Riemann-Hilbert characterization for matrix orthogonal polynomials[1]

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[1]joint work with F. A. Grünbaum and A. Martínez-Finkelshtein

Outline

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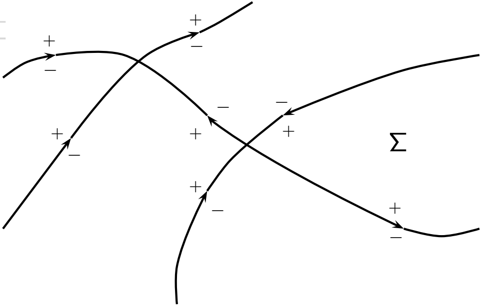
Riemann-Hilbert problem

Let $\Sigma \subset \mathbb{C}$ be an oriented contour and $\Sigma^0 = \Sigma \setminus \{\text{points of self-intersection of } \Sigma\}$. Suppose that there exists a matrix-valued smooth map $\mathbf{G} : \Sigma^0 \to GL(m, \mathbb{C})$. The Riemann-Hilbert problem (RHP) determined by a pair $(\Sigma, \mathbf{G})$ consists of finding an $m \times m$ matrix-valued function $\mathbf{Y}(z)$ s.t.

1. $\mathbf{Y}(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
2. $\mathbf{Y}_+(z) = \mathbf{Y}_-(z)\mathbf{G}(z)$ when $z \in \Sigma$
   $\mathbf{Y}_\pm(z) = \lim_{z' \to z, \pm\text{side}} \mathbf{Y}(z)$
3. $\mathbf{Y}(z) \to \mathbf{I}_m$ as $z \to \infty$



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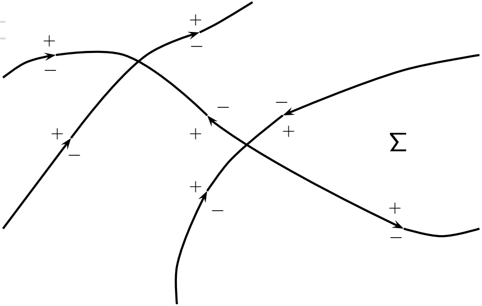

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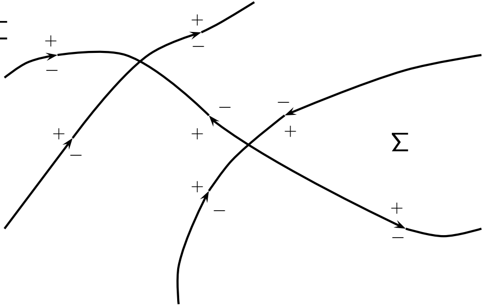

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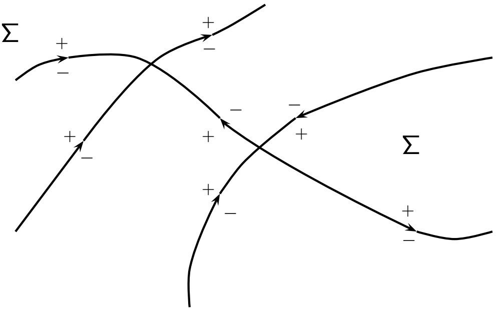

Scalar and additive RHP ($m = 1$)

Let $\omega : \mathbb{R} \to \mathbb{R}$ be a $L^1(\mathbb{R})$ and Hölder continuous function. The Riemann-Hilbert problem determined by $(\mathbb{R}, \omega)$ consists of finding a function $f : \mathbb{C} \to \mathbb{C}$ such that

1. $f(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $f_+(x) = f_-(x) + \omega(x)$ when $x \in \mathbb{R}$
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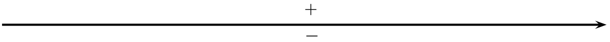

$\Sigma = \mathbb{R}$

The unique solution of this RHP is the Stieltjes or Cauchy transform of ω

$$f(z) = C(\omega)(z) \doteq \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\omega(t)}{t - z} dt$$

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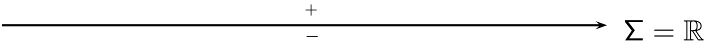

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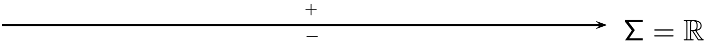

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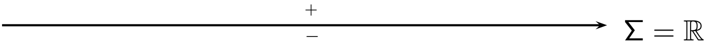

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Applications

- Integrable models. Inverse scattering of some nonlinear differential and difference equations like the nonlinear Schrödinger equation, the Korteweg-de Vries equation or the Toda equations.
- Orthogonal polynomials and random matrices. The distribution of eigenvalues of random matrices in several ensembles is reduced to computations involving orthogonal polynomials.
- Combinatorial probability. On the distribution of the length of the longest increasing subsequence of a random permutation.

Advantages for orthogonal polynomials

1. Algebraic properties: three term recurrence relation, ladder operators, second order differential equation
2. Uniform asymptotics: *steepest descent analysis for RHP* (Deift-Zhou,1993).

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Orthogonal polynomials

Let $d\mu$ be a positive Borel measure supported on $\mathbb{R}$.
We will assume $d\mu(x) = \omega(x)dx$, $\omega \geq 0$ and $x^i\omega, x^j\omega' \in L^1(\mathbb{R})$.
We can then construct a family of orthonormal polynomials $(p_n)_n$ s.t.

$$(p_n, p_m)_\omega = \int_{\mathbb{R}} p_n(x)p_m(x)\omega(x)dx = \delta_{n,m}, \quad n, m \geq 0$$

$$p_n(x) = \kappa_n(x^n + a_{n,n-1}x^{n-1} + \cdots) = \kappa_n \widehat{p}_n(x)$$

The monic polynomials $\widehat{p}_n(x)$ satisfy a three-term recurrence relation

$$x\widehat{p}_n(x) = \widehat{p}_{n+1}(x) + \alpha_n \widehat{p}_n(x) + \beta_n \widehat{p}_{n-1}(x)$$

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Solution of the RHP for orthogonal polynomials ($m = 2$)

We try to find a 2×2 matrix-valued function $\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2\times 2}$ such that

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
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3. $\mathbf{Y}^n(z) = (\mathbf{I}_2 + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RHP above is given by

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where $C(f)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} dt$ and $\gamma_n = \kappa_n^2$ (Fokas-Its-Kitaev, 1990).
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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

$$\mathbf{Y}^{n+1}(z) = \underbrace{\begin{pmatrix} z-\alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix}}_{\mathbf{E}_n(z)} \mathbf{Y}^n(z)$$

$$\frac{d}{dz}\mathbf{Y}^n(z) = \underbrace{\begin{pmatrix} -\mathfrak{B}_n(z) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathfrak{A}_n(z) \\ 2\pi i\,\mathfrak{A}_{n-1}(z)\gamma_{n-1} & \mathfrak{B}_n(z) \end{pmatrix}}_{\mathbf{F}_n(z)} \mathbf{Y}^n(z)$$

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Compatibility conditions

Cross-differentiating the Lax pair yield

$$\mathbf{E}_n'(z) + \mathbf{E}_n(z)\mathbf{F}_n(z) = \mathbf{F}_{n+1}(z)\mathbf{E}_n(z)$$

also known as string equations. In our situation, the compatibility conditions entry-wise are

$$1 + (z - \alpha_n)(\mathfrak{B}_{n+1}(z) - \mathfrak{B}_n(z)) = \beta_{n+1}\mathfrak{A}_{n+1}(z) - \beta_n\mathfrak{A}_{n-1}(z)$$
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Problem. Typically, the coefficients $\mathfrak{A}_n(z)$ and $\mathfrak{B}_n(z)$ are difficult to obtain. We can avoid that by transforming the RHP in another RHP with constant jump.

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Transformation of the RHP

Consider the transformation

$$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

We observe that $\mathbf{X}^n$ is invertible and that

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That means that $\mathbf{X}^n$ has a constant jump

$\Rightarrow$ $\mathbf{E}_n(z)$ and $\mathbf{F}_n(z)$ are completely determined by their behavior at $z \to \infty$.

Transformation of the RHP

Consider the transformation

$$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

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&= \mathbf{Y}_-^n(x) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \begin{pmatrix} \omega^{-1/2} & 0 \\ 0 & \omega^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \\
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Example: Hermite polynomials

The solution $\mathbf{Y}^n(z)$ of the RHP

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} 1 & e^{-x^2} \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_2 + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{H}_n(z) & C(\widehat{H}_n e^{-t^2})(z) \\ -2\pi i \gamma_{n-1} \widehat{H}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\widehat{H}_{n-1} e^{-t^2})(z) \end{pmatrix}$$

where $(\widehat{H}_n)_n$ is the family of monic Hermite polynomials.

The Lax pair and compatibility conditions

$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} e^{-z^2/2} & 0 \\ 0 & e^{z^2/2} \end{pmatrix}$ satisfies the following Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z-\alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i\gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z), \quad \frac{d}{dz}\mathbf{X}^n(z) = \begin{pmatrix} -z & -\frac{1}{\pi i}\gamma_n^{-1} \\ 4\pi i\gamma_{n-1} & z \end{pmatrix} \mathbf{X}^n(z)$$

The difference equation gives (using $\beta_n = \gamma_n/\gamma_{n+1}$) the TTRR

$$x\widehat{H}_n(x) = \widehat{H}_{n+1}(x) + \alpha_n\widehat{H}_n(x) + \beta_n\widehat{H}_{n-1}(x),$$

while the differential equation gives the ladder operators

$$\widehat{H}_n'(x) = 2\beta_n\widehat{H}_{n-1}(x), \quad \widehat{H}_n'(x) - 2z\widehat{H}_n(x) = -2\widehat{H}_{n+1}(x).$$

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$$\alpha_n = 0, \quad \beta_{n+1} - \beta_n = \frac{1}{2} \Rightarrow \beta_n = \frac{n}{2}$$

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Outline

Matrix orthogonal polynomials

The theory of matrix orthogonal polynomials on the real line (MOP) was introduced by Krein in 1949.

A $N \times N$ matrix polynomial on the real line is

$$\mathbf{P}(x) = \mathbf{A}_n x^n + \mathbf{A}_{n-1} x^{n-1} + \cdots + \mathbf{A}_0, \quad x \in \mathbb{R} \quad \mathbf{A}_i \in \mathbb{C}^{N \times N}$$

Let $\mathbf{W}$ be a $N \times N$ a matrix of measures or weight matrix.

We will assume $d\mathbf{W}(x) = \mathbf{W}(x)dx$ and $\mathbf{W}$ smooth and positive definite on $\mathbb{R}$.

We can construct a family of MOP with respect to the inner product

$$(\mathbf{P}, \mathbf{Q})_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{P}(x)\mathbf{W}(x)\mathbf{Q}^*(x)dx \in \mathbb{C}^{N \times N}$$

such that

$$(\mathbf{P}_n, \mathbf{P}_m)_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{P}_n(x)\mathbf{W}(x)\mathbf{P}_m^*(x)dx = \delta_{n,m}\mathbf{I}_N, \quad n, m \geq 0$$

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Solution of the RHP for MOP ($m = 2N$)

$\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2N\times 2N}$ such that

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} \mathbf{I}_N & \mathbf{W}(x) \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_{2N} + \mathcal{O}(1/z)) \begin{pmatrix} z^n\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & z^{-n}\mathbf{I}_N \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RH problem above is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{\mathbf{P}}_n(z) & C(\widehat{\mathbf{P}}_n\mathbf{W})(z) \\ -2\pi i\gamma_{n-1}\widehat{\mathbf{P}}_{n-1}(z) & -2\pi i\gamma_{n-1}C(\widehat{\mathbf{P}}_{n-1}\mathbf{W})(z) \end{pmatrix}$$

where $C(\mathbf{F})(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mathbf{F}(t)}{t-z}dt$ and $\gamma_n = \boldsymbol{\kappa}_n^*\boldsymbol{\kappa}_n$.

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The Lax pair

The solution of the RHP for orthogonal polynomials satisfy the Lax pair

$$\mathbf{Y}^{n+1}(z) = \underbrace{\begin{pmatrix} z-\alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i\gamma_n & 0 \end{pmatrix}}_{\mathbf{E}_n(z)} \mathbf{Y}^n(z),$$

$$\frac{d}{dz}\mathbf{Y}^n(z) = \underbrace{\begin{pmatrix} -\mathfrak{B}_n(z) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathfrak{A}_n(z) \\ 2\pi i\,\mathfrak{A}_{n-1}(z)\gamma_{n-1} & \mathfrak{B}_n^*(z) \end{pmatrix}}_{\mathbf{F}_n(z)} \mathbf{Y}^n(z),$$

where

$$\mathfrak{A}_n(z) = -\gamma_n \int_{\mathbb{R}} \frac{\widehat{\mathbf{P}}_n(t)\mathbf{W}'(t)\widehat{\mathbf{P}}_n^*(t)}{t-z}\,dt, \quad \mathfrak{B}_n(z) = -\left(\int_{\mathbb{R}} \frac{\widehat{\mathbf{P}}_n(t)\mathbf{W}'(t)\widehat{\mathbf{P}}_{n-1}^*(t)}{t-z}\,dt\right)\gamma_{n-1}$$

Compatibility conditions

Cross-differentiating the Lax pair yield

$$\mathbf{E}_n'(z) + \mathbf{E}_n(z)\mathbf{F}_n(z) = \mathbf{F}_{n+1}(z)\mathbf{E}_n(z)$$

also known as string equations. In our situation, the compatibility conditions entry-wise are

$$\mathbf{I}_N + \mathfrak{B}_{n+1}(z)(z\mathbf{I}_N - \alpha_n) - (z\mathbf{I}_N - \alpha_n)\mathfrak{B}_n(z) = \mathfrak{A}^*_{n+1}(z)\beta_{n+1} - \beta_n\mathfrak{A}^*_{n-1}(z)$$

$$\mathfrak{B}_{n+1}(z) + \gamma_n^{-1}\mathfrak{B}^*_n(z)\gamma_n = (z\mathbf{I}_N - \alpha_n)\mathfrak{A}^*_n(z)$$

Problem. Again, the coefficients $\mathfrak{A}_n(z)$ and $\mathfrak{B}_n(z)$ are difficult to obtain. We need to tranform the RHP in another RHP with constant jump.

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Transformation of the RHP

Goal: obtain an invertible transformation $\mathbf{Y}^n \to \mathbf{X}^n$ such that $\mathbf{X}^n$ has a constant jump across $\mathbb{R}$. Consider $\mathbf{X}^n(z) = \mathbf{Y}^n(z)\mathbf{V}(z)$ where

$$\mathbf{V}(z) = \begin{pmatrix} \mathbf{T}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-*}(z) \end{pmatrix}$$

where $\mathbf{T}$ is an invertible $N \times N$ smooth matrix function.

This motivates to consider a factorization of the weight in the form

$$\mathbf{W}(x) = \mathbf{T}(x)\mathbf{T}^*(x), \quad x \in \mathbb{R}.$$

This factorization is not unique since

$$\mathbf{T}(x) = \widehat{\mathbf{T}}(x)\mathbf{S}(x), \quad x \in \mathbb{R},$$

where $\widehat{\mathbf{T}}(x)$ is an upper triangular matrix and $\mathbf{S}(x)$ is an arbitrary smooth and unitary matrix.

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Transformation of the RHP II

We additionally assume

$$\mathbf{T}'(z) = \mathbf{G}(z)\mathbf{T}(z),$$

where **G** is a matrix polynomial of degree m (most of our examples)

Then

$$\frac{d}{dz}\mathbf{X}^n(z) = \mathbf{F}_n(z;\mathbf{G})\mathbf{X}^n(z),$$

$$\mathbf{F}_n(z;\mathbf{G}) = \begin{pmatrix} -\mathcal{B}_n(z;\mathbf{G}) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathcal{A}_n(z;\mathbf{G}) \\ 2\pi i\,\mathcal{A}_{n-1}(z;\mathbf{G})\gamma_{n-1} & \mathcal{B}_n^*(z;\mathbf{G}) \end{pmatrix}$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are matrix polynomials of degree $m-1$ and m respectively.

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Transformation of the RHP III

If there exists a non-trivial matrix-valued function $\mathbf{S}$, non-singular on $\mathbb{C}$, smooth and unitary on $\mathbb{R}$, s.t.

$$\mathbf{H}(z) = \mathbf{T}(z)\mathbf{S}'(z)\mathbf{S}^*(z)\mathbf{T}^{-1}(z)$$

is also a polynomial, then $\widetilde{\mathbf{T}} = \mathbf{T}\mathbf{S}$ satisfies

$$\mathbf{W}(x) = \widetilde{\mathbf{T}}(x)\widetilde{\mathbf{T}}^*(x), \quad x \in \mathbb{R}, \qquad \widetilde{\mathbf{T}}'(z) = \widetilde{\mathbf{G}}(z)\widetilde{\mathbf{T}}(z), \quad z \in \mathbb{C},$$

with $\widetilde{\mathbf{G}}(z) = \mathbf{G}(z) + \mathbf{H}(z)$ and the matrix $\mathbf{X}^n$ satisfies

$$\frac{d}{dz}\mathbf{X}^n(z) = \left(\mathbf{F}_n(z;\mathbf{G}) + \mathbf{F}_n(z;\mathbf{H})\right)\mathbf{X}^n(z) - \mathbf{X}^n(z)\begin{pmatrix} \chi(z) & \mathbf{0} \\ \mathbf{0} & -\chi^*(z) \end{pmatrix}$$

with $\chi(z) = \mathbf{S}'(z)\mathbf{S}^*(z)$.

Consequences: We have a class of ladder operators.

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$$\mathbf{W}(x) = \widetilde{\mathbf{T}}(x)\widetilde{\mathbf{T}}^*(x), \quad x \in \mathbb{R}, \qquad \widetilde{\mathbf{T}}'(z) = \widetilde{\mathbf{G}}(z)\widetilde{\mathbf{T}}(z), \quad z \in \mathbb{C},$$

with $\widetilde{\mathbf{G}}(z) = \mathbf{G}(z) + \mathbf{H}(z)$ and the matrix $\mathbf{X}^n$ satisfies

$$\frac{d}{dz}\mathbf{X}^n(z) = \left(\mathbf{F}_n(z;\mathbf{G}) + \mathbf{F}_n(z;\mathbf{H})\right)\mathbf{X}^n(z) - \mathbf{X}^n(z)\begin{pmatrix} \chi(z) & \mathbf{0} \\ \mathbf{0} & -\chi^*(z) \end{pmatrix}$$

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Consequences: We have a class of ladder operators.

Example: Hermite type MOP

Let us consider $\mathbf{T}(x) = e^{-x^2/2} e^{\mathbf{A}x}$ and

$$\mathbf{W}(x) = e^{-x^2} e^{\mathbf{A}x} e^{\mathbf{A}^* x}, \quad \mathbf{A} \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}.$$

Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z\mathbf{I}_N - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z)$$

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Compatibility conditions

$$\alpha_n = (\mathbf{A} + \gamma_n^{-1}\mathbf{A}^*\gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = \mathbf{A}\alpha_n - \alpha_n\mathbf{A} + \mathbf{I}_N$$

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Ladder operators

$$\widehat{\mathbf{P}}'_n(x) + \widehat{\mathbf{P}}_n(x)\mathbf{A} - \mathbf{A}\widehat{\mathbf{P}}_n(x) = 2\beta_n \widehat{\mathbf{P}}_{n-1}(x),$$

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Combining them we get a second order differential equation

Second order differential equation

$$\widehat{\mathbf{P}}''_n(x) + 2\widehat{\mathbf{P}}'_n(x)(\mathbf{A} - x\mathbf{I}_N) + \widehat{\mathbf{P}}_n(x)(\mathbf{A}^2 - 2x\mathbf{A})$$

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The matrix $\mathbf{H}$ can be written as

$$\mathbf{H}(x) = e^{\mathbf{A}x}\chi e^{-\mathbf{A}x} = \chi + \mathrm{ad}_{\mathbf{A}}(\chi)x + \mathrm{ad}_{\mathbf{A}}^2(\chi)\frac{x^2}{2} + \cdots,$$

where $\chi(x) = \mathbf{S}'(x)\mathbf{S}^*(x)$ is skew-Hermitian on $\mathbb{R}$.

This matrix equation was considered already by Durán-Grünbaum (2004), when χ is a constant matrix.

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First case $\mathbf{A} = \mathbf{L}$

New compatibility conditions

$$\mathbf{J}\alpha_n - \alpha_n\mathbf{J} + \alpha_n = \mathbf{L} + \frac{1}{2}(\mathbf{L}^2\alpha_n - \alpha_n\mathbf{L}^2), \quad \mathbf{J} - \gamma_n^{-1}\mathbf{J}\gamma_n = \mathbf{L}\alpha_n + \alpha_n\mathbf{L} - 2\alpha_n^2$$

New ladder operators (0-th order)

$$\widehat{\mathbf{P}}_n(x)\mathbf{J} - \mathbf{J}\widehat{\mathbf{P}}_n(x) - x(\widehat{\mathbf{P}}_n(x)\mathbf{L} - \mathbf{L}\widehat{\mathbf{P}}_n(x)) + 2\beta_n\widehat{\mathbf{P}}_n(x) - n\widehat{\mathbf{P}}_n(x) = 2(\mathbf{L} - \alpha_n)\beta_n\widehat{\mathbf{P}}_{n-1}(x)$$

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First-order differential equation

$$(\mathbf{L} - \alpha_n)\widehat{\mathbf{P}}_n'(x) + (\mathbf{L} - \alpha_n + x\mathbf{I}_N)(\widehat{\mathbf{P}}_n(x)\mathbf{L} - \mathbf{L}\widehat{\mathbf{P}}_n(x)) - 2\beta_n\widehat{\mathbf{P}}_n(x) = \widehat{\mathbf{P}}_n(x)\mathbf{J} - \mathbf{J}\widehat{\mathbf{P}}_n(x) - n\widehat{\mathbf{P}}_n(x)$$

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Sturm-Liouville type differential equation

Finally, something remarkable happens. Combining the second and the first order differential equation will give surprisingly

Sturm-Liouville type differential equation

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This is a second-order differential equation of Sturm-Liouville type satisfied by the MOP, already given by Durán-Grünbaum (2004)

Conclusions

1. The ladder operators method gives more insight about the differential properties of MOP and new phenomena
2. This method works for every weight matrix $\mathbf{W}$. The corresponding MOP satisfy differential equations, but not necessarily of Sturm-Liouville type

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Open problems I

RHP for Laguerre and Jacobi type matrix orthogonal polynomials

Find $\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2N\times 2N}$ with (Y1), (Y2), (Y3) and

(Y4) As $z \to c$, $c \in \{a, b\}$, $c \neq \pm\infty$, we have

$$\mathbf{Y}^n(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(h(z)) \\ \mathcal{O}(1) & \mathcal{O}(h(z)) \end{pmatrix},$$

where

$$h(z) = \begin{cases} |z-c|^{\gamma_c}, & \text{if } -1 < \gamma_c < 0, \\ \log|z-c|, & \text{if } \gamma_c = 0, \\ 1, & \text{if } \gamma_c > 0. \end{cases}$$

Calculate ladder operators and differential equations for examples of Laguerre and Jacobi type.

Open problems II

Ladder operators for multiple orthogonal polynomials of mixed type

Find $\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$. The main difference is the jump condition

$$\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} \mathbf{I}_p & \vec{w}_1^t \vec{w}_2 \\ \mathbf{0} & \mathbf{I}_q \end{pmatrix}$$

where $\vec{w}_1 = (w_{1,1}, \dots, w_{1,p})$ and $\vec{w}_2 = (w_{2,1}, \dots, w_{2,q})$.
If we perform the transformation $\mathbf{X}^n(z) = \mathbf{Y}^n(z)\mathbf{V}(z)$ where

$$\mathbf{V}(z) = \begin{pmatrix} w_{1,1} & & & & & \\ & \ddots & & & & \\ & & w_{1,p} & & & \\ & & & w_{2,1}^{-1} & & \\ & & & & \ddots & \\ & & & & & w_{2,q}^{-1} \end{pmatrix}$$

then $\mathbf{X}^n$ has a constant jump, and we could get explicit formulas for ladder operators of multiple orthogonal polynomials of mixed type.

Open problems III

Modification of the weight matrix $\mathbf{W}(x)$ by a parameter t

The corresponding MOP will depend of an extra parameter $\mathbf{P}_n(x,t)$.
In the scalar case, it is known that for some special modifications of the weight, the coefficients of the new three-term recurrence relation are related to Painlevé equations. One way of proving this is by using ladder operators.
Something similar can be done in the matrix case for some families where we can compute explicitly the coefficients of the three-term recurrence relation. It would lead to non-commutative Painlevé equations. RH methods could help since it is a straightforward way of getting ladder operators for MOP.