Spectral methods for bivariate Markov processes

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Outline

1. Markov processes
   - Preliminaries
   - Spectral methods
2. Bivariate Markov processes
   - Preliminaries
   - Spectral methods
3. An example
   - A quasi-birth-and-death process
   - A variant of the Wright-Fisher model

Outline

1-D Markov processes

A *Markov process* with state space $\mathcal{S} \subset \mathbb{R}$ is a collection of random variables $\{X_t \in \mathcal{S} : t \in \mathcal{T}\}$ indexed by time $\mathcal{T}$ (discrete or continuous) such that they have the Markov property: the future event only depends on the present, not on the past (no memory).

$\mathcal{S}$ discrete (Markov chains)

The transition probabilities come in terms of a matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad P_{ij}(t) \equiv \Pr(X_t = j | X_0 = i)$$

$\mathcal{S}$ continuous (Markov processes)

The probabilities are described in terms of a density

$$p(t; x, y) \equiv \frac{\partial}{\partial y} \Pr(X_t \leq y | X_0 = x), \quad x, y \in \mathcal{S}$$

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Three important cases

1. Random walks: $\mathcal{S}=\{0,1,2,\ldots\}$, $\mathcal{T}=\{0,1,2,\ldots\}$.

$$P=\begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_i\geq 0, a_i, c_i>0, \quad a_i+b_i+c_i=1$$

2. Birth-and-death processes: $\mathcal{S}=\{0,1,2,\ldots\}$, $\mathcal{T}=[0,\infty)$.
$P'(t)=\mathcal{A}P(t)$ y $P'(t)=P(t)\mathcal{A}$ where

$$\mathcal{A}=\begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \lambda_i,\mu_i>0$$

3. Diffusion processes: $\mathcal{S}=(a,b)\subseteq\mathbb{R}$, $\mathcal{T}=[0,\infty)$.
$\frac{\partial}{\partial t}p(t;x,y)=\mathcal{A}p(t;x,y)$ y $\frac{\partial}{\partial t}p(t;x,y)=\mathcal{A}^*p(t;x,y)$ donde

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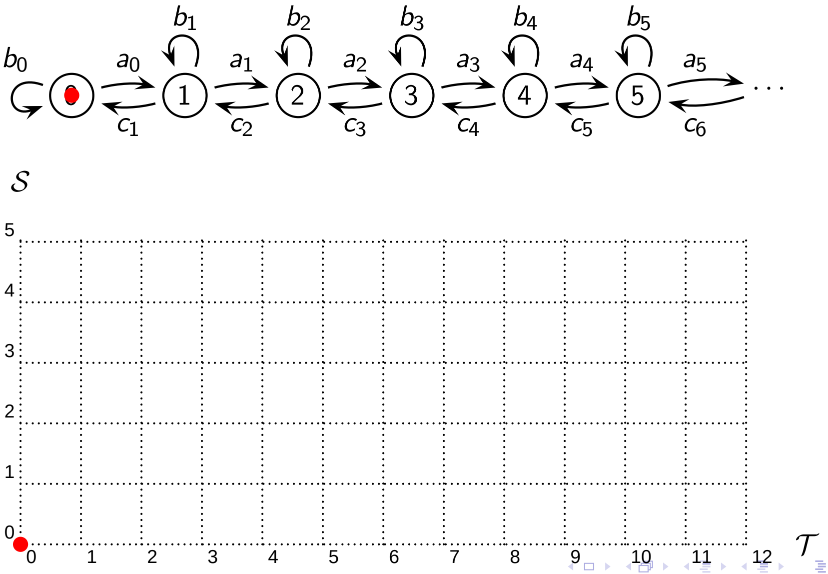
Random walks



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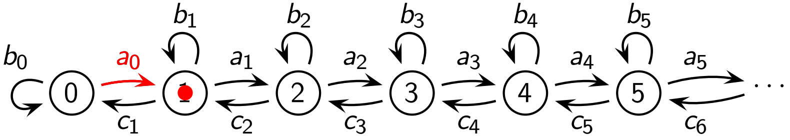

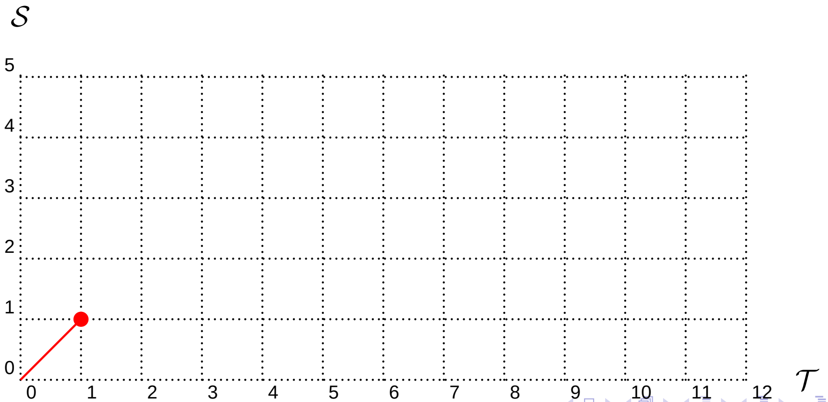

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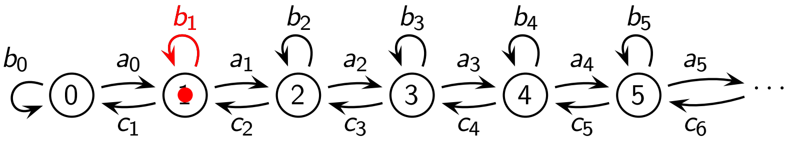

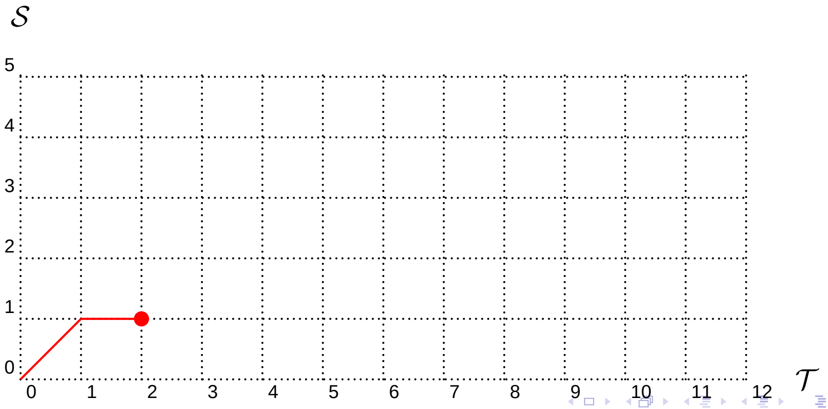

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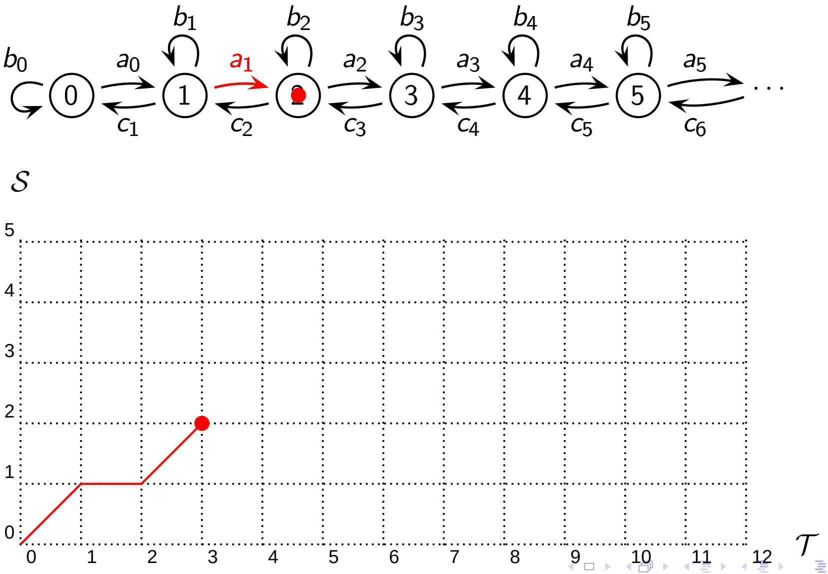

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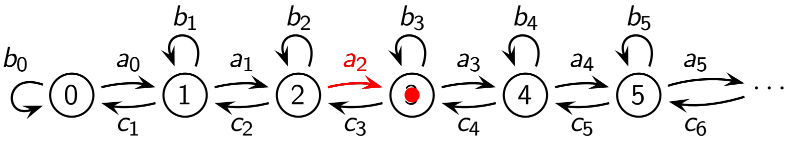

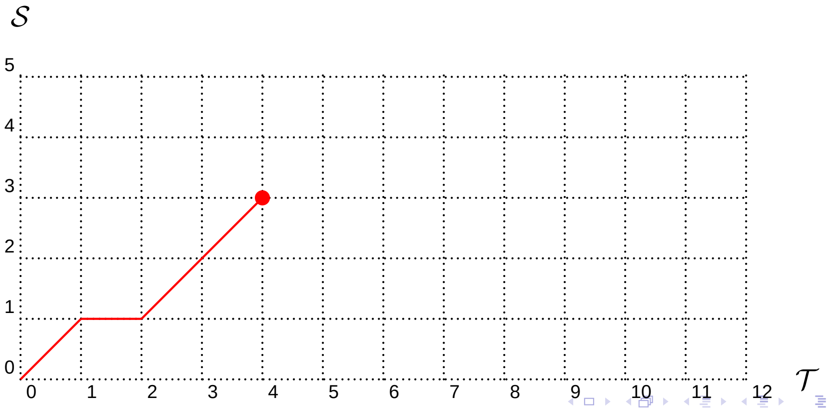

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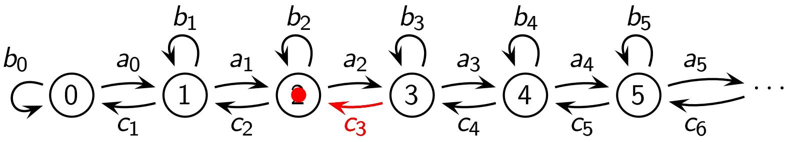

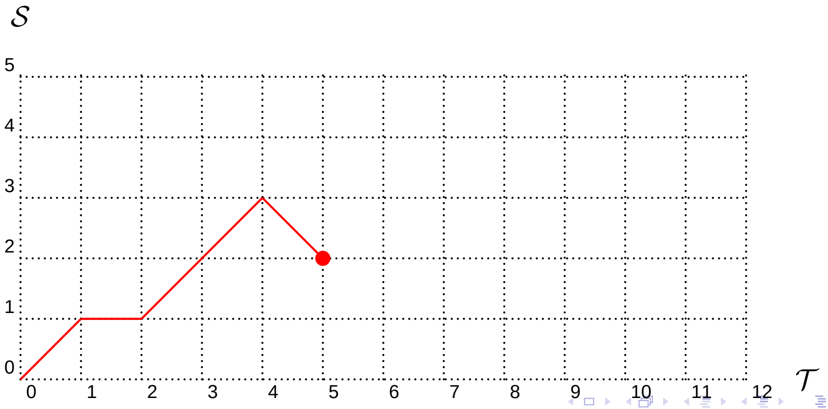

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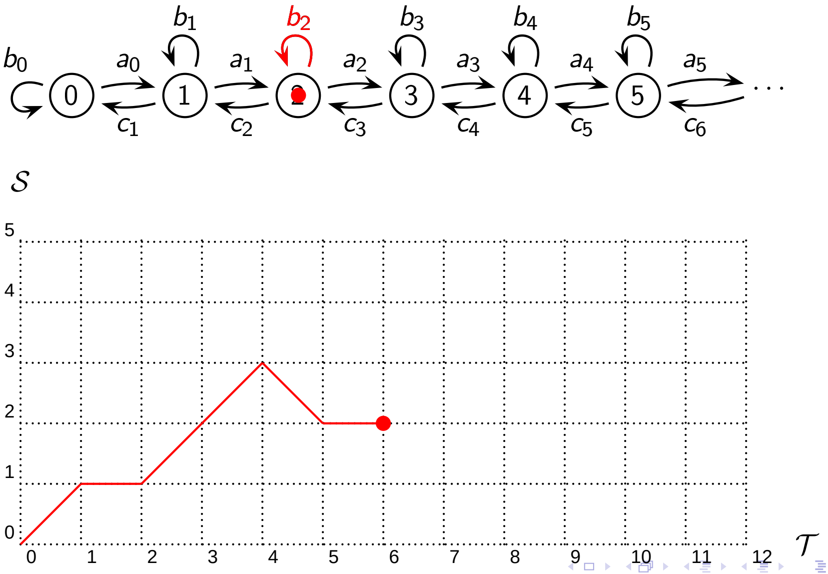

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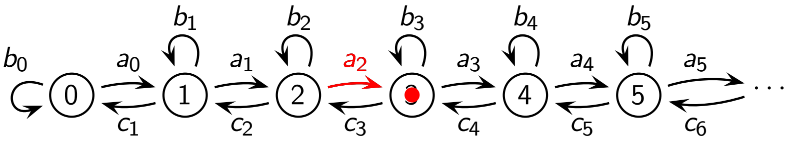

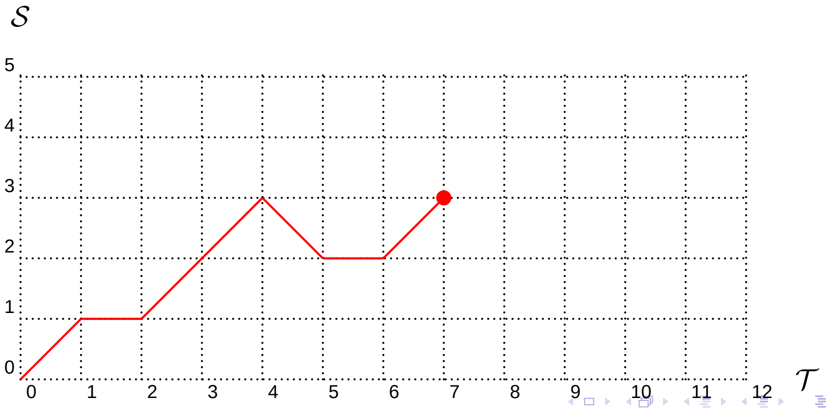

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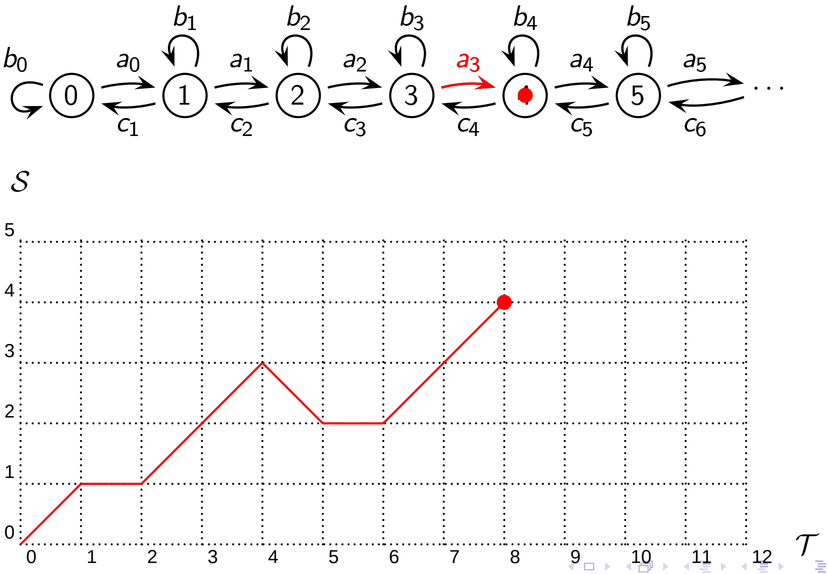

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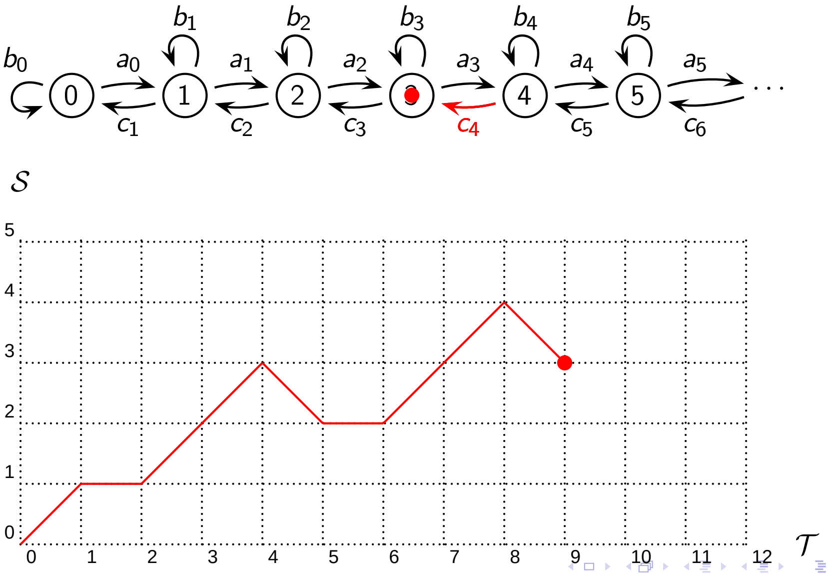

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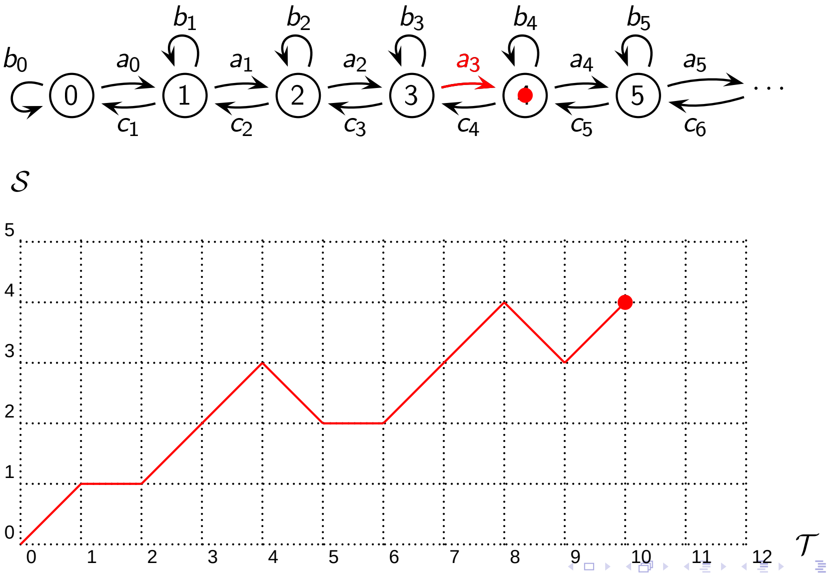

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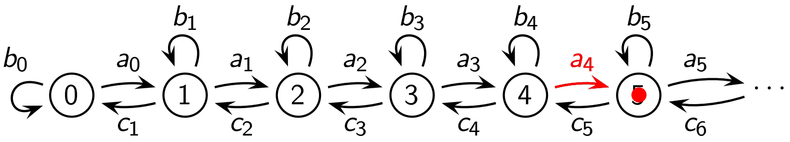

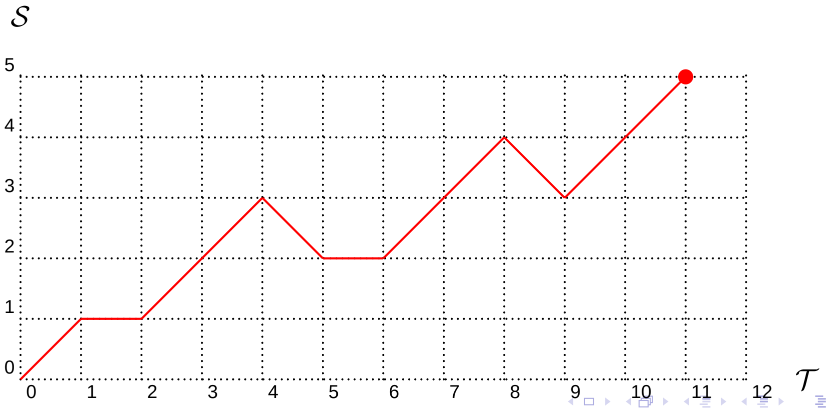

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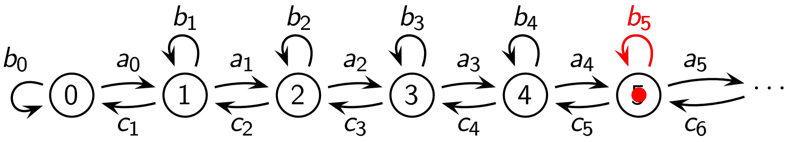
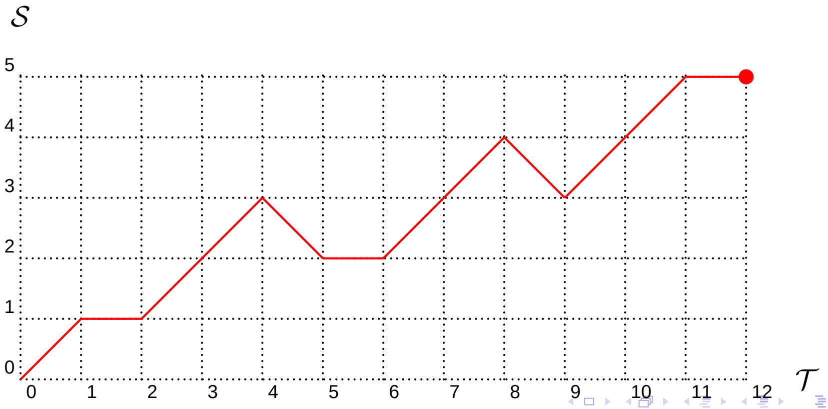
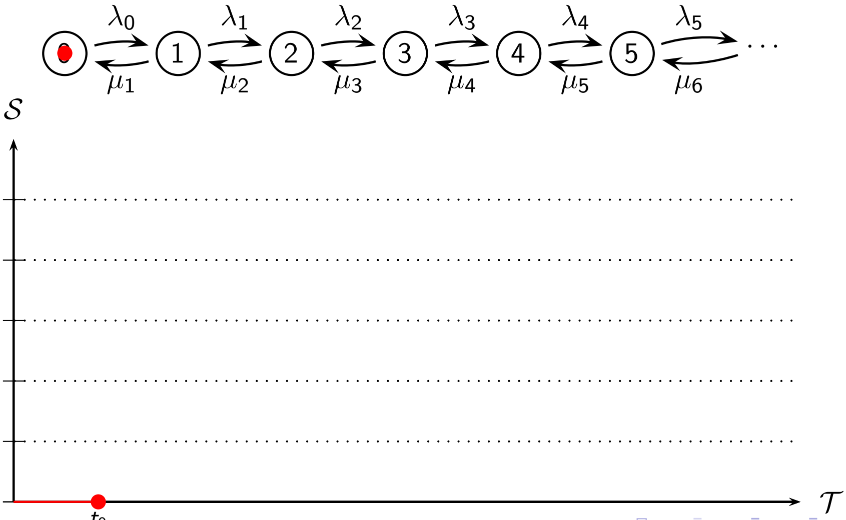




Birth-and-death processes



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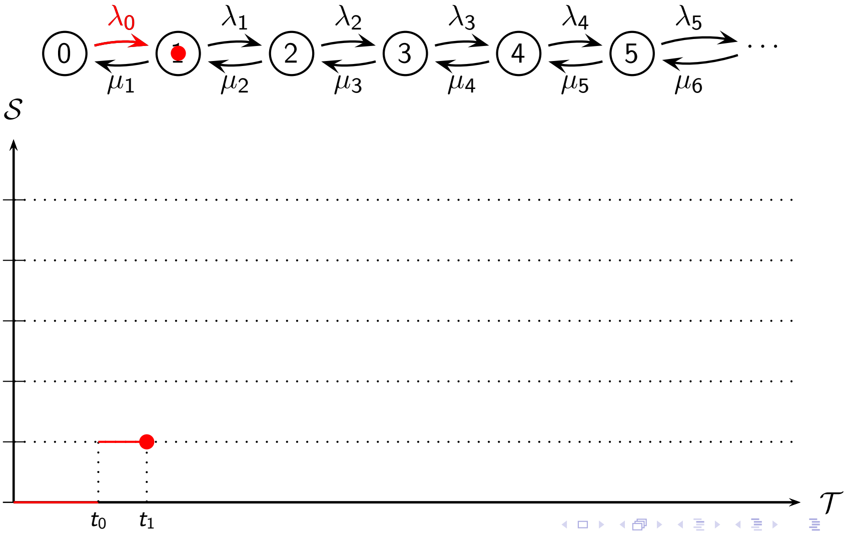

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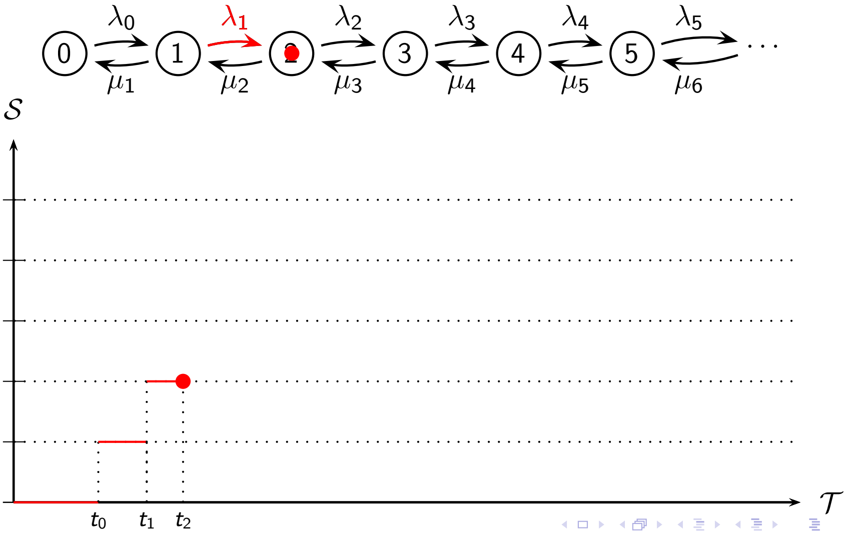

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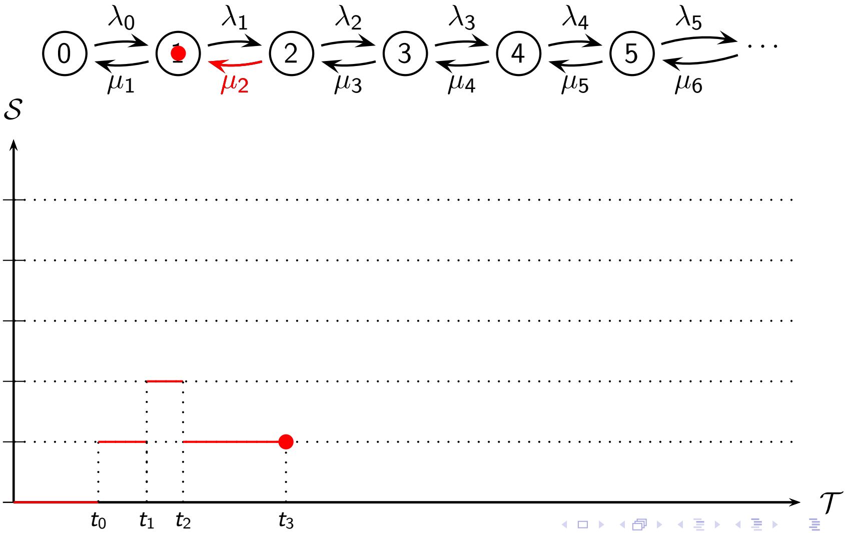

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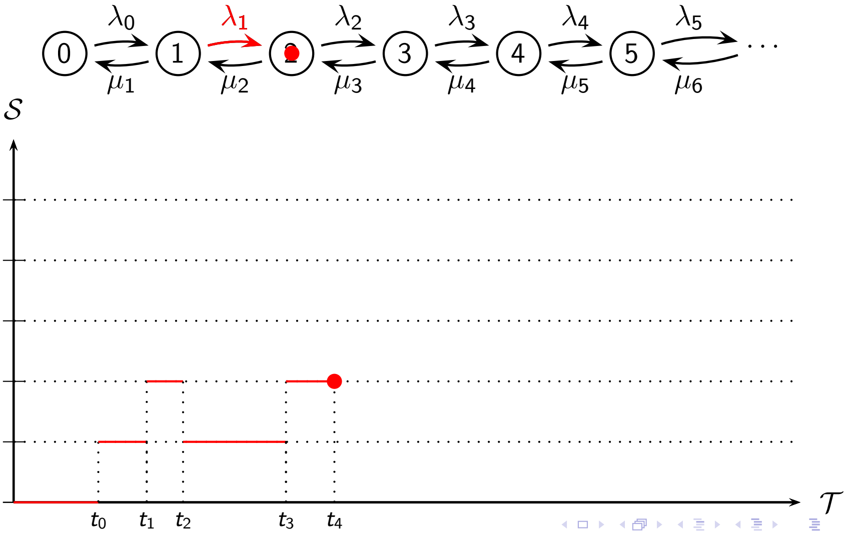

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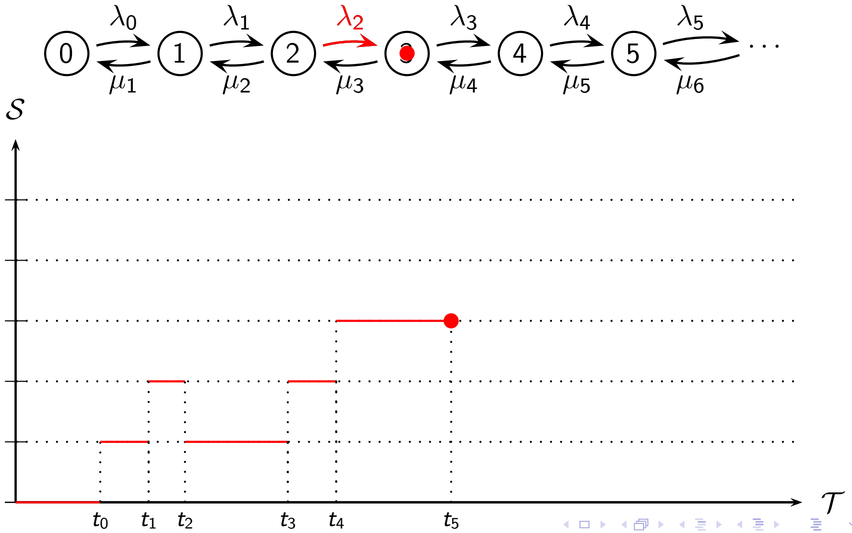

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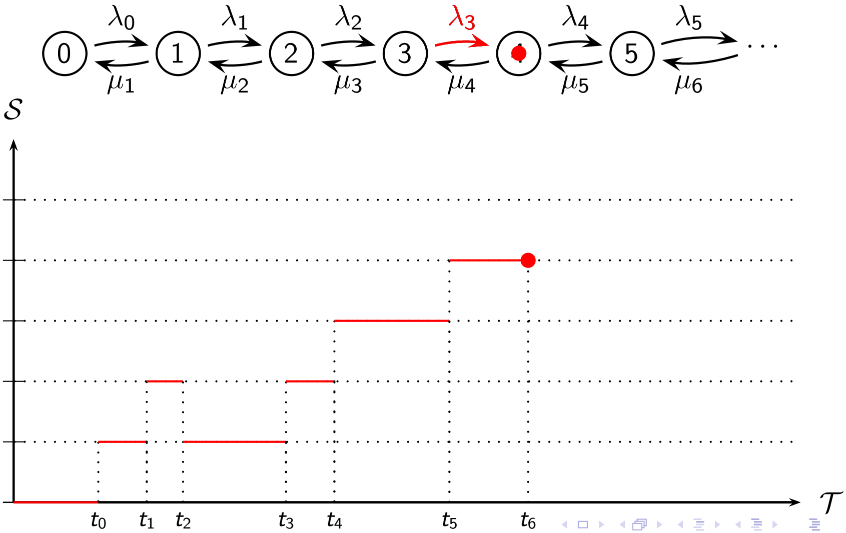

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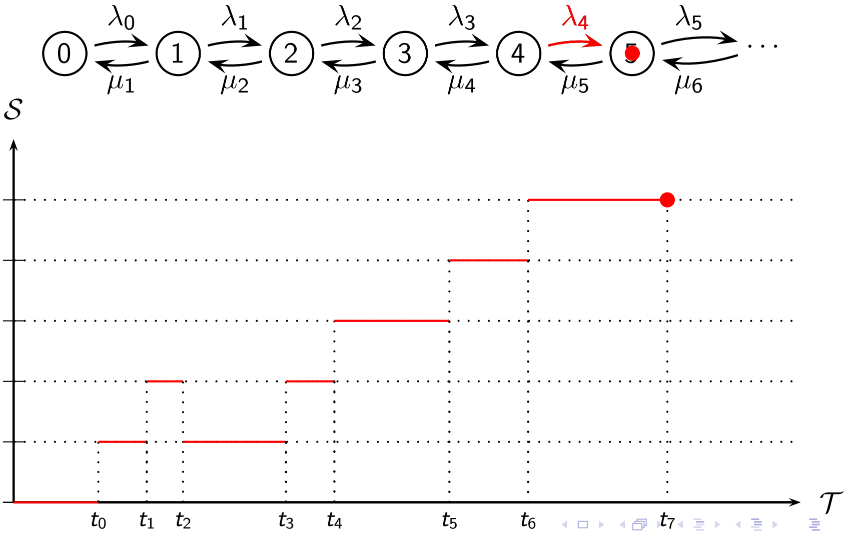

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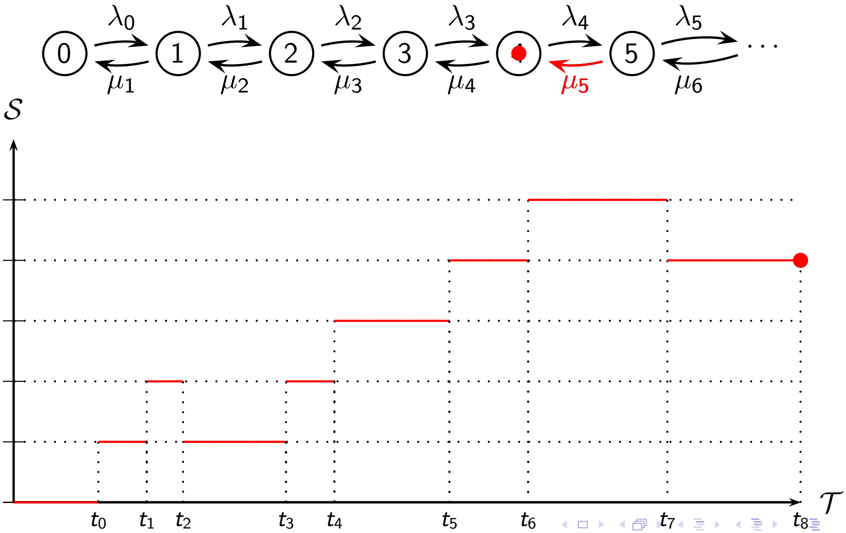
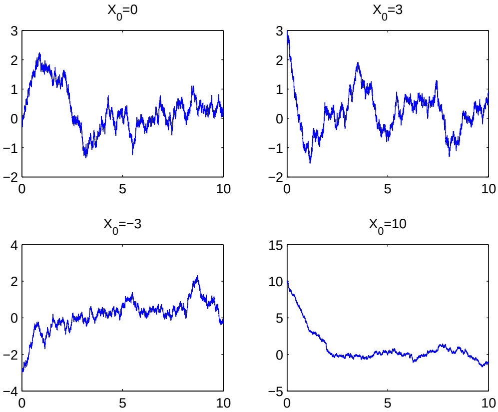


Diffusion processes

Ornstein-Uhlenbeck diffusion process: $\mathcal{S} = \mathbb{R}$ and $\sigma^2(x) = 1$, $\tau(x) = -x$
It describes the velocity of a massive Brownian particle under the influence of friction. It is the only nontrivial process which is stationary, Gaussian and Markovian.



Spectral methods

Given a infinitesimal operator $\mathcal{A}$, if we can find a measure $\omega(x)$ associated with $\mathcal{A}$, and a set of orthogonal eigenfunctions $f(i,x)$ such that

$$\mathcal{A}f(i,x) = \lambda(i,x)f(i,x),$$

then it is possible to find spectral representations of

- Transition probabilities $P_{ij}(t)$ (discrete case) or densities $p(t;x,y)$ (continuous case).
- Invariant measure or distribution $\boldsymbol{\pi} = (\pi_j)$ (discrete case) with

$$\pi_j = \lim_{t\to\infty} P_{ij}(t)$$

or $\psi(y)$ (continuous case) with

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Random walks

$\mathcal{S} = \mathcal{T} = \{0, 1, 2, \ldots\}$.

Spectral theorem: there exists a measure ω associated with P which orthogonal polynomials $(q_n)_n$ satisfy

$$Pq = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix}, \quad x \in [-1, 1]$$

Transition probabilities

$$\Pr(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

Invariant measure

Non-null vector $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ such that

$$\pi P = \pi \quad \Rightarrow \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\|q_i\|^2}$$

Examples: Jacobi polynomials (Legendre, Gegenbauer)

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Birth-and-death processes

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Non-null vector $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ such that

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Examples: Laguerre, Hahn, Krawtchouk, Charlier polynomials

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$$\mathcal{A}q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix} = -x \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix}$$

Transition probabilities

$$\Pr(X_t = j | X_0 = i) = P_{ij}(t) = \frac{1}{\|q_i\|^2} \int_0^\infty e^{-xt} q_i(x) q_j(x) d\omega(x)$$

Invariant measure

Non-null vector $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ such that

$$\pi\mathcal{A} = 0 \quad \Rightarrow \pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} = \frac{1}{\|q_i\|^2}$$

Examples: Laguerre, Hahn, Krawtchouk, Charlier polynomials

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$\mathcal{S} = (a, b) \subseteq \mathbb{R}$, $\mathcal{T} = [0, \infty)$

If there exists a positive measure ω symmetric with respect to $\mathcal{A}$ and the corresponding family of orthogonal functions $(\phi_n)_n$ satisfy

$$\mathcal{A}\phi_n(x) = \frac{1}{2}\sigma^2(x)\phi_n''(x) + \tau(x)\phi_n'(x) = \alpha_n\phi_n(x)$$

Transition probability density

$$p(t; x, y) = \sum_{n=0}^{\infty} e^{\alpha_n t}\phi_n(x)\phi_n(y)\omega(y)$$

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$$\psi(y) \quad \text{tal que} \quad \mathcal{A}^*\psi(y) = 0 \Rightarrow \psi(y) = \frac{1}{\int_S \omega(x)dx}\omega(y)$$

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Outline

1. Markov processes
   - Preliminaries
   - Spectral methods
2. Bivariate Markov processes
   - Preliminaries
   - Spectral methods
3. An example
   - A quasi-birth-and-death process
   - A variant of the Wright-Fisher model

2-D Markov processes

Now we have a bivariate or 2-component Markov process of the form $\{(X_t, Y_t) : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) and with state space $\mathcal{C} = \mathcal{S} \times \{1, 2, \ldots, N\}$, where $\mathcal{S} \subset \mathbb{R}$. The first component is the level while the second component is the phase.

Now the transition probabilities can be written in terms of a matrix-valued function $\mathbf{P}(t; x, A)$, defined for every $t \in \mathcal{T}, x \in \mathcal{S}$, and any Borel set A of $\mathcal{S}$, whose entry (i, j) gives

$$\mathbf{P}_{ij}(t; x, A) = \Pr\{X_t \in A, Y_t = j | X_0 = x, Y_0 = i\}.$$

Every entry must be nonnegative and

$$\mathbf{P}(t; x, A)\mathbf{e}_N \leq \mathbf{e}_N, \quad \mathbf{e}_N = (1, 1, \ldots, 1)^T$$

The infinitesimal operator $\mathcal{A}$ is now matrix-valued.

Ideas behind: *random evolutions*
(Griego-Hersh-Papanicolaou-Pinsky-Kurtz...60's and 70's).

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Discrete time quasi-birth-and-death processes

Now we have $\mathcal{C} = \{0,1,2,\dots\} \times \{1,2,\dots,N\}$, $\mathcal{T} = \{0,1,2,\dots\}$ and

$$(\mathbf{P}_{ii'})_{jj'} = \Pr(X_{n+1} = i, Y_{n+1} = j | X_n = i', Y_n = j') = 0 \quad \text{for} \quad |i - i'| > 1$$

i.e. a $N \times N$ block tridiagonal transition probability matrix

$$\mathbf{P} = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$(A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geq 0, \ \det(A_n), \det(C_n) \neq 0$$

$$\sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1, \ i = 1,\dots,N$$

Similar for continuous time quasi-birth-and-death processes but now we have $\mathcal{C} = \{0,1,2,\dots\} \times \{1,2,\dots,N\}$, $\mathcal{T} = [0,+\infty)$ and the transition probability matrix $\mathcal{A}$ satisfies

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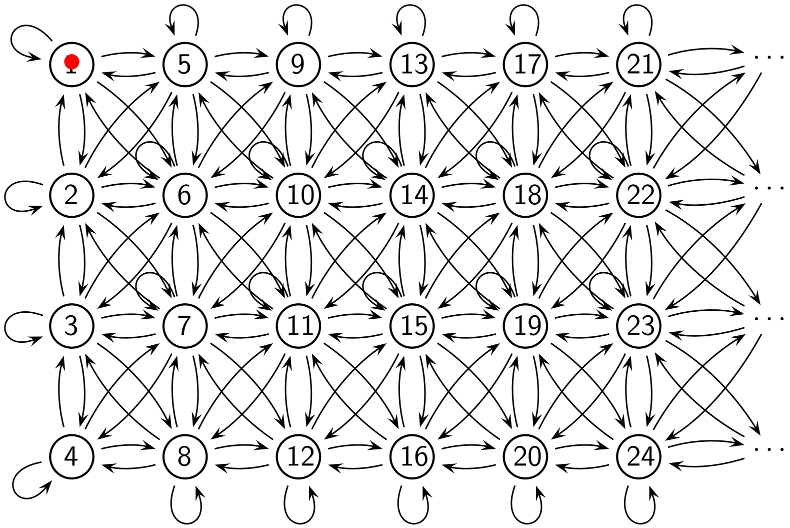

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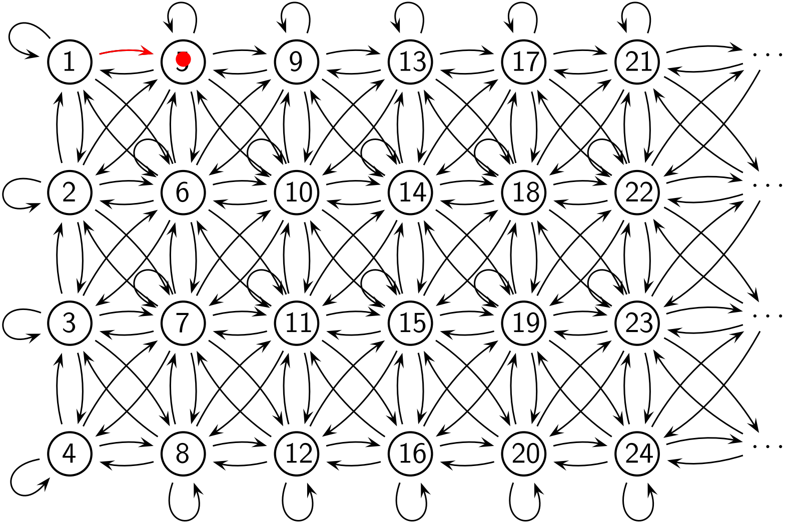

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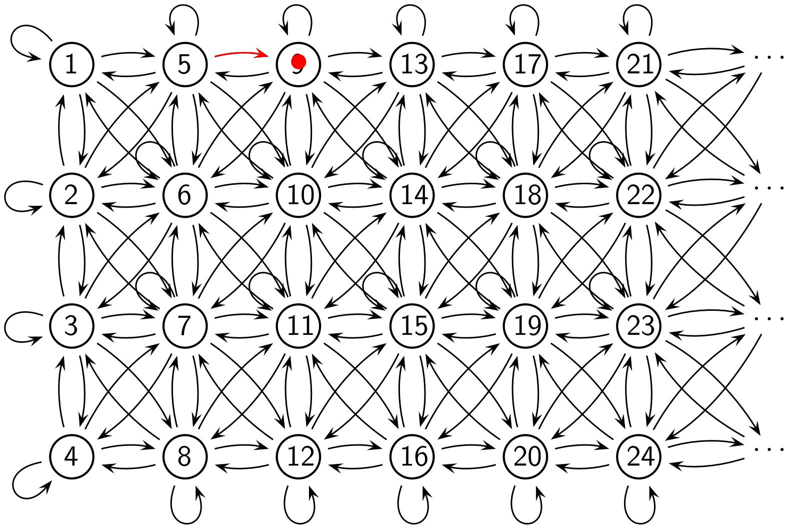

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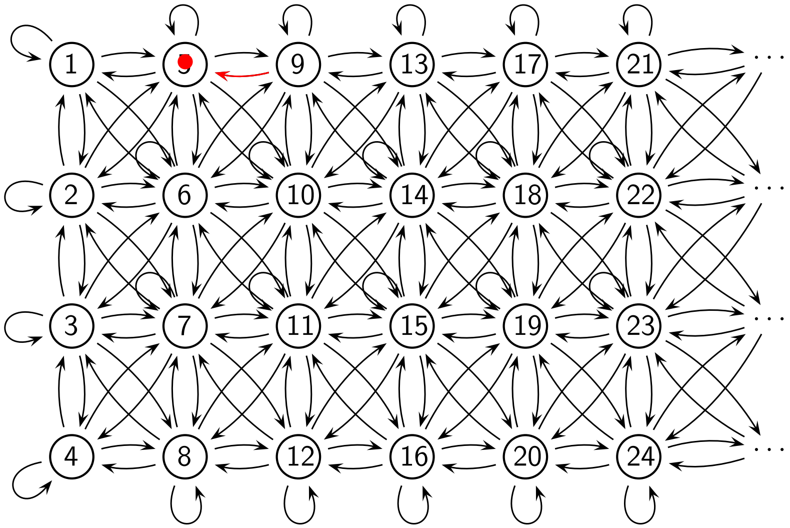

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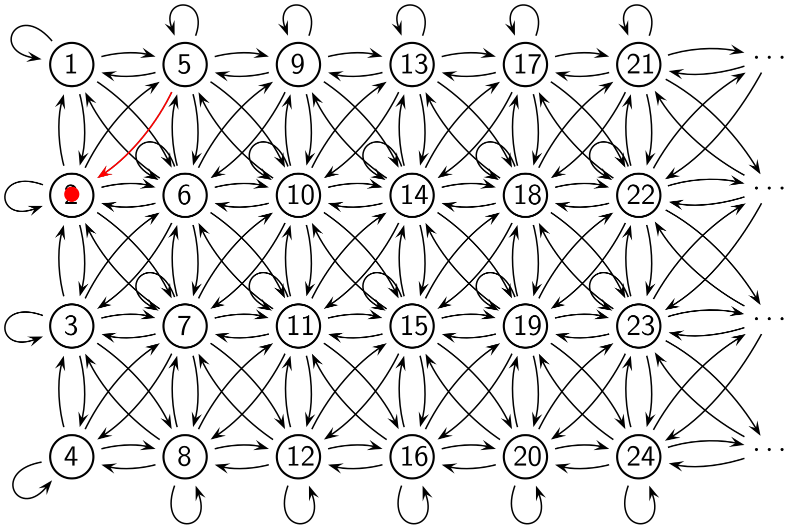

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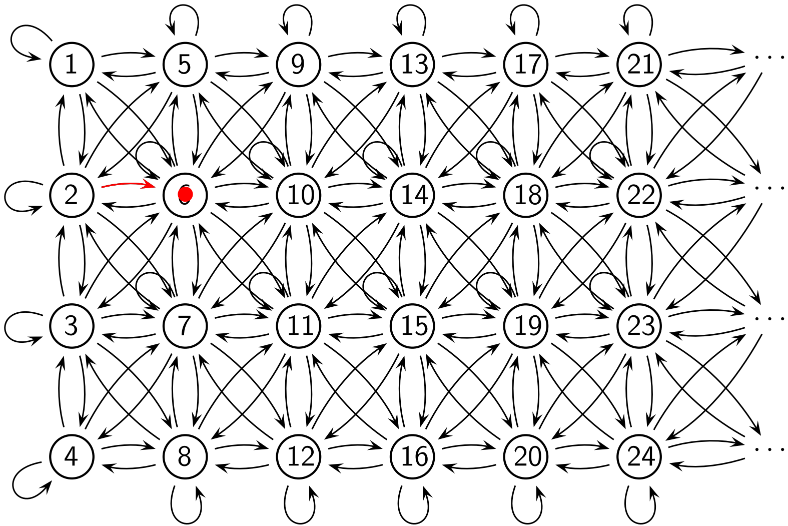

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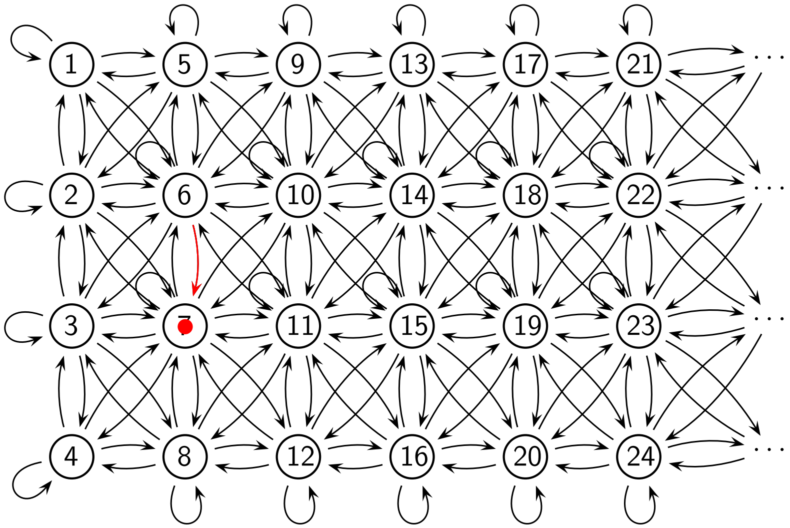

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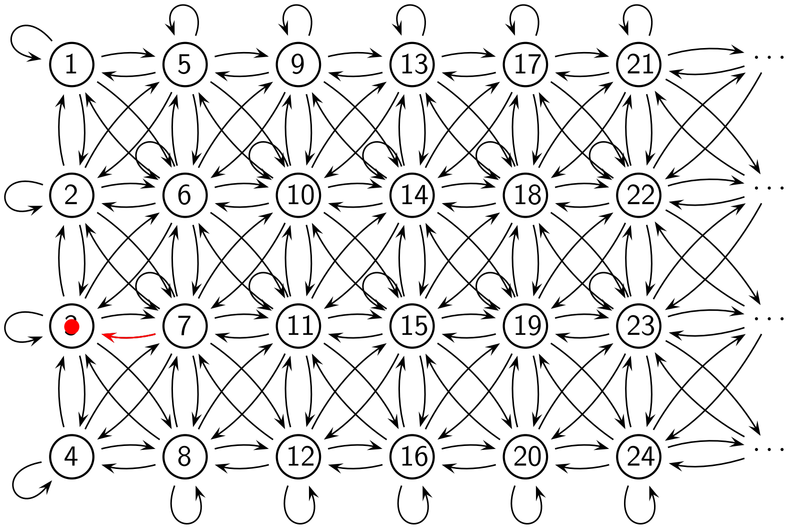

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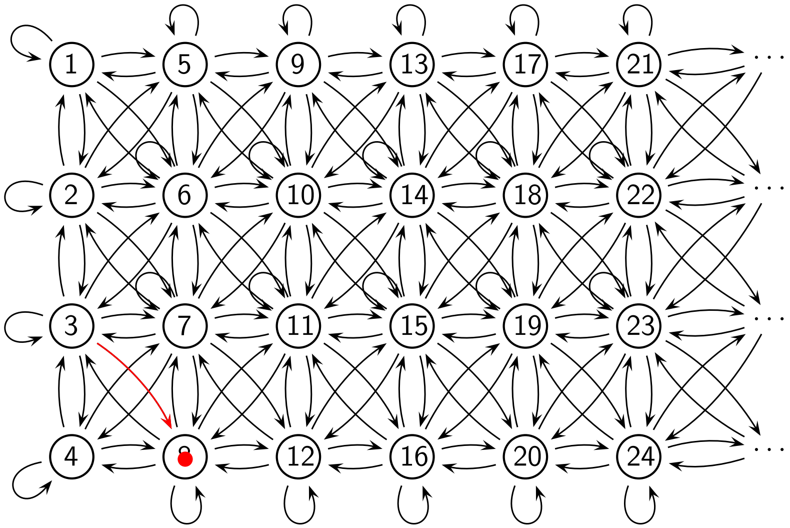

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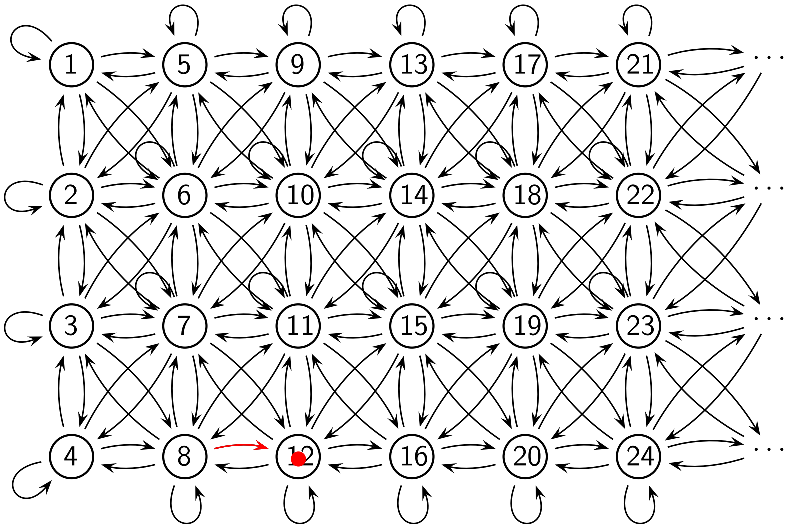

Switching diffusion processes

We have $\mathcal{C} = (a, b) \times \{1, 2, \ldots, N\}$, $\mathcal{T} = [0, \infty)$. The transition probability density is now a matrix which entry (i, j) gives

$$\mathbf{P}_{ij}(t; x, A) = \Pr(X_t \in A, Y_t = j | X_0 = x, Y_0 = i)$$

for any $t > 0$, $x \in (a, b)$ and A any Borel set.

The infinitesimal operator $\mathcal{A}$ is now a matrix-valued differential operator (Berman, 1994)

$$\mathcal{A} = \frac{1}{2}\mathbf{A}(x)\frac{d^2}{dx^2} + \mathbf{B}(x)\frac{d^1}{dx^1} + \mathbf{Q}(x)\frac{d^0}{dx^0}$$

We have that $\mathbf{A}(x)$ and $\mathbf{B}(x)$ are diagonal matrices and $\mathbf{Q}(x)$ is the infinitesimal operator of a continuous time Markov chain, i.e.

$$\mathbf{Q}_{ii}(x) \leq 0, \quad \mathbf{Q}_{ij}(x) \geq 0, i \neq j, \quad \mathbf{Q}(x)\mathbf{e}_N = \mathbf{0}$$

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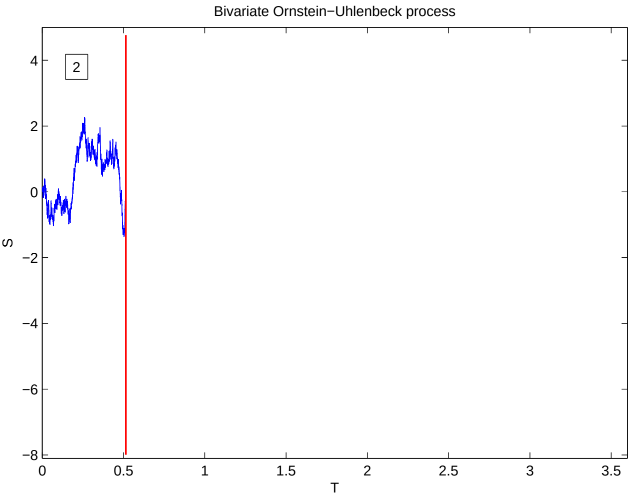
An illustrative example

$N = 3$ phases and $\mathcal{S} = \mathbb{R}$ with

$$\mathbf{A}_{ii}(x) = i^2, \quad \mathbf{B}_{ii}(x) = -ix, \quad i = 1, 2, 3.$$



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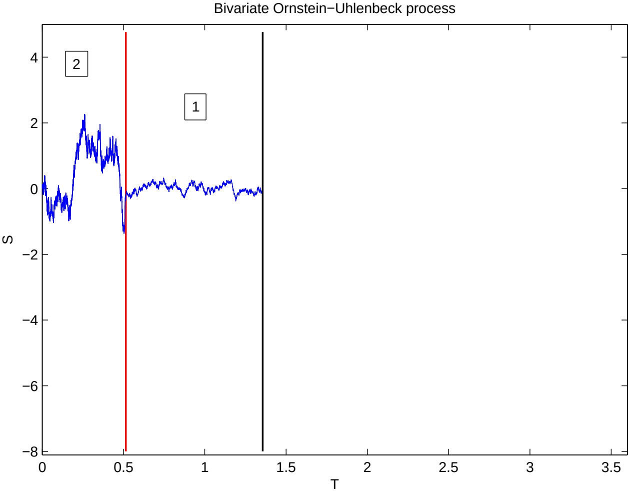

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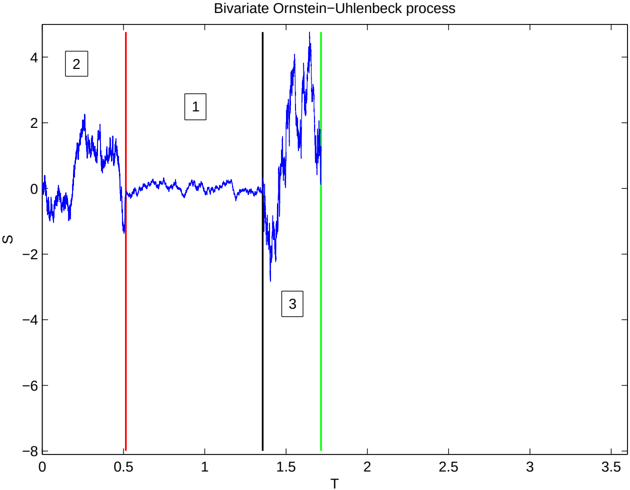

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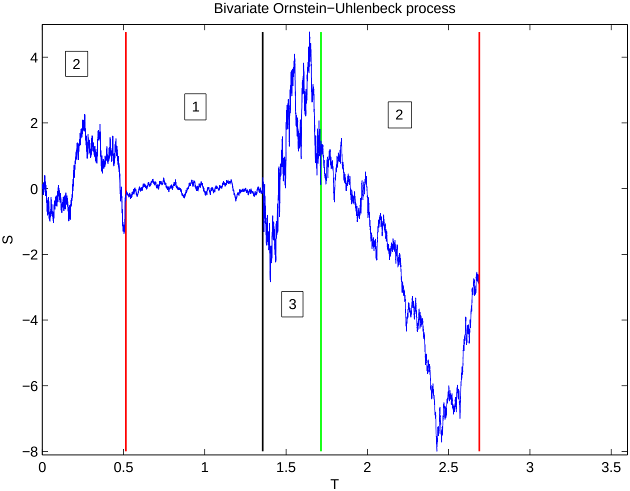

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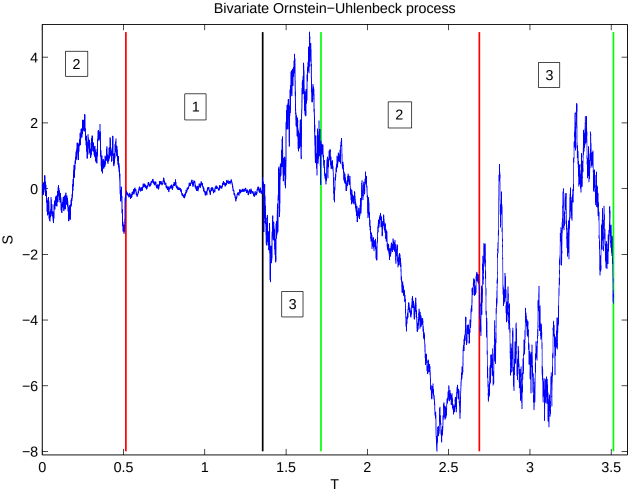

Spectral methods

Now, given a matrix-valued infinitesimal operator $\mathcal{A}$, if we can find a weight matrix $\mathbf{W}(x)$ associated with $\mathcal{A}$, and a set of orthogonal matrix eigenfunctions $\mathbf{F}(i,x)$ such that

$$\mathcal{A}\mathbf{F}(i,x) = \mathbf{\Lambda}(i,x)\mathbf{F}(i,x),$$

then it is possible to find spectral representations of

- Transition probabilities $\mathbf{P}(t;x,y)$.
- Invariant measure or distribution $\pi = (\pi_j)$ (discrete case) with

$$\pi_j = \lim_{t\to\infty} \mathbf{P}_{\cdot j}(t) \in \mathbb{R}^N$$

or $\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y))$ (continuous case) with

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or $\psi(y) = (\psi_1(y), \psi_2(y), \dots, \psi_N(y))$ (continuous case) with

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Spectral methods

Now, given a matrix-valued infinitesimal operator $\mathcal{A}$, if we can find a weight matrix $\mathbf{W}(x)$ associated with $\mathcal{A}$, and a set of orthogonal matrix eigenfunctions $\mathbf{F}(i,x)$ such that

$$\mathcal{A}\mathbf{F}(i,x) = \mathbf{\Lambda}(i,x)\mathbf{F}(i,x),$$

then it is possible to find spectral representations of

- Transition probabilities $\mathbf{P}(t;x,y)$.
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Discrete time quasi-birth-and-death processes

$\mathcal{C} = \{0,1,2,\ldots\} \times \{1,2,\ldots,N\}$, $\mathcal{T} = \{0,1,2,\ldots\}$
(Grünbaum y Dette-Reuther-Studden-Zygmunt, 2007)
Spectral theorem: there exists a weight matrix **W** associate with **P** which matrix-valued orthogonal polynomials $(\mathbf{\Phi}_n)_n$ satisfy

$$\mathbf{P}\mathbf{\Phi} = \begin{pmatrix} B_0 & A_0 & & \\ C_1 & B_1 & A_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{\Phi}_0(x) \\ \mathbf{\Phi}_1(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} \mathbf{\Phi}_0(x) \\ \mathbf{\Phi}_1(x) \\ \vdots \end{pmatrix}, \quad x \in [-1,1]$$

Transition probabilities

$$\mathbf{P}_{ij}^n = \left(\int_{-1}^{1} x^n \mathbf{\Phi}_i(x) d\mathbf{W}(x) \mathbf{\Phi}_j^*(x) \right) \left(\int_{-1}^{1} \mathbf{\Phi}_j(x) d\mathbf{W}(x) \mathbf{\Phi}_j^*(x) \right)^{-1}$$

Invariant measure (MdI, 2011)

$\boldsymbol{\pi} = (\boldsymbol{\pi}_0; \boldsymbol{\pi}_1; \cdots) \equiv (\Pi_0 \mathbf{e}_N; \Pi_1 \mathbf{e}_N; \cdots)$ tal que $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ where $\mathbf{e}_N = (1,\ldots,1)^T$ and

$$\Pi_n = (C_1^T \cdots C_n^T)^{-1} \Pi_0 (A_0 \cdots A_{n-1}) = \left(\int_{-1}^{1} \mathbf{\Phi}_n(x) d\mathbf{W}(x) \mathbf{\Phi}_n^*(x) \right)^{-1}$$

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Switching diffusion models

$\mathcal{C} = (a, b) \times \{1, 2, \dots, N\}$, $\mathcal{T} = [0, \infty)$

If there exists a weight matrix $\mathbf{W}$ symmetric w.r.t. $\mathcal{A}$ which matrix-valued orthogonal functions $(\mathbf{\Phi}_n)_n$ satisfies

$$\mathcal{A}\mathbf{\Phi}_n(x) = \frac{1}{2}\mathbf{A}(x)\mathbf{\Phi}_n''(x) + \mathbf{B}(x)\mathbf{\Phi}_n'(x) + \mathbf{Q}(x)\mathbf{\Phi}_n(x) = \mathbf{\Phi}_n(x)\mathbf{\Gamma}_n$$

Transition probability density matrix (MdI, 2012)

$$\mathbf{P}(t; x, y) = \sum_{n=0}^{\infty} \mathbf{\Phi}_n(x) e^{\mathbf{\Gamma}_n t} \mathbf{\Phi}_n^*(y) \mathbf{W}(y)$$

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$\psi(y) = (\psi_1(y), \psi_2(y), \dots, \psi_N(y))$ such that $\mathcal{A}^*\psi(y) = \mathbf{0}$

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Outline

An example coming from group representation

Let $N \in \{1,2,\ldots\}$, $\alpha,\beta > -1$, $0 < k < \beta+1$ and $\mathbf{E}_{ij}$ will denote the matrix with 1 at entry (i,j) and 0 otherwise.
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$$\mathbf{W}(x) = x^{\alpha}(1-x)^{\beta}\sum_{i=1}^{N}\binom{\beta-k+i-1}{i-1}\binom{N+k-i-1}{N-i}x^{N-i}\mathbf{E}_{ii}$$

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$$\mathbf{Q}(x) = \sum_{i=2}^{N}\mu_i(x)\mathbf{E}_{i,i-1} - \sum_{i=1}^{N}(\lambda_i(x)+\mu_i(x))\mathbf{E}_{ii} + \sum_{i=1}^{N-1}\lambda_i(x)\mathbf{E}_{i,i+1},$$

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- The orthogonal eigenfunctions $\Phi_i(x)$ of $\mathcal{A}$ are called matrix-valued spherical functions associated with the complex projective space. There are many structural formulas available studied in the last years (Grünbaum-Pacharoni-Tirao-Román-MdI).
- Bispectrality: $\Phi_i(x)$ satisfy a three-term recurrence relation

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A quasi-birth-and-death process $(N = 2)$

Conjugation

$$\widetilde{\mathbf{W}}(x) = \mathbf{T}^*(x)\mathbf{W}(x)\mathbf{T}(x)$$

where

$$\mathbf{T}(x) = \begin{pmatrix} 1-x & 1-x \\ 1-x & -x - \dfrac{\alpha+1}{\beta-k+1} \end{pmatrix}$$

Grünbaum-MdI (2008) and consider a family of matrix-valued orthogonal polynomials $(\mathbf{Q}_n(x))_n$ with respect to $\widetilde{\mathbf{W}}(x)$.
This transformation allows us to have a second-order differential operator of *Sturm-Liouville* type

We choose the family of OMP $(\mathbf{Q}_n(x))_n$ such that

- Three term recurrence relation

$$x\mathbf{Q}_n(x) = A_n\mathbf{Q}_{n+1}(x) + B_n\mathbf{Q}_n(x) + C_n\mathbf{Q}_{n-1}(x), \quad n = 0, 1, \ldots$$

where the Jacobi matrix is stochastic

- Choosing $\mathbf{Q}_0(x) = \mathbf{I}$ the leading coefficient of $\mathbf{Q}_n$ is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)} \begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|\mathbf{Q}_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)} \times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

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- The choice of the leading coefficient is motivated by the fact that

$$\mathbf{Q}_n(1)\mathbf{e}_N = \mathbf{e}_N$$

where $\mathbf{e}_N = (1, 1, \cdots, 1)^T$.

- Consequently, the Jacobi matrix is stochastic:

$$\begin{array}{ccc} 1 \cdot \mathbf{Q}_n(1)\mathbf{e}_N & = & A_n\mathbf{Q}_{n+1}(1)\mathbf{e}_N + B_n\mathbf{Q}_n(1)\mathbf{e}_N + C_n\mathbf{Q}_{n-1}(1)\mathbf{e}_N \\ \parallel & & \parallel \\ \mathbf{e}_N & = & (A_n + B_n + C_n)\mathbf{e}_N \end{array}$$

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PARTICULAR CASE $\alpha = \beta = 0, \ k = 1/2$

$$A_n = \begin{pmatrix} \dfrac{(2n+1)(n+2)^2}{2(2n+3)^2(n+1)} & 0 \\ \dfrac{2(n+2)}{(2n+5)(2n+3)^2} & \dfrac{n+3}{2(2n+5)} \end{pmatrix}$$

$$B_n = \begin{pmatrix} \dfrac{1}{2} - \dfrac{4n^2+8n-1}{2(2n+1)^2(2n+3)^2} & \dfrac{n+2}{(2n+3)^2(n+1)} \\ \dfrac{2(n+1)}{(2n+1)(2n+3)^2} & \dfrac{1}{2} - \dfrac{1}{(2n+3)^2} \end{pmatrix}$$

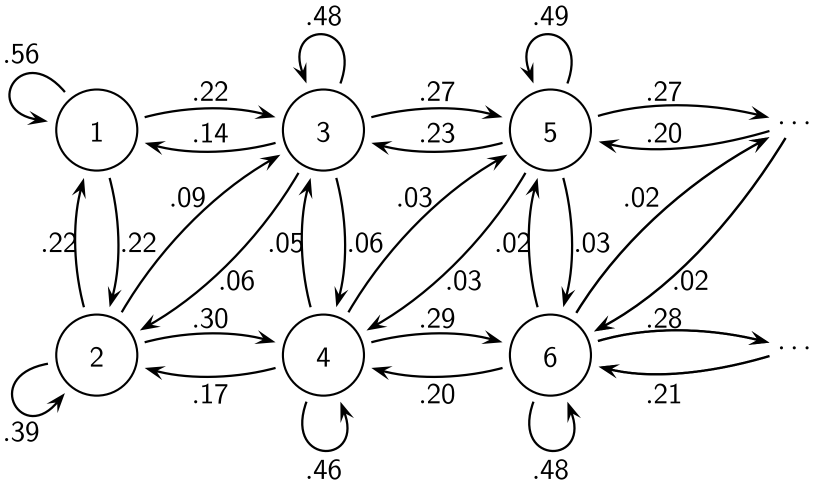
$$C_n = \begin{pmatrix} \dfrac{n^2(2n+3)}{2(2n+1)^2(n+1)} & \dfrac{n}{(n+1)(2n+1)^2} \\ 0 & \dfrac{n}{2(2n+1)} \end{pmatrix}$$

Particular case $\alpha = \beta = 0,\ k = 1/2$

Pentadiagonal Jacobi matrix:

$$P = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{100} & & & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & & & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} & & \\ & & & & \frac{81}{392} & \frac{3}{196} & \frac{1955}{3969} & \frac{5}{324} & \frac{175}{648} & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Associated network



The invariant measure

Invariant measure

The row vector

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots)$$

$$\boldsymbol{\pi}^n = \left(\frac{1}{(\|\mathbf{Q}_n\|^2_{\widetilde{\mathbf{W}}})_{1,1}}, \frac{1}{(\|\mathbf{Q}_n\|^2_{\widetilde{\mathbf{W}}})_{2,2}}, \cdots, \frac{1}{(\|\mathbf{Q}_n\|^2_{\widetilde{\mathbf{W}}})_{N,N}} \right), \quad n \geq 0$$

is an invariant measure of P

Particular case $N = 2$, $\alpha = \beta = 0$, $k = 1/2$:

$$\pi^n = \left(\frac{2(n+1)^3}{(2n+3)(2n+1)}, \frac{(n+1)(n+2)}{2n+3} \right), \quad n \geq 0$$

$$\pi = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \cdots \right)$$

The invariant measure

Invariant measure

The row vector

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots)$$

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The Wright-Fisher diffusion model involving only mutation effects considers a big population of constant size M composed of two types A and B.

$$A \xrightarrow{\frac{1+\beta}{2}} B, \quad B \xrightarrow{\frac{1+\alpha}{2}} A, \quad \alpha, \beta > -1$$

As $M \to \infty$, this model can be described by a diffusion process whose state space is $S = [0,1]$ with drift and diffusion coefficient

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The N phases of our bivariate Markov process are variations of the Wright-Fisher model in the drift coefficients:

$$\mathbf{B}_{ii}(x) = \alpha + 1 + N - i - x(\alpha + \beta + 2 + N - i), \quad \mathbf{A}_{ii}(x) = 2x(1-x)$$

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Waiting times and tendency

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We have to take a look to the *diagonal entries* of $\mathbf{Q}(x)$:

$$\mathbf{Q}_{ii}(x) = -\frac{1}{1-x}\left[(N-i)(i+\beta-k) + x(i-1)(N-i+k)\right]$$

- If $x \to 1^- \Rightarrow$ all phases are *instantaneous*.
- If $x \to 0^+$ or $k \to 0^+ \Rightarrow$ phase N is *absorbing*.
- If $k \to \beta + 1 \Rightarrow$ phase 1 is *absorbing*.

Tendency

- If $k \to \beta + 1 \Rightarrow$ *Backward tendency*

  Meaning: The parameter k helps the population of A's to survive against the population of B's.

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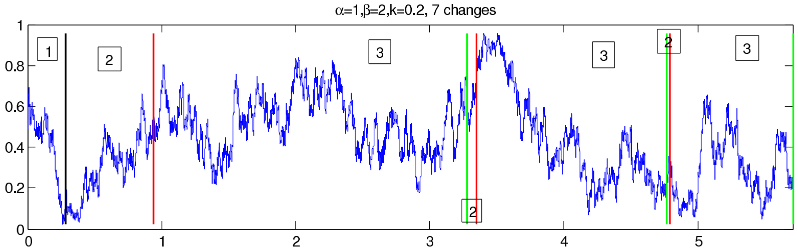
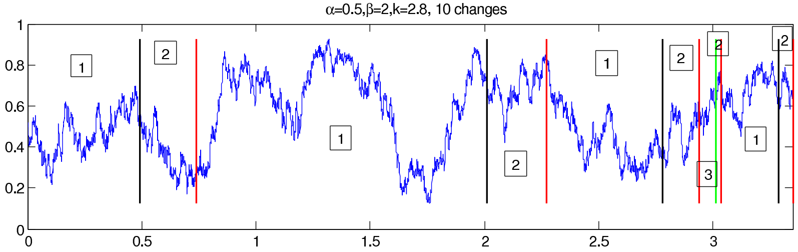
Example of tendency





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The invariant distribution $\psi(y)$ $(\alpha, \beta \geq 0)$ comes from the study of

$$\lim_{t\to\infty} \mathbf{P}(t;x,y) = \sum_{n=0}^{\infty} \mathbf{\Phi}_n(x) e^{\mathbf{\Gamma}_n t} \mathbf{\Phi}_n^*(y) \mathbf{W}(y)$$

This should be independent of the initial state and phase.
Therefore we should expect a row vector invariant distribution

$$\boldsymbol{\psi}(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y))$$

with $0 \leq \psi_j(y) \leq 1$ and

$$\sum_{j=1}^{N} \int_0^1 \psi_j(y) dy = 1$$

Explicit formula (MdI, 2012)

$$\Rightarrow \psi(y) = \left(\int_0^1 \mathbf{e}_N^T \mathbf{W}(x) \mathbf{e}_N dx \right)^{-1} \mathbf{e}_N^T \mathbf{W}(y)$$

where $\mathbf{e}^T = (1, 1, \ldots, 1)$. In particular

$$\psi_j(y) = y^{\alpha+N-j}(1-y)^{\beta} \binom{N-1}{j-1} \binom{\alpha+\beta+N}{\alpha} \frac{(\beta+N)(k)_{N-j}(\beta-k+1)_{j-1}}{(\alpha+\beta-k+2)_{N-1}}$$

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Study of the invariant distribution