Differential properties of orthogonal matrix polynomials

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Outline

Scalar orthogonality

Let ω be a positive measure on $\mathbb{R}$ with $\int_{\mathbb{R}} x^n \mathrm{d}\omega(x) < \infty, n \geq 0$.
A function $f \in L^2_\omega(\mathbb{R})$ if

$$\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}\omega(x) < \infty$$

Applying Gram-Schmidt to the set $\{1, x, x^2, \cdots\}$ it is possible to construct a family of orthonormal polynomials $(p_n)_n$ of the form

$$p_n(x) = \kappa_n x^n + \kappa_{n-1} x^{n-1} + \cdots$$

such that

$$\langle p_n, p_m \rangle_\omega = \int_{\mathbb{R}} p_n(x) p_m(x) \mathrm{d}\omega(x) = \delta_{nm}, \quad n, m \geq 0$$

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TTRR and SODE

- This is equivalent to a three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad a_{n+1} \neq 0, \quad b_n \in \mathbb{R}$$

Jacobi operator (tridiagonal):

$$x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}$$

- Bochner (1929): characterize $(p_n)_n$ satisfying second-order differential equations of Sturm-Liouville type:

$$\mathcal{A}p_n \equiv \underbrace{(\alpha_2x^2 + \alpha_1x + \alpha_0)}_{\sigma(x)} p_n''(x) + \underbrace{(\beta_1x + \beta_0)}_{\tau(x)} p_n'(x) = \lambda_np_n(x)$$

$\Rightarrow$ Hermite, Laguerre and Jacobi (Bessel) polynomials

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Classical families

Hermite: $\sigma(x) = 1, \omega(x) = e^{-x^2}$, $x \in (-\infty, \infty)$:

$$H_n''(x) - 2xH_n'(x) = -2nH_n(x), \quad H_{n+1}(x) - 2xH_n(x) = -2nH_{n-1}(x)$$

Laguerre: $\sigma(x) = x, \omega(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x \in (0, \infty)$:

$$xL_n^\alpha(x)'' + (\alpha + 1 - x)L_n^\alpha(x)' = -nL_n^\alpha(x)$$

$$(n+1)L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x)L_n^\alpha(x) = -(n+\alpha)L_{n-1}^\alpha(x)$$

Jacobi: $\sigma(x) = x(1-x), \omega(x) = x^\alpha(1-x)^\beta$, $\alpha, \beta > -1$, $x \in (0, 1)$:

$$x(1-x)P_n^{(\alpha,\beta)}(x)'' + (\alpha + 1 - (\alpha + \beta + 2)x)P_n^{(\alpha,\beta)}(x)' = -n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x)$$

$$xP_n^{(\alpha,\beta)}(x) = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x) + \left(1 + \frac{n(n+\beta)}{2n+\alpha+\beta} - \frac{(n+1)(n+\beta+1)}{2n+\alpha+\beta+2}\right)P_n^{(\alpha,\beta)}(x) + \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x)$$

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1-D Markov processes

Let $(\Omega, \mathcal{F}, \Pr)$ a probability space, a (1-D) Markov process with state space $\mathcal{S} \subset \mathbb{R}$ is a collection of $\mathcal{S}$-valued random variables $\{X_t : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) such that

$$\Pr(X_{t_0+t_1} \leq y | X_{t_0} = x, X_\tau, 0 \leq \tau < t_0) = \Pr(X_{t_0+t_1} \leq y | X_{t_0} = x)$$

for all $t_0, t_1 > 0$. This is what is called the Markov property.
The main goal is to find a description of the transition density

$$p(t; x, y) \equiv \frac{\partial}{\partial y} \Pr(X_t \leq y | X_0 = x)$$

if $\mathcal{S}$ is continuous or transition probabilities

$$p(t; x, y) \equiv \Pr(X_t = y | X_0 = x)$$

if $\mathcal{S}$ is discrete.

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1-D Markov processes

On the set $\mathfrak{B}(\mathcal{S})$ of all real-valued, bounded, Borel measurable functions define the transition operator

$$(T_t f)(x) = E[f(X_t)|X_0 = x], \quad t \geq 0$$

The family $\{T_t, t > 0\}$ has the *semigroup property* $T_{s+t} = T_s T_t$
The infinitesimal operator $\mathcal{A}$ of the family $\{T_t, t > 0\}$ is

$$(\mathcal{A}f)(x) = \lim_{s \downarrow 0} \frac{(T_s f)(x) - f(x)}{s}$$

and $\mathcal{A}$ determines all $\{T_t, t > 0\}$.
There are 3 important cases related to orthogonal polynomials

1. Random walks: $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = \{0, 1, 2, \ldots\}$.
2. Birth and death processes: $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = [0, \infty)$.
3. Diffusion processes: $\mathcal{S} = (a, b) \subseteq \mathbb{R}$, $\mathcal{T} = [0, \infty)$.

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Random walks

We have $\mathcal{S} = \{0, 1, 2, \dots\}$, $\mathcal{T} = \{0, 1, 2, \dots\}$ and

$$\Pr(X_{n+1} = j | X_n = i) = 0 \quad \text{for} \quad |i - j| > 1$$

i.e. a tridiagonal transition probability matrix (stochastic)

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_i \geq 0, a_i, c_i > 0, \quad a_i + b_i + c_i = 1$$

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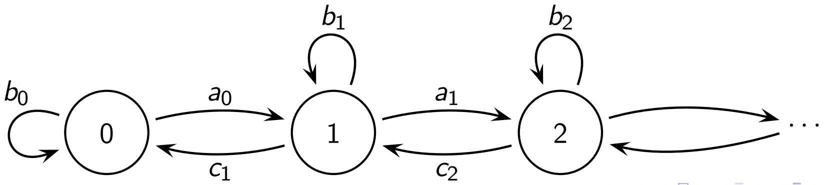

Introducing the polynomials $(q_i)_i$ by the conditions $q_{-1}(x) = 0$, $q_0(x) = 1$ and the recursion relation

$$xq_i(x) = a_i q_{i+1}(x) + b_i q_i(x) + c_i q_{i-1}(x), \quad i = 0, 1, \ldots$$

there exists a unique measure $\mathrm{d}\omega(x)$ supported in $[-1, 1]$ such that $(q_i)_i$ are orthogonal w.r.t $\mathrm{d}\omega(x)$.

Karlin-McGregor formula

$$\Pr(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2} \int_{-1}^{1} x^n q_i(x) q_j(x) \mathrm{d}\omega(x)$$

Invariant measure or distribution

A non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \geq 0$ such that

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

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Birth and death processes

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- $P_{i,i+1}(h) = \lambda_i h + o(h)$, as $h \downarrow 0, \lambda_i > 0, i \in \mathcal{S}$;
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$$\mathcal{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \lambda_i, \mu_i > 0$$

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We have $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = [0, \infty)$ and $P_{ij}(t) = \Pr(X_{t+s} = j | X_s = i)$ independent of s with the properties

- $P_{i,i+1}(h) = \lambda_i h + o(h)$, as $h \downarrow 0, \lambda_i > 0, i \in \mathcal{S}$;
- $P_{i,i-1}(h) = \mu_i h + o(h)$ as $h \downarrow 0, \mu_i > 0, i \in \mathcal{S}$;
- $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$ as $h \downarrow 0, i \in \mathcal{S}$;
- $P_{i,j}(h) = o(h)$ for $|i - j| > 1$.

$P(t)$ satisfies the backward and forward equation $P'(t) = \mathcal{A}P(t)$ and $P'(t) = P(t)\mathcal{A}$ with initial condition $P(0) = I$ where $\mathcal{A}$ is the tridiagonal infinitesimal operator matrix

$$\mathcal{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \lambda_i, \mu_i > 0$$

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Introducing the polynomials $(q_i)_i$ by the conditions $q_{-1}(x) = 0$, $q_0(x) = 1$ and the recursion relation

$$-xq_i(x) = \lambda_i q_{i+1}(x) - (\lambda_i + \mu_i)q_i(x) + \mu_i q_{i-1}(x), \quad i = 0, 1, \ldots$$

there exists a unique measure $d\omega(x)$ supported in $[0, \infty)$ such that $(q_i)_i$ are orthogonal w.r.t $d\omega(x)$.

Karlin-McGregor formula

$$\Pr(X_t = j | X_0 = i) = P_{ij}(t) = \frac{1}{\|q_i\|^2} \int_0^\infty e^{-xt} q_i(x) q_j(x) d\omega(x)$$

Invariant measure or distribution

A non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \geq 0$ such that

$$\pi \mathcal{A} = 0$$

$$\Rightarrow \pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} = \frac{1}{\|q_i\|^2}$$

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Diffusion processes

We have $\mathcal{S} = (a, b), -\infty \leq a < b \leq \infty$, $\mathcal{T} = [0, \infty)$.

Suppose that as $t \downarrow 0$

- $E[X_{s+t} - X_s | X_s = x] = t\mu(x) + o(t)$;
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$\mu(x)$ is the drift coefficient and $\sigma^2(x) > 0$ the diffusion coefficient.

Infinitesimal generator

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

Suppose we have a set of eigenfunctions $(\phi_n)_n$ of $\mathcal{A}$, i.e. $\mathcal{A}\phi_n = \alpha_n\phi_n$ and $(\phi_n)_n$ are normalized w.r.t a weight $\rho(x)$.

$$p(t; x, y) = \sum_{n=0}^{\infty} e^{\alpha_n t}\phi_n(x)\phi_n(y)\rho(y)$$

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Matrix orthogonality

Matrix valued polynomials on $\mathbb{R}$:

$$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_0, \quad x \in \mathbb{R}, \quad K_i \in \mathbb{C}^{N\times N}$$

Orthogonality: weight matrix W with $\int_{\mathbb{R}} x^n \mathrm{d}W(x) < \infty, n \geq 0$.

A matrix valued function $F \in L^2_W(\mathbb{R}; \mathbb{C}^{N\times N})$ if

$$\int_{\mathbb{R}} F(x)\mathrm{d}W(x)F^*(x) < \infty$$

Applying Gram-Schmidt to the set $\{I, xI, x^2I, \cdots\}$ it is possible to construct a family of orthonormal matrix polynomials $(P_n)_n$ (OMP) with nonsingular leading coefficient such that

$$\langle P_n, P_m\rangle_W = \int_{\mathbb{R}} P_n(x)\mathrm{d}W(x)P_m^*(x) = \delta_{nm} I$$

Other inner products

$$(P, Q)_W = \int_{\mathbb{R}} Q^*(x)\mathrm{d}W(x)P(x) \in \mathbb{C}^{N\times N}, \quad P, Q \in \mathbb{C}^{N\times N}[x]$$

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TTRR AND SODE

- This is equivalent to a three term recurrence relation

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x), \quad |A_n| \neq 0, \quad B_n = B_n^*$$

Jacobi operator (block tridiagonal)

$$x\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & A_2^* & B_2 & A_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix}$$

- Durán (1997): characterize orthonormal $(P_n)_n$ satisfying matrix second-order differential equations of Sturm-Liouville type:

$$P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n P_n(x), \quad n \geq 0$$
$$\text{grad } F_i \leq i, \quad \Lambda_n \quad \text{Hermitian}$$

It has not been until very recently when the first examples appeared: Grünbaum-Pacharoni-Tirao (2003) and Durán-Grünbaum (2004)

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- Matrix spherical functions associated with $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$
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- Durán-Grünbaum (2004): Symmetry equations

New phenomena

- For a fixed family of OMP there exist several linearly independent second-order differential operators having them as eigenfunctions
- OMP satisfying odd-order differential equations
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2-D Markov processes

Now we have a bivariate or 2-component Markov process of the form $\{Z_t = (X_t, Y_t) : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T} \subset [0, \infty)$ (time) and with state space $\mathcal{C} = \mathcal{S} \times \{1, 2, \ldots, N\}$, where $\mathcal{S} \subset \mathbb{R}$. The first component is called the level while the second component is the phase.

The transition or density probabilities, as well as the infinitesimal operator $\mathcal{A}$ are going to be matrix-valued.

There are 3 important cases related to OMP

1. Discrete time quasi-birth-and-death processes: $\mathcal{C} = \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}$, $\mathcal{T} = \{0, 1, 2, \ldots\}$.
2. Continuous time quasi-birth-and-death processes: $\mathcal{C} = \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}$, $\mathcal{T} = [0, \infty)$.
3. Bivariate processes with diffusion and discrete components: $\mathcal{C} = (a, b) \times \{1, 2, \ldots, N\}$, $\mathcal{T} = [0, \infty)$.

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Now we have a bivariate or 2-component Markov process of the form $\{Z_t = (X_t, Y_t) : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T} \subset [0, \infty)$ (time) and with state space $\mathcal{C} = \mathcal{S} \times \{1, 2, \dots, N\}$, where $\mathcal{S} \subset \mathbb{R}$. The first component is called the level while the second component is the phase.

The transition or density probabilities, as well as the infinitesimal operator $\mathcal{A}$ are going to be matrix-valued.

There are 3 important cases related to OMP

1. Discrete time quasi-birth-and-death processes:
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2. Continuous time quasi-birth-and-death processes:
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Discrete time quasi-birth-and-death processes

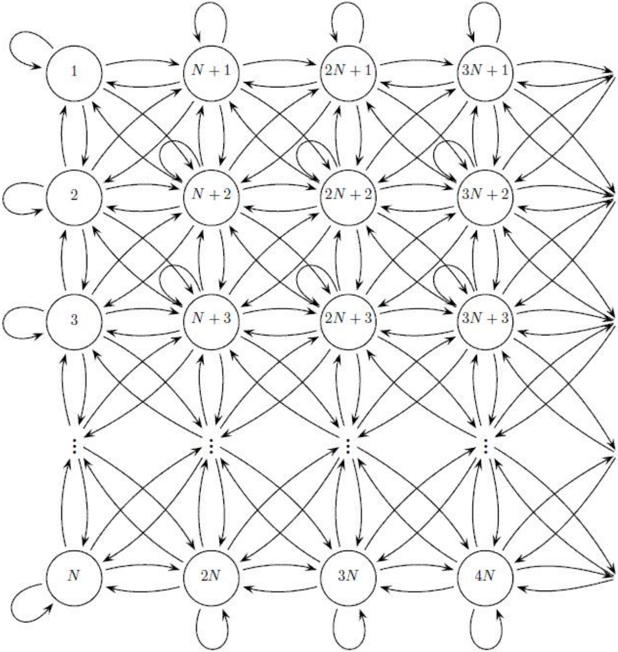
Now we have $\mathcal{C} = \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}$, $\mathcal{T} = \{0, 1, 2, \ldots\}$ and

$$(P_{ii'})_{jj'} = \Pr(X_{n+1} = i, Y_{n+1} = j | X_n = i', Y_n = j') = 0 \quad \text{for} \quad |i - i'| > 1$$

i.e. a $N \times N$ block tridiagonal transition probability matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$(A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geq 0, \ \det(A_n), \det(C_n) \neq 0$$
$$\sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1, \ i = 1, \ldots, N$$


1
N + 1
2N + 1
3N + 1
2
N + 2
2N + 2
3N + 2
3
N + 3
2N + 3
3N + 3
N
2N
3N
4N

OMP: Grünbaum and Dette-Reuther-Studden-Zygmunt (2007):
Introducing the matrix polynomials $(Q_i)_i$ by the conditions $Q_{-1}(x) = 0$, $Q_0(x) = I$ and the recursion relation

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and under certain technical conditions over A_i, B_i, C_i, there exists an unique weight matrix $dW(x)$ supported in $[-1, 1]$ such that $(Q_i)_i$ are orthogonal w.r.t $dW(x)$.

Karlin-McGregor formula

$$P_{ij}^n = \left(\int_{-1}^{1} x^n Q_i(x) dW(x) Q_j^*(x) \right) \left(\int_{-1}^{1} Q_j(x) dW(x) Q_j^*(x) \right)^{-1}$$

Invariant measure or distribution (MdI, 2010)

Non-null vector with non-negative components

$$\pi = (\pi^0; \pi^1; \cdots) \equiv (\Pi_0 e_N; \Pi_1 e_N; \cdots)$$

such that $\pi P = \pi$ where $e_N = (1, \ldots, 1)^T$ and

$$\Pi_n = (C_1^T \cdots C_n^T)^{-1} \Pi_0 (A_0 \cdots A_{n-1}) = \left(\int_{-1}^{1} Q_n(x) dW(x) Q_n^*(x) \right)^{-1}$$

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For the continuous time is similar. Now the matrix polynomials $(Q_i)_i$ are orthogonal w.r.t a weight matrix $dW(x)$ supported in $[0,\infty)$.

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We have $\mathcal{C} = (a, b) \times \{1, 2, \ldots, N\}$, $\mathcal{T} = [0, \infty)$. The transition probability density is now a matrix which entry (i, j) gives

$$P_{ij}(t; x, A) = \Pr(X_t \in A, Y_t = j | X_0 = x, Y_0 = i)$$

for any $t > 0$, $x \in (a, b)$ and A any Borel set.

Suppose that for any $\epsilon > 0$

- $\lim_{t\downarrow 0} \frac{1}{t} \int_{\{y:|y-x|\leq\epsilon\}} (y - x) P(t; x, dy) = B(x)$;
- $\lim_{t\downarrow 0} \frac{1}{t} \int_{\{y:|y-x|\leq\epsilon\}} (y - x)^2 P(t; x, dy) = A(x)$;
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Under these conditions we have (Berman, 1994) that $A(x)$ and $B(x)$ are diagonal matrices and $C(x)$ satisfies

$$C_{ii}(x) \leq 0, \quad C_{ij}(x) \geq 0, i \neq j, \quad -C_{ii}(x) = \sum_{j \neq i} C_{ij}(x)$$

Infinitesimal generator (Berman, 1994)

$$\mathcal{A} = \frac{1}{2} A(x) \frac{d^2}{dx^2} + B(x) \frac{d^1}{dx^1} + C(x) \frac{d^0}{dx^0}$$

If we are able to find a set of matrix-valued eigenfunctions $(\Phi_n)_n$ of $\mathcal{A}$, i.e. $\mathcal{A}\Phi_n = \Phi_n \Gamma_n$ and $(\Phi_n)_n$ are normalized w.r.t a weight matrix $W(x)$, then we find

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An example

Conjugation

$$W(x) = T^* \widetilde{W}(x) T$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & -\dfrac{\alpha+\beta-k+2}{\beta-k+1} \end{pmatrix}$$

Grünbaum-MdI (2008)

$$\widetilde{W}(x) = x^{\alpha}(1-x)^{\beta} \begin{pmatrix} kx+\beta-k+1 & (1-x)(\beta-k+1) \\ (1-x)(\beta-k+1) & (1-x)^2(\beta-k+1) \end{pmatrix}$$

$x \in (0,1)$, $\alpha, \beta > -1$, $0 < k < \beta + 1$

Pacharoni-Tirao (2006)

We consider the family of OMP $(Q_n(x))_n$ such that

- Three term recurrence relation

$$xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x), \quad n = 0, 1, \ldots$$

where the Jacobi matrix is stochastic

- Choosing $Q_0(x) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)} \begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)} \times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

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Pentadiagonal Jacobi matrix

Particular case $\alpha = \beta = 0$, $k = 1/2$:

$$P = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{50} & & & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & & & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} & & \\ & & & & \frac{81}{392} & \frac{3}{196} & \frac{1955}{3969} & \frac{5}{324} & \frac{175}{648} & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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Invariant measure

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The row vector

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is an invariant measure of P

Particular case $N = 2$, $\alpha = \beta = 0$, $k = 1/2$:

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$$\pi = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \cdots \right)$$

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2-D diffusion process

Now, with the same notation as before, consider the matrix

$$\Psi(x) = \begin{pmatrix} 1 & 1 \\ 1 & \frac{x(\alpha+\beta-k+2)-\alpha-1}{\beta-k+1} \end{pmatrix}$$

Then, the family of polynomials $\Phi_n(x) = \Psi(x) Q_n^*(x)$ is orthogonal w.r.t the weight matrix

$$\hat{W}(x) = x^\alpha (1-x)^\beta \begin{pmatrix} kx & 0 \\ 0 & \beta-k+1 \end{pmatrix}$$

The family $(\Phi_n)_n$ is eigenfunction of the following second order differential operator (Grünbaum, Pacharoni, Tirao, 2002)

$$\mathcal{A} = x(1-x)\frac{d^2}{dx^2} + \begin{pmatrix} \alpha+2-x(\alpha+\beta+3) & 0 \\ 0 & \alpha+1-x(\alpha+\beta+2) \end{pmatrix} \frac{d^1}{dx^1} + \begin{pmatrix} \frac{\beta-k+1}{x-1} & \frac{\beta-k+1}{1-x} \\ \frac{kx}{1-x} & \frac{kx}{x-1} \end{pmatrix} \frac{d^0}{dx^0}$$

2-D diffusion process

Now, with the same notation as before, consider the matrix

$$\Psi(x) = \begin{pmatrix} 1 & 1 \\ 1 & \frac{x(\alpha+\beta-k+2)-\alpha-1}{\beta-k+1} \end{pmatrix}$$

Then, the family of polynomials $\Phi_n(x) = \Psi(x)Q_n^*(x)$ is orthogonal w.r.t the weight matrix

$$\hat{W}(x) = x^\alpha(1-x)^\beta \begin{pmatrix} kx & 0 \\ 0 & \beta-k+1 \end{pmatrix}$$

The family $(\Phi_n)_n$ is eigenfunction of the following second order differential operator (Grünbaum, Pacharoni, Tirao, 2002)

$$\mathcal{A} = x(1-x)\frac{d^2}{dx^2} + \begin{pmatrix} \alpha+2-x(\alpha+\beta+3) & 0 \\ 0 & \alpha+1-x(\alpha+\beta+2) \end{pmatrix}\frac{d^1}{dx^1} + \begin{pmatrix} \frac{\beta-k+1}{x-1} & \frac{\beta-k+1}{1-x} \\ \frac{kx}{1-x} & \frac{kx}{x-1} \end{pmatrix}\frac{d^0}{dx^0}$$

And diagonal eigenvalue

$$\Gamma_n = \begin{pmatrix} -n(n+\alpha+\beta+2) & \\ 0 & -n^2 - n(\alpha+\beta+3) - (\alpha+\beta-k+2) \end{pmatrix}$$

Therefore we have a bivariate process with diffusion and discrete components on the state space $\mathcal{C} = (0,1) \times \{1,2\}$ such that

Transition probability density matrix (MdI, 2010)

$$P(t;x,y) = \sum_{n=0}^{\infty} \Phi_n(x) e^{\Gamma_n t} \Phi_n^*(y) \hat{W}(y)$$
$$= \Psi(x) \left(\sum_{n=0}^{\infty} Q_n^*(x) e^{\Gamma_n t} Q_n(y) W(y) \right) \Psi^{-1}(y)$$

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Other applications

Quantum mechanics

[Durán–Grünbaum] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006)

Time-and-band limiting

[Durán–Grünbaum] *A survey on orthogonal matrix polynomials satisfying second order differential equations*, J. Comput. Appl. Math. (2005)

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