Some recent developments about birth-and-death models and orthogonal polynomials[1]

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[1] Joint work with Pablo Román

Outline

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1. Markov chains and OP
   - Markov chains
   - Bivariate Markov chains
2. The new example
   - Matrix-valued spherical functions
   - Two birth-and-death models

One dimensional Markov chains

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A (1-D) Markov chain with state space $\mathcal{S} \subset \mathbb{N}$ is a collection of $\mathcal{S}$-valued random variables $\{X_t : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) such that they have the Markov property, i.e. the behavior of the future only depends on the present and not the past.

The main goal is to find a description of the transition probabilities

$$P_{ij}(t) = \mathbb{P}(X_t = j | X_0 = i), \quad i, j \in \mathcal{S} \subset \mathbb{N}$$

1. Discrete time: the process is characterized in terms of the one-step transition probability matrix P, which entries are given by

$$P_{ij} = \mathbb{P}(X_1 = j | X_0 = i) \quad \Rightarrow \quad P_{ij}(n) = (P^n)_{ij}$$

2. Continuous time: in this case the process is characterized (among other properties) by the behavior near $t \to 0^+$, i.e.

$$P_{ij}(t) = t a_{ij} + o(t), \quad t \to 0^+, \quad a_{ij} \geq 0,$$

$$P_{ii}(t) = 1 - t a_i + o(t), \quad t \to 0^+, \quad a_i = \sum_{j \neq i} a_{ij}$$

If we denote by $\mathcal{A} = (a_{ij})$, then we have $P'(t) = \mathcal{A}P(t) = P(t)\mathcal{A}$.

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EXAMPLES RELATED TO OP

1. Random walks: $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = \{0, 1, 2, \ldots\}$.
Transitions are only allowed between adjacent states, i.e.

$$P_{ij} = \begin{cases} p_i, & \text{if} \quad j = i + 1 \\ r_i, & \text{if} \quad j = i \\ q_i, & \text{if} \quad j = i - 1 \\ 0, & \text{elsewhere} \end{cases}$$

Therefore, P is a semi-infinite tridiagonal matrix (Jacobi matrix)

$$P = \begin{pmatrix} r_0 & p_0 & 0 & & \\ q_1 & r_1 & p_1 & 0 & \\ 0 & q_2 & r_2 & p_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad r_i \geq 0, p_i, q_i > 0, \quad p_i + r_i + q_i = 1$$

The n-step transition probability matrix is then given by $P^{(n)} = P^n$.
Some examples related to OP are the gambler's ruin (Jacobi) and urn models like the Ehrenfest model (Krawtchouk) or the Laplace-Bernoulli model (Hahn).

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Therefore we have a semi-infinite tridiagonal matrix $\mathcal{A}$ (also a Jacobi matrix)

$$\mathcal{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & & \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & 0 & \\ 0 & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \lambda_i, \mu_i > 0$$

The transition probability matrix $P(t)$ satisfies the backward and forward Kolmogorov equations $P'(t) = \mathcal{A}P(t) = P(t)\mathcal{A}$, $P(0) = I$.
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Spectral methods

Spectral or Favard's Theorem: there exist a unique measure associated with P or $\mathcal{A}$. Therefore it is possible to find spectral representations of the transition probabilities (Karlin-McGregor formulas):

1. Random walks: the measure ω is supported on $[-1,1]$. If we denote by (q_i) the sequence of OP generated by P then we have

$$\mathbb{P}(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

2. Birth-and-death processes: now the measure ω is supported on $[0,\infty)$. If we denote by (q_i) the sequence of OP generated by $\mathcal{A}$ then we have

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**The spectral measure can either be discrete (finite or infinite) or continuous.

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BIVARIATE MARKOV CHAINS

Now consider a bivariate or 2-component Markov chain of the form

$$\{(X_t, Y_t) : t \in \mathcal{T}\}, \quad X_t \in \mathcal{S} \subset \mathbb{N}, \quad Y_t \in \{1, 2, \dots, N\}$$

The first component is the level and the second component is the phase.

Now the transition probabilities can be written in terms of block matrix (each block of dimension $N \times N$)

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1. Discrete time: as in one dimensional case the process is characterized by its one-step transition probability block matrix P.
2. Continuous time: as in one dimensional case the process is characterized by the Kolmogorov equations $P'(t) = \mathcal{A}P(t) = P(t)\mathcal{A}$ but now $\mathcal{A}$ is a block matrix with same properties as before.

Ideas behind: *random evolutions*

(Griego-Hersh-Papanicolaou-Pinsky-Kurtz...60's and 70's).

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Processes related to matrix-valued OP

Quasi-birth-and-death processes:
$\boldsymbol{P}$ and $\mathcal{A}$ are now block-tridiagonal matrices of the form

$$\begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0 & & & \\ \boldsymbol{C}_1 & \boldsymbol{B}_1 & \boldsymbol{A}_1 & & \\ & \boldsymbol{C}_2 & \boldsymbol{B}_2 & \boldsymbol{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Depending if time is discrete or continuous we will have special stochastic properties for the block entries of $\boldsymbol{P}$ or $\mathcal{A}$:

1. Discrete time: all entries of $\boldsymbol{B}_n, \boldsymbol{A}_n, \boldsymbol{C}_{n+1}$ are nonnegative and $(\boldsymbol{B}_0 + \boldsymbol{A}_0)\boldsymbol{e}_N = \boldsymbol{e}_N$, $(\boldsymbol{C}_n + \boldsymbol{B}_n + \boldsymbol{A}_n)\boldsymbol{e}_N = \boldsymbol{e}_N$, $n \geq 1$, where $\boldsymbol{e}_N = (1, \ldots, 1)^T$.
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Depending if time is discrete or continuous we will have special stochastic properties for the block entries of $\boldsymbol{P}$ or $\mathcal{A}$:

1. Discrete time: all entries of $\boldsymbol{B}_n, \boldsymbol{A}_n, \boldsymbol{C}_{n+1}$ are nonnegative and $(\boldsymbol{B}_0 + \boldsymbol{A}_0)\boldsymbol{e}_N = \boldsymbol{e}_N$, $(\boldsymbol{C}_n + \boldsymbol{B}_n + \boldsymbol{A}_n)\boldsymbol{e}_N = \boldsymbol{e}_N, n \geq 1$, where $\boldsymbol{e}_N = (1, \ldots, 1)^T$.
2. Continuous time: all off-diagonal entries of $\mathcal{A}$ are nonnegative and $(\boldsymbol{B}_0 + \boldsymbol{A}_0)\boldsymbol{e}_N = \boldsymbol{0}_N$, $(\boldsymbol{C}_n + \boldsymbol{B}_n + \boldsymbol{A}_n)\boldsymbol{e}_N = \boldsymbol{0}_N, n \geq 1$.

The main tool to study spectral methods will be the theory of matrix-valued orthogonal polynomials.

Spectral methods

Spectral or Favard's Theorem: under certain symmetry conditions on the coefficients of the block entries of $\boldsymbol{P}$ or $\mathcal{A}$ there exists a unique weight matrix $\boldsymbol{W}$. Therefore we will have again spectral representations of the transition probabilities:

1. Quasi-birth-and-death processes (discrete time): the weight matrix $\boldsymbol{W}$ is supported on $[-1,1]$. If we denote by $(\boldsymbol{Q}_i)$ the sequence of MVOP generated by $\boldsymbol{P}$ then we have (Grünbaum and Dette-Reuther-Studden-Zygmunt, 2007).

$$\boldsymbol{P}_{ij}^n = \left(\int_{-1}^1 x^n \boldsymbol{Q}_i(x) d\boldsymbol{W}(x) \boldsymbol{Q}_j^*(x)\right)\left(\int_{-1}^1 \boldsymbol{Q}_j(x) d\boldsymbol{W}(x) \boldsymbol{Q}_j^*(x)\right)^{-1}$$

2. Quasi-birth-and-death processes (continuous time): now the weight matrix $\boldsymbol{W}$ is supported on $[0,\infty)$. If we denote by $(\boldsymbol{Q}_i)$ the sequence of MVOP generated by $\mathcal{A}$ then we have (Dette-Reuther, 2010):

$$\boldsymbol{P}_{ij}(t) = \left(\int_0^\infty e^{-xt} \boldsymbol{Q}_i(x) d\boldsymbol{W}(x) \boldsymbol{Q}_j^*(x)\right)\left(\int_0^\infty \boldsymbol{Q}_j(x) d\boldsymbol{W}(x) \boldsymbol{Q}_j^*(x)\right)^{-1}$$

**Again, the spectral weight matrix can either be discrete (finite or infinite) or continuous.

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Outline

1. Markov chains and OP
   - Markov chains
   - Bivariate Markov chains

2. The new example
   - Matrix-valued spherical functions
   - Two birth-and-death models

Matrix-valued spherical functions

Spherical functions associated with groups of the form G/K where (G, K) is a Gel'fand pair are very much related with OP (Helgason, Vilenkin, Klimyk). They are eigenfunctions of the Casimir operator associated with the group. The extension to the matrix-valued case was started by Tirao (1977). The connection with MVOP was discovered by Grünbaum-Pacharoni-Tirao (2003).

1. Complex projective space: $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$. Grünbaum-Pacharoni-Tirao (2002). Later it was found the relation with stochastic processes by Grünbaum-MdI (2008), Grünbaum-Pacharoni-Tirao (2012) and MdI (2012).

2. Complex hyperbolic plane: $H_2(\mathbb{C}) = \mathrm{SU}(2,1)/\mathrm{U}(2)$. Pacharoni-Román-Tirao (2006). Dual to the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$.

3. Real sphere: $S^n = \mathrm{SO}(n+1)/\mathrm{O}(n)$. Tirao-Zurrián (2013). Also connected with the real projective space $P_n(\mathbb{R}) = \mathrm{SO}(n+1)/\mathrm{O}(n)$.

In all cases (and others not mentioned) an explicit expression of the weight matrix, the second-order differential operator, the three-term recurrence relation and other structural formulas were derived for the matrix-valued spherical functions. In most of the cases the relation with MVOP was also given.

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The pair $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag}\ \mathrm{SU}(2))$

Koornwinder (1985) studied spherical functions associated with pairs of the form $(K\times K, K)$, where the subgroup is diagonally embedded and $K=\mathrm{SU}(2)$.

More recently Koelink-van Pruijssen-Román (2012) studied with a different approach this example and give the relation with MVOP.

For $\ell\in\mathbb{N}$ and $N=2\ell+1$ they produced a one-parameter family of $N\times N$ MVOP where the weight matrix is

$$W(y)=[y(1-y)]^{\nu-1/2}\Psi_0(y)T(\Psi_0(y))^*,\quad T_{ij}=\delta_{ij}\binom{2\ell}{i}\frac{(\nu)_i}{(\nu+2\ell-i)_i}$$

where $\Psi_0(y)$ is certain matrix-valued function containing spherical functions.

The corresponding symmetric second-order differential operator is given by

$$D=y(1-y)\partial_y^2+(C+\nu-y(2\ell+2\nu+1))\partial_y-(V-(\nu-1)(2\ell+\nu+1))$$

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Two important facts

1. The structure of the group induces the existence of a constant matrix Y such that we can decompose by blocks the weight matrix W in the form

$$\widetilde{W}(y) = YW(y)Y^* = \left(\begin{array}{c|c} W_1(y) & 0 \\ \hline 0 & W_2(y) \end{array}\right)$$

where W_1 is $(\ell+1)\times(\ell+1)$ and W_2 is $\ell \times \ell$. So we will study the probabilistic aspects of these two independent processes ($\ell = 1$).

2. We look for certain family of MVOP such that the corresponding block tridiagonal Jacobi matrix $\mathcal{A}$ has a "stochastic" interpretation, meaning that the sum of each row of $\mathcal{A}$ is ≤ 0 and the off-diagonal entries of $\mathcal{A}$ are ≥ 0 (therefore the infinitesimal operator of a continuous-time Markov chain).

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Two birth-and-death models ($\ell = 1$)

Let $W_1(y)$ (2×2) and $w_2(y)$ (scalar) be the corresponding block weight matrices and denote by $Q_{n,1}$ and $q_{n,2}$ the corresponding families of MVOP satisfying $Q_{n,1}(0)\boldsymbol{e}_2 = \boldsymbol{e}_2, \boldsymbol{e}_2 = (1,1)^T$ and $q_{n,2}(0) = 1$.

1. A birth-and-death process: The polynomials $q_{n,2}$ satisfy the three-term recurrence relation

$$-yq_{n,2}(y) = a_n q_{n+1,2}(y) - (a_n + c_n)q_{n,2}(y) + c_n q_{n-1,2}(y)$$

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$$a_n = \frac{2\nu + n + 2}{4(\nu + n + 1)}, \quad c_n = \frac{n}{4(\nu + n + 1)}$$

Therefore the Jacobi matrix is

$$\mathcal{A}_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & & \\ \frac{1}{4(\nu+2)} & -\frac{1}{2} & \frac{2\nu+3}{4(\nu+2)} & 0 & \\ 0 & \frac{1}{2(\nu+3)} & -\frac{1}{2} & \frac{\nu+2}{2(\nu+3)} & 0 \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \nu > -3/2$$

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The potential coefficients (inverse of the norms of $q_{n,2}$) are

$$\pi_0 = 1, \quad \pi_n = \frac{2(\nu+n+1)(2\nu+3)_{n-1}}{n!}, \quad n \geq 1$$

while the (normalized) weight is given by

$$w_2(y) = \frac{4^{\nu+1}\Gamma(\nu+2)}{\sqrt{\pi}\Gamma(\nu+3/2)}\left[y(1-y)\right]^{\nu+1/2}, \quad y \in (0,1), \quad \nu > -3/2$$

Therefore we have the Karlin-McGregor representation

$$\begin{aligned} P_{ij}^{(2)}(t) = \mathbb{P}(X_t = j | X_0 = i) &= \pi_j \int_0^1 e^{-yt} q_{i,2}(y) q_{j,2}(y) w_2(y) dy \\ &= \frac{2(\nu+j+1)(2\nu+3)_{j-1} 4^{\nu+1}\Gamma(\nu+2)}{j!\sqrt{\pi}\Gamma(\nu+3/2)} \int_0^1 e^{-yt} q_{i,2} q_{j,2} \left[y(1-y)\right]^{\nu+1/2} dy \end{aligned}$$

Since we have the explicit expression of the weight $w_2(y)$ we can study the recurrence of the process. For $-3/2 < \nu \leq -1/2$ the process is *null recurrent* (since $\sum \pi_n = \infty$), while if $\nu > -1/2$ then the process is *transient*.

This birth-and-death process can be seen as a *rational variant of the one-server queue* as the length of the queue increases.

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2. A quasi-birth-and-death process: The polynomials $Q_{n,1}(y)$ satisfy the three-term recurrence relation

$$-yQ_{n,1}(y) = A_n Q_{n+1,1}(y) + B_n Q_{n,1}(y) + C_n Q_{n-1,1}(y)$$

where the coefficients are given by

$$A_n = \begin{pmatrix} \frac{2\nu+n+2}{4(\nu+n+2)} & 0 \\ 0 & \frac{(n+\nu)(2\nu+n+2)}{4(\nu+n+1)^2} \end{pmatrix}, B_n = \begin{pmatrix} -\frac{1}{2} & \frac{\nu}{2(\nu+n)(\nu+n+2)} \\ \frac{1+\nu}{2(\nu+n+1)^2} & -\frac{1}{2} \end{pmatrix}, C_n = \begin{pmatrix} \frac{n}{4(\nu+n)} & 0 \\ 0 & \frac{n(\nu+n+2)}{4(\nu+n+1)^2} \end{pmatrix}$$

Therefore the Jacobi matrix (pentadiagonal) is

$$\mathcal{A}_1 = \left(\begin{array}{cc|cc|cc|cc|c} -\frac{1}{2} & \frac{1}{2(\nu+2)} & \frac{\nu+1}{2(\nu+2)} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2(\nu+1)} & -\frac{1}{2} & 0 & \frac{\nu}{2(\nu+1)} & 0 & 0 & 0 & 0 & \cdots \\ \hline \frac{1}{4(\nu+1)} & 0 & -\frac{1}{2} & \frac{\nu}{2(\nu+1)(\nu+3)} & \frac{2\nu+3}{4(\nu+3)} & 0 & 0 & 0 & \cdots \\ 0 & \frac{\nu+3}{4(\nu+2)^2} & \frac{1+\nu}{2(\nu+2)^2} & -\frac{1}{2} & 0 & \frac{(1+\nu)(2\nu+3)}{4(\nu+2)^2} & 0 & 0 & \cdots \\ \hline 0 & 0 & \frac{1}{2(\nu+2)} & 0 & -\frac{1}{2} & \frac{\nu}{2(\nu+2)(\nu+4)} & \frac{\nu+2}{2(\nu+4)} & 0 & \cdots \\ 0 & 0 & 0 & \frac{\nu+4}{2(\nu+3)^2} & \frac{1+\nu}{2(\nu+3)^2} & -\frac{1}{2} & 0 & \frac{(2+\nu)^2}{2(\nu+3)^2} & \cdots \\ \hline & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array} \right)$$

and it is the infinitesimal operator of a quasi-birth-and-death process ($\nu \geq 0$).

The (normalized) weight matrix is given by

$$W_1(y) = \frac{4^{\nu+1/2}\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)}\left[y(1-y)\right]^{\nu-1/2}\begin{pmatrix} 1-\frac{2(1+\nu)}{\nu+1/2}y(1-y) & \frac{\nu+1}{\nu+2}(1-2y) \\ \frac{\nu+1}{\nu+2}(1-2y) & \frac{\nu+1}{\nu+2}\left(1-\frac{2\nu}{\nu+1/2}y(1-y)\right)\end{pmatrix}, \quad \nu \geq 0$$

Each block entry (i,j) of $P^{(1)}(t)$ admits a Karlin-McGregor representation

$$P_{ij}^{(1)}(t) = \left(\int_0^1 e^{-yt}Q_{i,1}(y)W_1(y)Q_{j,1}^*(y)dx\right)\Pi_j$$

$$\Pi_0 = I, \quad \Pi_n = \left(\|Q_{n,1}\|_{W_1}^2\right)^{-1} = \frac{2(2\nu+3)_{n-1}}{n!}\begin{pmatrix} \frac{(\nu+1)^2}{\nu+n+1} & 0 \\ 0 & \frac{\nu(\nu+2)(\nu+n+1)}{(\nu+n)(\nu+n+2)}\end{pmatrix}$$

We can also compute *explicitly* the invariant measure of the process

$$\begin{aligned}\pi &= \left((\Pi_0 e_2)^T;(\Pi_1 e_2)^T;(\Pi_2 e_2)^T;\cdots\right), \quad e_2^T = (1,1),\\ &= \left(1,1;\frac{2(\nu+1)^2}{\nu+2},\frac{2\nu(\nu+2)^2}{(\nu+1)(\nu+3)};\frac{(2\nu+3)(\nu+1)^2}{\nu+3},\frac{(2\nu+3)\nu(\nu+3)}{\nu+4};\cdots\right)\end{aligned}$$

In a similar way studied in the scalar case, the process is *null recurrent* for $0 \leq \nu \leq 1/2$, while if $\nu > 1/2$ then the process will be *transient*.

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Interpretation: We have a 2 phases quasi-birth-and-death process. If the process moves along any of the phases, then the process can add (or remove) 2 elements to the queue. On the contrary, if the process moves from one phase to another, then the process add (or remove) 1 element to the queue. As the length of the queue increases, it is very unlikely that a transition between phases occurs. Therefore this quasi-birth-and-death process may be viewed as a *rational variation of a couple of one-server queues where the interaction between them is significant in the first states of the queue.*