Some examples of orthogonal matrix polynomials satisfying odd order differential equations [a]

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Introduction and outline of the talk

- The theory of orthogonal matrix polynomials (OMP) was introduced by M. G. Krein in 1949.
- Systematically studied in the last 15 years.
- In 1997, Durán raised the problem of characterizing OMP satisfying *second order differential equations.*

  Scalar case: Bochner (1929): Hermite-Laguerre-Jacobi

  ⇒**New (non-trivial) matrix examples** (2003): Durán-Grünbaum-Pacharoni-Tirao.
- New phenomena:
  - Several linearly independent second–order differential operators
  - OMP satisfying odd order differential operators.
  - Richer behavior of the algebra of differential operators.

Background I

- Given a self adjoint positive definite matrix weight function W we define a skew symmetric bilinear form (*inner product*):

$$(P,Q)=\int_a^b P(t)\,W(t)\,Q^*(t)\,dt.$$

- We say that two weight matrices $W_1(t)$ and $W_2(t)$ are *similar* if there exists a nonsingular matrix T with $W_1=TW_2T^*$. If W_2 is diagonal, we say that W_1 *reduces to scalar weights*.
- This leads to a family of OMP $\{P_n\}_{n\geq 0}$ with deg $P_n=n$, non–singular leading coefficient and $(P_n,P_m)=\Theta, n\neq m$.
- Examples considered here satisfy

$$\ell_2(P_n)\equiv P_n''(t)A_2(t)+P_n'(t)A_1(t)+P_n(t)A_0=\Lambda_nP_n(t).$$

ℓ_2 is *symmetric* if $(\ell_2(P),Q)=(P,\ell_2(Q))$.

Background II

Generating examples:

- Solving *symmetry equations.* [Durán–Grünbaum]:

$$A_2 W = W A_2^*$$

$$2(A_2 W)' = A_1 W + W A_1^*,$$

$$(A_2 W)'' - (A_1 W)' + A_0 W = W A_0^*.$$

  with boundary conditions:

$$\lim_{t \to x} A_2(t) W(t) = 0 = \lim_{t \to x} (A_1(t) W(t) - W(t) A_1^*(t)), \quad \text{for} \quad x = a, b.$$

- *Matrix spherical functions* associated with $P_N(\mathbb{C}) = \mathrm{SU}(N+1)/\mathrm{U}(N)$. [Grünbaum–Pacharoni–Tirao].

The family of examples

- $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^{\alpha} e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^*t}, \quad \alpha > -1, \quad t \in (0, +\infty)$

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \nu_i \in \mathbb{C}, \; J = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

- $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}(t) = t^{\alpha} e^{-t} T(t) T^*(t)$, where

$$T'(t) = \frac{1}{2}\left(A + \frac{J}{t}\right) T(t), \quad \mathrm{ad}_A J = [A, J] = -A$$

Second order differential operators for $W_{\alpha,\nu_1,\cdots,\nu_{N-1}}$

- $\ell_{2,1} = D^2 tI + D^1[(\alpha+1)I + J + t(A-I)] + D^0[(J+\alpha I)A - J]$
- Assuming

$$i(N-i)|\nu_{N-1}|^2 = (N-1)|\nu_i|^2 + (N-i-1)|\nu_i|^2|\nu_{N-1}|^2, \quad i = 1,\cdots,N-2.$$

$\Rightarrow \ell_{2,2} = D^2 A_2 + D^1 A_1 + D^0 A_0$, where

$$A_2 = t(J - At),$$
$$A_1 = (1+\alpha+J)J + Y - t(J + (\alpha+2)A + Y^* - \mathrm{ad}_A Y),$$
$$A_0 = \frac{N-1}{|\nu_{N-1}|^2}[J - (\alpha+J)A].$$

where $(Y)_{i,i-1} = \frac{i(N-i)}{\nu_i}$, $i = 2,\cdots,N$, and $(Y)_{i,j} = 0$ otherwise.

- They commute and satisfy the following relation:

$$\prod_{i=1}^{N}\left((i-1)\ell_{2,1} - \ell_{2,2} + \left[\frac{(N-1)(N-i)}{|\nu_{N-1}|^2} + (i-1)(N-i)\right] I\right) = 0.$$

The algebra of differential operators associated with $W_{\alpha,a}$

- Given a fixed family of OMP $(\mathcal{P}_n)_n$, we study the algebra over $\mathbb{C}$

$$\mathcal{D}(W_{\alpha,a}) = \left\{ \ell = \sum_{i=0}^{k} D^i A_i(t) : \ell(\mathcal{P}_n(t)) = \Gamma_n(\ell)\mathcal{P}_n(t),\ n \geq 0 \right\},$$

where now we study the case when $N = 2$ and

$$W_{\alpha,a}(t) = t^\alpha e^{-t} \begin{pmatrix} t(1+|a|^2 t) & at \\ \bar{a}t & 1 \end{pmatrix}, \quad \alpha > -1, t > 0,$$

- The map

$$\Gamma_n : \mathcal{D}(W_{\alpha,a}) \longrightarrow \mathbb{C}^{2\times 2}, \quad n = 0, 1, 2, \ldots$$

is a *faithful representation*, i. e., $\Gamma_n(\ell_1\ell_2) = \Gamma_n(\ell_1)\Gamma_n(\ell_2)$ with $\ell_1, \ell_2 \in \mathcal{D}(W_{\alpha,a})$ and if $\Gamma_n(\ell) = 0$ for all n, then $\ell = 0$.

Scalar case

If $\mathcal{L}$ is the corresponding second order differential operator of the classical orthogonal polynomials (Hermite, Laguerre or Jacobi):

$$H_n(t)'' - 2tH_n(t)' = -2nH_n(t)$$

$$tL_n^\alpha(t)'' + (\alpha + 1 - t)L_n^\alpha(t)' = -nL_n^\alpha(t)$$

$$t(1-t)P_n^{(\alpha,\beta)}(t)'' + (\alpha + 1 - (\alpha+\beta+2)t)P_n^{(\alpha,\beta)}(t)' =$$
$$-n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(t)$$

then

$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{L}^i,$$

where $c_i \in \mathbb{C}$ and $\mathcal{U}$ is an even order differential operator.

Then $\mathcal{D} =< \mathcal{L} >$

The algebra

- Number of **new** linearly independent differential operators:

order	0	1	2	3	4	5	6	7	8
dimension	1	0	2	2	2	2	2	2	2

- Basis for the second order differential operators:

$$\mathcal{L}_1 = D^2 tI + D^1 \begin{pmatrix} \alpha+2-t & at \\ 0 & \alpha+1-t \end{pmatrix} + D^0 \begin{pmatrix} -\frac{1+|a|^2}{|a|^2} & (1+\alpha)a \\ 0 & -\frac{1}{|a|^2} \end{pmatrix}$$

$$\mathcal{L}_2 = D^2 \begin{pmatrix} t & -2at^2 \\ 0 & -t \end{pmatrix} + D^1 \begin{pmatrix} \alpha+2+t & -\frac{(2+|a|^2(2\alpha+5))t}{\bar{a}} \\ \frac{2}{a} & -t-\alpha-1 \end{pmatrix} + D^0 \begin{pmatrix} \frac{1+|a|^2}{|a|^2} & -\frac{(1+\alpha)(2+|a|^2)}{\bar{a}} \\ 0 & -\frac{1}{|a|^2} \end{pmatrix}$$

Third order differential operators

- At order 3 a possible basis for the 2 new symmetric *non-commutative differential operators* is

$$\mathcal{L}_3 = D^3 A_{3,1}(t) + D^2 A_{2,1}(t) + D^1 A_{1,1}(t) + A_{0,1}$$

and

$$\mathcal{L}_4 = D^3 A_{3,2}(t) + D^2 A_{2,2}(t) + D^1 A_{1,2}(t) + A_{0,2},$$

where the leading differential coefficients are give by:

$$A_{3,1}(t) = t\left(At - \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{ad}_A^n A^*\right) = \begin{pmatrix} -|a|^2t^2 & at^2(1+|a|^2t) \\ -\bar{a}t & |a|^2t^2 \end{pmatrix}$$

$$A_{3,2}(t) = t\left(At + \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{ad}_A^n A^*\right) = \begin{pmatrix} |a|^2t^2 & at^2(-1+|a|^2t) \\ \bar{a}t & -|a|^2t^2 \end{pmatrix}$$

A sample of relations I

- Four quadratic relations:

$$\mathcal{L}_1^2 = \mathcal{L}_2^2, \qquad \mathcal{L}_3^2 = -\mathcal{L}_4^2,$$
$$\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1, \quad \mathcal{L}_3\mathcal{L}_4 = -\mathcal{L}_4\mathcal{L}_3.$$

- Four permutational relations:

$$\mathcal{L}_1\mathcal{L}_3 - \mathcal{L}_2\mathcal{L}_4 = 0, \quad \mathcal{L}_2\mathcal{L}_3 - \mathcal{L}_1\mathcal{L}_4 = 0,$$
$$\mathcal{L}_3\mathcal{L}_2 + \mathcal{L}_4\mathcal{L}_1 = 0, \quad \mathcal{L}_3\mathcal{L}_1 + \mathcal{L}_4\mathcal{L}_2 = 0.$$

- Four more quadratic relations:

$$\mathcal{L}_3 = \mathcal{L}_1\mathcal{L}_4 - \mathcal{L}_4\mathcal{L}_1, \quad \mathcal{L}_4 = \mathcal{L}_1\mathcal{L}_3 - \mathcal{L}_3\mathcal{L}_1$$
$$\mathcal{L}_3 = \mathcal{L}_2\mathcal{L}_3 + \mathcal{L}_3\mathcal{L}_2, \quad \mathcal{L}_4 = \mathcal{L}_2\mathcal{L}_4 + \mathcal{L}_4\mathcal{L}_2.$$

- Finally, we can present two interesting cubic relations:

$$\mathcal{L}_1\mathcal{L}_3^2 = \mathcal{L}_3^2\mathcal{L}_1, \quad \mathcal{L}_2\mathcal{L}_3^2 = \mathcal{L}_3^2\mathcal{L}_2.$$

A sample of relations II

- We can put $\mathcal{L}_2$ in terms of $\mathcal{L}_1$ and $\mathcal{L}_3$

$$\begin{aligned}\left[|a|^2(2+\alpha)-1\right]\left[|a|^2(\alpha-1)-1\right]\mathcal{L}_2 = {} & 2|a|^2\left[|a|^2(2\alpha+1)-2\right]\mathcal{L}_1 \\ & + \left[|a|^4(\alpha^2+\alpha-5)-|a|^2(2\alpha+1)+1\right]\mathcal{L}_1^2 \\ & - 2|a|^2\left[|a|^2(2\alpha+1)-2\right]\mathcal{L}_1^3 + 3|a|^4\mathcal{L}_1^4 \\ & - \frac{1}{2}\left[|a|^2(2\alpha+1)-2\right]\mathcal{L}_3^2 + \frac{15}{2}|a|^2\mathcal{L}_3^2\mathcal{L}_1 - \frac{9}{2}|a|^2\mathcal{L}_3\mathcal{L}_1\mathcal{L}_3.\end{aligned}$$

- **Conjecture**: $\mathcal{D}(W_{\alpha,a}) =< \mathcal{L}_1, \mathcal{L}_3 >$.
- For the exceptional values of $\alpha = 1 + \frac{1}{|a|^2}$ or $\alpha = -2 + \frac{1}{|a|^2}$, the left side of the last formula vanishes.

  $\Rightarrow$ **Conjecture**: $\mathcal{D}(W_{\alpha,a}) =< \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 >$

Closing remarks and future directions

- This is the *first example* of weight matrix which *does not reduce to scalar weights* having symmetric *odd* order differential operators.
- It is remarkable that weight matrices corresponding to $N = 3$ and 4 yield to *fifth* and *seventh* order (respectively) symmetric differential operators, but no third order.
- We hope to find theoretical proofs of these new phenomena and illustrate new ones in future publications.
- The matrix case is **much richer** than the scalar one.
- Possible applications:
  - **Quantum mechanics**: [Durán-Grünbaum] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006).
  - **Time-and-band limiting**: [Durán-Grünbaum]*A survey on orthogonal matrix polynomials satisfying second order differential equations*, J. Comput. Appl. Math. (2005).