatrix}
B_0 & A_0 & & \\
C_1 & B_1 & A_1 & \\
& C_2 & B_2 & A_2 \\
& & \ddots & \ddots & \ddots \\
\end{pmatrix}
$$

Depending if time is discrete or continuous we will have special stochastic properties for the block entries of $P$ or $A$:

- Discrete time: all entries of $B_n$, $A_n$, $C_{n+1}$ are nonnegative and $(B_0 + A_0)e_N = e_N$, $(C_n + B_n + A_n)e_N = e_N$, $n \geq 1$, where $e_N = (1, \ldots, 1)^T$.

- Continuous time: all off-diagonal entries of $A$ are nonnegative and $(B_0 + A_0)e_N = 0_N$, $(C_n + B_n + A_n)e_N = 0_N$, $n \geq 1$.

The main tool to study spectral methods will be the theory of matrix-valued orthogonal polynomials.
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   where $e_N = (1, \ldots, 1)^T$.

2. **Continuous time:** all off-diagonal entries of $A$ are nonnegative and 
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Depending if time is discrete or continuous we will have special stochastic properties for the block entries of $P$ or $A$:

1. **Discrete time:** all entries of $B_n, A_n, C_{n+1}$ are nonnegative and
   \[(B_0 + A_0)e_N = e_N, \quad (C_n + B_n + A_n)e_N = e_N, \quad n \geq 1,\]
   where $e_N = (1, \ldots, 1)^T$.

2. **Continuous time:** all off-diagonal entries of $A$ are nonnegative and
   \[(B_0 + A_0)e_N = 0_N, \quad (C_n + B_n + A_n)e_N = 0_N, \quad n \geq 1.\]

The main tool to study spectral methods will be the theory of matrix-valued orthogonal polynomials.
**SPECTRAL METHODS**

**Spectral or Favard’s Theorem:** under certain symmetry conditions on the coefficients of the block entries of \( P \) or \( A \) there exists a unique weight matrix \( W \). Therefore we will have again spectral representations of the transition probabilities:

1. **Quasi-birth-and-death processes** (discrete time): the weight matrix \( W \) is supported on \([-1, 1]\). If we denote by \((Q_i)\) the sequence of MVOP generated by \( P \) then we have (Grüenbaum and Dette-Reuther-Studden-Zygmunt, 2007):

\[
P^n_{ij} = \left( \int_{-1}^{1} x^n Q_i(x) dW(x) Q_j^*(x) \right) \left( \int_{-1}^{1} Q_j(x) dW(x) Q_j^*(x) \right)^{-1}
\]

2. **Quasi-birth-and-death processes** (continuous time): now the weight matrix \( W \) is supported on \([0, \infty)\). If we denote by \((Q_i)\) the sequence of MVOP generated by \( A \) then we have (Dette-Reuther, 2010):

\[
P_{ij}(t) = \left( \int_{0}^{\infty} e^{-xt} Q_i(x) dW(x) Q_j^*(x) \right) \left( \int_{0}^{\infty} Q_j(x) dW(x) Q_j^*(x) \right)^{-1}
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**Again, the spectral weight matrix can either be discrete (finite or infinite) or continuous.**
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Outline

1. Markov chains and OP
   - Markov chains
   - Bivariate Markov chains

2. The new example
   - Matrix-valued spherical functions
   - Two birth-and-death models
Matrix-valued spherical functions

Spherical functions associated with groups of the form $G/K$ where $(G, K)$ is a Gel’fand pair are very much related with OP (Helgason, Vilenkin, Klimyk). They are eigenfunctions of the Casimir operator associated with the group. The extension to the matrix-valued case was started by Tirao (1977). The connection with MVOP was discovered by Grünbaum-Pacharoni-Tirao (2003).

1. **Complex projective space:** $P_n(\mathbb{C}) = SU(n+1)/U(n)$. Grünbaum-Pacharoni-Tirao (2002). Later it was found the relation with stochastic processes by Grünbaum-MdI (2008), Grünbaum-Pacharoni-Tirao (2012) and MdI (2012).

2. **Complex hyperbolic plane:** $H_2(\mathbb{C}) = SU(2,1)/U(2)$. Pacharoni-Román-Tirao (2006). Dual to the complex projective plane $P_2(\mathbb{C}) = SU(3)/U(2)$.

3. **Real sphere:** $S^n = SO(n+1)/O(n)$. Tirao-Zurrián (2013). Also connected with the real projective space $P_n(\mathbb{R}) = SO(n+1)/O(n)$.

In all cases (and others not mentioned) an explicit expression of the weight matrix, the second-order differential operator, the three-term recurrence relation and other structural formulas were derived for the matrix-valued spherical functions. In most of the cases the relation with MVOP was also given.
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**The pair \((SU(2) \times SU(2), \text{diag } SU(2))\)**

Koornwinder (1985) studied spherical functions associated with pairs of the form \((K \times K, K)\), where the subgroup is **diagonally embedded** and \(K = SU(2)\).

More recently Koelink-van Pruijssen-Román (2012) studied with a different approach this example and give the relation with MVOP.

For \(\ell \in \mathbb{N}\) and \(N = 2\ell + 1\) they produced a one-parameter family of \(N \times N\) MVOP where the **weight matrix** is

\[
W(y) = [y(1 - y)]^{\nu-1/2} \Psi_0(y) T(\Psi_0(y))^*, \quad T_{ij} = \delta_{ij} \left(\frac{2\ell}{i}\right) \frac{(\nu)_i}{(\nu + 2\ell - i)_i}
\]

where \(\Psi_0(y)\) is certain matrix-valued function containing spherical functions.

The corresponding **symmetric second-order differential operator** is given by

\[
D = y(1 - y)\partial_y^2 + (C + \nu - y(2\ell + 2\nu + 1))\partial_y - (V - (\nu - 1)(2\ell + \nu + 1))
\]

where \(C\) is tridiagonal and \(V\) diagonal with **eigenvalue**

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\Lambda_n = -n(n - 1) - n(2\ell + 2\nu + 1) - (V - (\nu - 1)(2\ell + \nu + 1))
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**Two Important Facts**

1. The structure of the group induces the existence of a constant matrix $Y$ such that we can decompose by blocks the weight matrix $W$ in the form

$$
\tilde{W}(y) = YW(y)Y^* = \begin{pmatrix} W_1(y) & 0 \\ 0 & W_2(y) \end{pmatrix}
$$

where $W_1$ is $(\ell + 1) \times (\ell + 1)$ and $W_2$ is $\ell \times \ell$. So we will study the probabilistic aspects of these two independent processes ($\ell = 1$).

2. We look for certain family of MVOP such that the corresponding block tridiagonal Jacobi matrix $A$ has a “stochastic” interpretation, meaning that the sum of each row of $A$ is $\leq 0$ and the off-diagonal entries of $A$ are $\geq 0$ (therefore the infinitesimal operator of a continuous-time Markov chain).
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**TWO BIRTH-AND-DEATH MODELS** \((\ell = 1)\)

Let \(W_1(y) (2 \times 2)\) and \(w_2(y)\) (scalar) be the corresponding block weight matrices and denote by \(Q_{n,1}\) and \(q_{n,2}\) the corresponding families of MVOP satisfying \(Q_{n,1}(0)e_2 = e_2, e_2 = (1, 1)^T\) and \(q_{n,2}(0) = 1\).

1. A birth-and-death process: The polynomials \(q_{n,2}\) satisfy the three-term recurrence relation

\[-yq_{n,2}(y) = a_n q_{n+1,2}(y) - (a_n + c_n)q_{n,2}(y) + c_n q_{n-1,2}(y)\]

where the coefficients are given by

\[a_n = \frac{2\nu + n + 2}{4(\nu + n + 1)}, \quad c_n = \frac{n}{4(\nu + n + 1)}\]

Therefore the Jacobi matrix is

\[A_2 = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{4(\nu+2)} & 0 & 0 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{2\nu+3}{4(\nu+2)} & 0 & 0 \\
0 & \frac{1}{2(\nu+3)} & -\frac{1}{2} & \frac{\nu+2}{2(\nu+3)} & 0 \\
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and it is the infinitesimal operator of a birth-and-death process.
The potential coefficients (inverse of the norms of $q_{n,2}$) are

$$
\pi_0 = 1, \quad \pi_n = \frac{2(\nu + n + 1)(2\nu + 3)n^{-1}}{n!}, \quad n \geq 1
$$

while the (normalized) weight is given by

$$
 w_2(y) = \frac{4^{\nu+1}\Gamma(\nu + 2)}{\sqrt{\pi}\Gamma(\nu + 3/2)} [y(1 - y)]^{\nu + 1/2}, \quad y \in (0, 1), \quad \nu > -3/2
$$

Therefore we have the Karlin-McGregor representation

$$
P^{(2)}_{ij}(t) = \mathbb{P}(X_t = j | X_0 = i) = \pi_j \int_0^1 e^{-yt} q_{i,2}(y) q_{j,2}(y) w_2(y) dy
$$

Since we have the explicit expression of the weight $w_2(y)$ we can study the recurrence of the process. For $-3/2 < \nu \leq -1/2$ the process is null recurrent (since $\sum \pi_n = \infty$), while if $\nu > -1/2$ then the process is transient.

This birth-and-death process can be seen as a rational variant of the one-server queue as the length of the queue increases.
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This birth-and-death process can be seen as a rational variant of the one-server queue as the length of the queue increases.
2. A quasi-birth-and-death process: The polynomials $Q_{n,1}(y)$ satisfy the three-term recurrence relation

$$-yQ_{n,1}(y) = A_n Q_{n+1,1}(y) + B_n Q_{n,1}(y) + C_n Q_{n-1,1}(y)$$

where the coefficients are given by

$$A_n = \begin{pmatrix} \frac{2\nu+n+2}{4(\nu+n+2)} & 0 & 0 & \cdots \\ 0 & \frac{1+\nu}{2(\nu+n+1)^2} & 0 & \cdots \\ 0 & 0 & \frac{2(\nu+n)(\nu+n+2)}{4(\nu+n+1)^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B_n = \begin{pmatrix} -\frac{1}{2} & 0 & \cdots \\ 0 & \frac{2(\nu+n)(\nu+n+2)}{4(\nu+n+1)^2} & \cdots \\ \nu+1 & 0 & \cdots \\ \frac{1+\nu}{2(\nu+n+1)^2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad C_n = \begin{pmatrix} \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & \cdots \\ \frac{\nu+3}{4(\nu+2)^2} & 0 & \cdots \\ \frac{\nu}{2(\nu+2)^2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

Therefore the Jacobi matrix (pentadiagonal) is

$$A_1 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2(\nu+2)} & \frac{\nu+1}{2(\nu+2)} & 0 & \cdots \\ \frac{1}{2(\nu+1)} & 0 & 0 & 0 & \cdots \\ \frac{1}{4(\nu+1)} & \frac{\nu+3}{4(\nu+2)^2} & 0 & 0 & \cdots \\ 0 & \frac{1+\nu}{2(\nu+2)^2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{\nu+4}{2(\nu+3)^2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

and it is the infinitesimal operator of a quasi-birth-and-death process ($\nu \geq 0$).
The (normalized) weight matrix is given by

\[
W_1(y) = \frac{4^{\nu+1/2} \Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} [y(1 - y)]^{\nu-1/2} \left( 1 - \frac{2(1+\nu)}{\nu+1/2} y(1 - y) \right) \frac{\nu+1}{\nu+2} \left( 1 - 2y \right) \frac{\nu+1}{\nu+2} \left( 1 - 2\frac{\nu}{\nu+1/2} y(1 - y) \right), \quad \nu \geq 0
\]

Each block entry \((i, j)\) of \(P^{(1)}(t)\) admits a Karlin-McGregor representation

\[
P_{ij}^{(1)}(t) = \left( \int_0^1 e^{-yt} Q_{i,1}(y) W_1(y) Q_{j,1}^*(y) dx \right) \Pi_j
\]

\[
\Pi_0 = I, \quad \Pi_n = \left( \|Q_{n,1}\|_{W_1}^2 \right)^{-1} = \frac{2(2\nu + 3)n^{-1}}{n!} \begin{pmatrix} \frac{(\nu+1)^2}{\nu+n+1} & 0 \\ 0 & \frac{\nu(\nu+2)(\nu+n+1)}{(\nu+n)(\nu+n+2)} \end{pmatrix}
\]

We can also compute explicitly the invariant measure of the process

\[
\pi = \left( (\Pi_0 e_2)^T; (\Pi_1 e_2)^T; (\Pi_2 e_2)^T; \cdots \right), \quad e_2^T = (1, 1),
\]

\[
= \left( 1, 1; \frac{2(\nu+1)^2}{\nu+2}, \frac{2\nu(\nu+2)^2}{(\nu+1)(\nu+3)}; \frac{(2\nu+3)(\nu+1)^2}{\nu+3}, \frac{(2\nu+3)\nu(\nu+3)}{\nu+4}; \cdots \right)
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In a similar way studied in the scalar case, the process is null recurrent for \(0 \leq \nu \leq 1/2\), while if \(\nu > 1/2\) then the process will be transient.
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In a similar way studied in the scalar case, the process is null recurrent for \(0 \leq \nu \leq 1/2\), while if \(\nu > 1/2\) then the process will be transient.
**Interpretation:** We have a 2 phases quasi-birth-and-death process. If the process moves along any of the phases, then the process can add (or remove) 2 elements to the queue. On the contrary, if the process moves from one phase to another, then the process add (or remove) 1 element to the queue. As the length of the queue increases, it is very unlikely that a transition between phases occurs. Therefore this quasi-birth-and-death process may be viewed as a *rational variation of a couple of one-server queues where the interaction between them is significant in the first states of the queue.*