Methods and new phenomena of orthogonal matrix polynomials satisfying differential equations\textsuperscript{1}

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13th International Conference on Approximation Theory
San Antonio, March 7-10, 2010

\textsuperscript{1} joint work with F. A. Grünbaum and A. Martínez-Finkelshtein
Outline

1. Preliminaries
2. Methods and new phenomena
3. Applications
Outline

1 Preliminaries

2 Methods and new phenomena

3 Applications
A $N \times N$ matrix polynomial on the real line is

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0, \quad x \in \mathbb{R}, \quad A_i \in \mathbb{C}^{N \times N}$$

Krein (1949): Orthogonal matrix polynomials (OMP)

Let $W$ be a $N \times N$ self adjoint positive definite weight matrix

We can construct a family $(P_n)_n$ of OMP with respect to the inner product

$$(P, Q)_W = \int_a^b P(x) W(x) Q^*(x) \, dx \in \mathbb{C}^{N \times N}$$

such that

$$(P_n, P_m)_W = \int_a^b P_n(x) W(x) P_m^*(x) \, dx = \delta_{n,m} I, \quad n, m \geq 0$$

$$P_n(x) = \kappa_n (x^n + a_{n-1} x^{n-1} + \cdots) = \kappa_n \hat{P}_n(x)$$
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Three-term recurrence relation

Orthonormality of \((P_n)_n\) is equivalent to a three term recurrence relation

\[
xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x), \quad n \geq 0
\]

\[
det(A_{n+1}) \neq 0, \quad B_n = B_n^*
\]

Jacobi operator (block tridiagonal)

\[
\begin{pmatrix}
P_0(x) \\
P_1(x) \\
P_2(x) \\
\vdots
\end{pmatrix}
= 
\begin{pmatrix}
B_0 & A_1 & A_2 & \cdots \\
A_1^* & B_1 & A_2 & \cdots \\
A_2^* & B_2 & A_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
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Or equivalently for the monic family

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x\hat{P}_n(x) = \hat{P}_{n+1}(x) + \alpha_n\hat{P}_n(x) + \beta_n\hat{P}_{n-1}(x), \quad n \geq 0
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Second-order differential equations

Durán (1997): characterize orthonormal \((P_n)_n\) satisfying second-order differential equations of Sturm-Liouville (hypergeometric) type

\[ P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n P_n(x), \quad n \geq 0 \]
\[ \text{grad } F_i \leq i, \quad \Lambda_n \quad \text{Hermitian} \]

Equivalent to the symmetry (i.e. \((PD, Q)_W = (P, QD)_W\)) of

\[ D = \partial^2 F_2(x) + \partial^1 F_1(x) + \partial^1 F_0, \quad \partial = \frac{d}{dx} \]

Scalar case: Bochner (1929): Hermite, Laguerre and Jacobi

Typically the weight matrices are of the form \(W = \omega TT^*\)
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2 Methods and new phenomena

3 Applications
Methods and new phenomena

Methods

- Matrix spherical functions associated with $P_n(\mathbb{C}) = SU(n+1)/U(n)$
- Durán-Grünbaum (2004): Symmetry equations

New phenomena

- For a fixed family of OMP there exist several linearly independent second-order differential operators having them as eigenfunctions
- OMP satisfying odd-order differential equations
- For a fixed second-order differential operator, there can be more than one family of lin. ind. OMP having them as eigenfunctions
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The Riemann-Hilbert problem

The Riemann-Hilbert problem (RHP) for orthogonal polynomials was introduced by Fokas-Its-Kitaev (1990)

For a given \( \omega \) with \( x' \omega, x' \omega' \in L^1(\mathbb{R}) \) we try to find \( Y^n : \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2} \) s.t.

1. \( Y^n \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \)
2. \( Y^n_+(x) = Y^n_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix} \) when \( x \in \mathbb{R} \)
3. \( Y^n(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \) as \( z \rightarrow \infty \)

Advantages

1. Algebraic properties: three term recurrence relation, ladder operators, second order differential equation
2. Uniform asymptotics: steepest descent analysis for RHP (Deift-Zhou, 1993). Very useful for functions which do not have an integral representation form
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The RHP for OMP

The unique solution of the RHP for OMP is given by

\[ Y^n(z) = \begin{pmatrix} \hat{P}_n(z) & C(\hat{P}_n W)(z) \\ -2\pi i \gamma_{n-1} \hat{P}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\hat{P}_{n-1} W)(z) \end{pmatrix}, \quad n \geq 1 \]

where \( \gamma_n = \kappa_n^* \kappa_n \) and \( C(F)(z) = \frac{1}{2\pi i} \int_a^b \frac{F(t)}{t-z} \, dt \)

\( Y^n(z) \) satisfies the following pair of first-order difference-differential relations (also known as Lax pair)

\[ Y^{n+1}(z) = E_n(z) Y^n(z), \quad \frac{d}{dz} Y^n(z) = F_n(z) Y^n(z) \]

Cross-differentiation gives compatibility conditions (or string equations)

\[ E'_n(z) + E_n(z) F_n(z) = F_{n+1}(z) E_n(z) \]

Problem: get explicit expression of \( F_n(z) \)
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Transformation of the RHP

Let \( W(x) = T(x)T^*(x), \ x \in \mathbb{R} \) and consider

\[
X^n(z) = Y^n(z) \begin{pmatrix} T(z) & 0 \\ 0 & T^{-*}(\bar{z}) \end{pmatrix}
\]

Therefore we have a class of Lax pairs

\[
X^{n+1}(z) = E^S_n(z)X^n(z), \quad \frac{d}{dz}X^n(z) = F^S_n(z)X^n(z)
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Example

Let us consider \((S = I)\)

\[W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R}\]

for any \(A \in \mathbb{C}^{N \times N}\) (Durán-Grüenbaum, 2004)

\[X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2} e^{Az} & 0 \\ 0 & e^{z^2/2} e^{-A^* z} \end{pmatrix}\]

Lax pair

\[X^{n+1}(z) = \begin{pmatrix} zI - \alpha_n & \frac{1}{2\pi i} \gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} X^n(z), \quad \frac{d}{dz} X^n(z) = \begin{pmatrix} -zI + A & -\frac{1}{\pi i} \gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & zI - A^* \end{pmatrix} X^n(z)\]

Compatibility conditions

\[\alpha_n = (A + \gamma_n^{-1} A^* \gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I\]
Example

Let us consider \((S = I)\)

\[
W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R}
\]

for any \(A \in \mathbb{C}^{N \times N}\) (Durán-Grünbaum, 2004)

\[
X^n(z) = Y^n(z) \begin{pmatrix}
e^{-z^2/2} e^{Az} & 0 \\
0 & e^{z^2/2} e^{-A^* z}
\end{pmatrix}
\]

Lax pair

\[
X^{n+1}(z) = \begin{pmatrix} zI - \alpha_n & \frac{1}{2\pi i} \gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} X^n(z), \quad \frac{d}{dz} X^n(z) = \begin{pmatrix} -zI + A & -\frac{1}{\pi i} \gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & zI - A^* \end{pmatrix} X^n(z)
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Compatibility conditions

\[
\alpha_n = (A + \gamma_n^{-1}A^*\gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I
\]
From block entries $(1, 1)$ and $(2, 1)$ of
\[
\frac{d}{dz} X^n(z) = \begin{pmatrix} -zl + A & -\frac{1}{\pi i} \gamma^{-1} \\ 4\pi i \gamma_{n-1} & zl - A^* \end{pmatrix} X^n(z)
\] we get ladder operators

Ladder operators

\[
\hat{P}'_n(z) + \hat{P}_n(z)A - A\hat{P}_n(z) = 2\beta_n\hat{P}_{n-1}(z)
\]

\[
-\hat{P}'_n(z) + 2(z - \alpha_n)\hat{P}_n(z) + A\hat{P}_n(z) - \hat{P}_n(z)A = 2\hat{P}_{n+1}(z)
\]

Combining them we get a second order differential equation

Second order differential equation

\[
\hat{P}''_n(z) + 2\hat{P}'_n(z)(A - zI) + \hat{P}_n(z)A^2 - A^2\hat{P}_n(z) + 4\beta_n\hat{P}_n(z) =
\]

\[
-2z(\hat{P}_n(z)A - A\hat{P}_n(z)) + 2(\alpha_n - A)(\hat{P}'_n(z) + \hat{P}_n(z)A - A\hat{P}_n(z))
\]
From block entries (1, 1) and (2, 1) of
\[ \frac{d}{dz} X^n(z) = \begin{pmatrix} -zI + A & -1/\pi i \gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & zI - A^* \end{pmatrix} X^n(z) \] we get ladder operators

Ladder operators

\[ \hat{P}'_n(z) + \hat{P}_n(z)A - A\hat{P}_n(z) = 2\beta_n\hat{P}_{n-1}(z) \]
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Second order differential equation

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Let us now consider the special case of

\[ W(x) = e^{-x^2} e^{Ax} e^{iJx} e^{-iJx} e^{A^*x}, \quad x \in \mathbb{R} \]

where \( A = \sum_{i=1}^{N} \nu_i E_{i,i+1}, \nu_i \in \mathbb{C} \setminus \{0\}, \) and \( J = \sum_{i=1}^{N}(N-i)E_{i,i} \)

**New compatibility conditions**

\[
\begin{align*}
J\alpha_n - \alpha_n J + \alpha_n &= A + \frac{1}{2}(A^2\alpha_n - \alpha_n A^2) \\
J - \gamma_n^{-1} J\gamma_n &= A\alpha_n + \alpha_n A - 2\alpha_n^2
\end{align*}
\]

**New ladder operators (0-th order)**

\[
\begin{align*}
\hat{P}_nJ - J\hat{P}_n - x(\hat{P}_nA - A\hat{P}_n) + 2\beta_n\hat{P}_n - n\hat{P}_n &= 2(A - \alpha_n)\beta_n\hat{P}_{n-1} \\
\hat{P}_n(J - xA) - \gamma_n^{-1}(J - xA^*)\gamma_n\hat{P}_n + 2\beta_{n+1}\hat{P}_n - (n+1)\hat{P}_n &= 2(\alpha_n - A)\hat{P}_{n+1}
\end{align*}
\]

**First-order differential equation**

\[
(\mathcal{A} - \alpha_n)\hat{P}_n' + (\mathcal{A} - \alpha_n + xI)(\hat{P}_nA - A\hat{P}_n) - 2\beta_n\hat{P}_n = \hat{P}_nJ - J\hat{P}_n - n\hat{P}_n
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Let us now consider the special case of

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$$\hat{P}_n J - J\hat{P}_n - x(\hat{P}_n A - A\hat{P}_n) + 2\beta_n \hat{P}_n - n\hat{P}_n = 2(A - \alpha_n)\beta_n \hat{P}_{n-1}$$

$$\hat{P}_n(J - xA) - \gamma_n^{-1}(J - xA^*)\gamma_n \hat{P}_n + 2\beta_{n+1} \hat{P}_n - (n + 1)\hat{P}_n = 2(\alpha_n - A)\hat{P}_{n+1}$$

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**First-order differential equation**

\[
(A - \alpha_n)\hat{P}'_n + (A - \alpha_n + xI)(\hat{P}_n A - A\hat{P}_n) - 2\beta_n \hat{P}_n = \hat{P}_n J - J\hat{P}_n - n\hat{P}_n
\]
Finally, something remarkable happens. Combining the second and the first order differential equation will give surprisingly

\[ \hat{P}_n''(x) + \hat{P}_n'(x)(2A - 2xI) + \hat{P}_n(x)(A^2 - 2J) = (-2nl + A^2 - 2J)\hat{P}_n(x) \]

This is a second-order differential equation of Sturm-Liouville type satisfied by the OMP, already given by Durán-Grünbaum (2004)

Conclusions

1. The ladder operators method gives more insight about the differential properties of OMP and new phenomena
2. This method works for every weight matrix $W$. The corresponding OMP satisfy differential equations, but not necessarily of Sturm-Liouville type
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Finally, something remarkable happens. Combining the second and the first order differential equation will give surprisingly

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Outline

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Quasi-birth-and-death processes

New applications

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