SECOND ORDER DIFFERENTIAL OPERATORS HAVING SEVERAL FAMILIES OF ORTHOGONAL MATRIX POLYNOMIALS AS EIGENFUNCTIONS

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Special Functions, Information Theory and Mathematical Physics
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Outline

Preliminaries I

- The theory of orthogonal matrix polynomials (OMP) was introduced by Kreĭn in 1949.
- Equivalence between $(P_n)_n$, orthonormal w. r. t. W

$$(P_n, P_m)_W = \int_{\mathbb{R}} P_n(t)\, W(t)\, P_m^*(t)\, dt = \delta_{nm} I, \quad n, m \geq 0$$

and a Jacobi (block tridiagonal) operator

$$\mathcal{L} = \begin{pmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & A_2^* & B_2 & A_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$tP_n(t) = A_{n+1} P_{n+1}(t) + B_n P_n(t) + A_n^* P_{n-1}(t), \quad n \geq 0$$

- Systematically studied in the last 15 years.

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Preliminaries II

- (Durán, 1997) Find all families of OMP $(P_n)_n$ such that

$$P_nD \equiv P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0(t) = \Gamma_n P_n(t), \quad n \geq 0$$

- D is symmetric if $(PD, Q)_W = (P, QD)_W$, P, Q matrix polynomials.
- Generating examples:

Symmetry Equations

$$F_2W = WF_2^*$$

$$2(F_2W)' = F_1W + WF_1^*,$$

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Matrix spherical functions associated with $P_N(\mathbb{C}) = \mathrm{SU}(N+1)/\mathrm{U}(N)$

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OMP satisfying odd order differential operators (even first order).

Goal

- For a fixed second-order differential operator D, there can be more than one family of linearly independent (monic) OMP $(P_{n,\gamma})_n$, $\gamma > 0$, having them as eigenfunctions

$$P_{n,\gamma} D = \Gamma_n P_{n,\gamma}, \quad n = 0, 1, \ldots, \quad \gamma > 0,$$

- It happens that

$$P_{n,\gamma} \perp W + \gamma\delta_{t_0} M, \ \gamma > 0.$$

- Scalar case: Not possible for second-order $\Rightarrow$ Fourth order:

Laguerre type $e^{-t} + M\delta_0$
Legendre type $1 + M\delta_{-1} + N\delta_1$
Jacobi type $(1-t)^{\alpha} + M\delta_0$

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then

$$D \text{ is symmetric w.r.t. } W(t)$$

$$\Leftrightarrow$$

$$D \text{ is symmetric w.r.t. } \widetilde{W}(t) = W(t) + \delta_{t_0}(t) M.$$

EXAMPLES WITH $t_0 \in \mathbb{R}$

$$W_a(t) = e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}.$$

$$M = \begin{pmatrix} \varphi^{\pm}_{t_0,a} & 1 \\ 1 & 1/\varphi^{\pm}_{t_0,a} \end{pmatrix}, \; \varphi^{\pm}_{a,t_0} = \frac{at_0 \pm \sqrt{4 + a^2 t_0^2}}{2} \; t_0 \in \mathbb{R}.$$

$$\begin{aligned} D_{a,t_0} = \partial^2 &\begin{pmatrix} -\varphi^{\mp}_{a,t_0} + at_0 - at & -1 - (a^2 t_0)t + a^2 t^2 \\ -1 & -\varphi^{\mp}_{a,t_0} + at \end{pmatrix} \\ +\partial^1 &\begin{pmatrix} -2a + 2\varphi^{\mp}_{a,t_0} t & -2t_0 - 2a\varphi^{\mp}_{a,t_0} + 2(2 + a^2)t \\ 2t_0 & 2(\varphi^{\mp}_{a,t_0} - at_0)t \end{pmatrix} \\ &+\partial^0 \begin{pmatrix} \varphi^{\mp}_{a,t_0} + 2\frac{t_0}{a} & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & -\varphi^{\mp}_{a,t_0} - 2\frac{t_0}{a} \end{pmatrix}. \end{aligned}$$

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Examples with $t_0 \in \mathbb{R}$

- The sequences $(P_{n,\gamma})_n$ orthogonal w.r.t.

$$W_a(t) + \gamma\delta_{t_0}(t)M, \quad \gamma > 0$$

satisfy

$$P_{n,\gamma}D_{a,t_0} = \Gamma_{n,a,t_0}P_{n,\gamma}, \quad n \geq 0$$

- In this group, 2 more weight matrices

$$W_{a,\alpha}(t) = t^{\alpha}e^{-t}\begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}$$

$$W_{\alpha,\beta,k}(t) = t^{\alpha}(1-t)^{\beta}\begin{pmatrix} kt^2 + \beta - k + 1 & (\beta - k + 1)(1-t) \\ (\beta - k + 1)(1-t) & (\beta - k + 1)(1-t)^2 \end{pmatrix}$$

$\alpha, \beta > -1$, $a \in \mathbb{R} \setminus \{0\}$, $0 < k < \beta + 1$.

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Example with $t_0 = 0$

$$W(t) = t^{\alpha} e^{-t} \begin{pmatrix} t(1+a^2 t) & at \\ at & 1 \end{pmatrix}, \; t \in (0,\infty), \; \alpha > -1, \; a \in \mathbb{R}\backslash\{0\}$$

$$M = \begin{pmatrix} a^2(1+\alpha)^2 & a(1+\alpha) \\ a(1+\alpha) & 1 \end{pmatrix}$$

$$D = \partial^2 \begin{pmatrix} 0 & -at^2 \\ 0 & -t \end{pmatrix} + \partial^1 \begin{pmatrix} t & -\frac{(1+a^2(\alpha+3))t}{a} \\ \frac{1}{a} & -(\alpha+1) \end{pmatrix} + \partial^0 \begin{pmatrix} \frac{a^2+1}{a^2} & -\frac{(1+a^2)(\alpha+1)}{a} \\ 0 & 0 \end{pmatrix}$$

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The one step example located at $t_0 = 0$

$$W(t) = t^{\alpha}(1-t)^{\beta}\begin{pmatrix} kt+\beta-k+1 & (1-t)(\beta-k+1) \\ (1-t)(\beta-k+1) & (1-t)^2(\beta-k+1) \end{pmatrix}$$

$t \in (0,1),\ \alpha,\beta > -1,\ 0 < k < \beta+1.$

$$M = \begin{pmatrix} \frac{(\alpha+\beta-k+2)^2}{(\beta-k+1)^2} & \frac{\alpha+\beta-k+2}{\beta-k+1} \\ \frac{\alpha+\beta-k+2}{\beta-k+1} & 1 \end{pmatrix}$$

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Applications

Quantum mechanics

[Durán and Grünbaum, 2006] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006).

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