Methods and new phenomena of orthogonal matrix polynomials satisfying differential equations[1]

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[1] joint work with F. A. Grünbaum and A. Martínez-Finkelshtein

Outline

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Preliminaries

A $N \times N$ matrix polynomial on the real line is

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0, \quad x \in \mathbb{R}, \quad A_i \in \mathbb{C}^{N \times N}$$

Krein (1949): Orthogonal matrix polynomials (OMP)
Let W be a $N \times N$ self adjoint positive definite weight matrix
We can construct a family $(P_n)_n$ of OMP with respect to the inner product

$$(P, Q)_W = \int_a^b P(x) W(x) Q^*(x) dx \in \mathbb{C}^{N \times N}$$

such that

$$(P_n, P_m)_W = \int_a^b P_n(x) W(x) P_m^*(x) dx = \delta_{n,m} I, \quad n, m \geq 0$$

$$P_n(x) = \kappa_n (x^n + a_{n,n-1} x^{n-1} + \cdots) = \kappa_n \widehat{P}_n(x)$$

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Three-term recurrence relation

Orthonormality of $(P_n)_n$ is equivalent to a three term recurrence relation

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x), \quad n \geq 0$$
$$\det(A_{n+1}) \neq 0, \quad B_n = B_n^*$$

Jacobi operator (block tridiagonal)

$$x\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 & A_1 & & & \\ A_1^* & B_1 & A_2 & & \\ & A_2^* & B_2 & A_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix}$$

Or equivalently for the monic family

$$x\widehat{P}_n(x) = \widehat{P}_{n+1}(x) + \alpha_n\widehat{P}_n(x) + \beta_n\widehat{P}_{n-1}(x), \quad n \geq 0$$

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Second-order differential equations

Durán (1997): characterize orthonormal $(P_n)_n$ satisfying second-order differential equations of Sturm-Liouville (hypergeometric) type

$$P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n P_n(x), \quad n \geq 0$$
$$\text{grad}\, F_i \leq i, \quad \Lambda_n \quad \text{Hermitian}$$

Equivalent to the symmetry (i.e. $(PD, Q)_W = (P, QD)_W$) of

$$D = \partial^2 F_2(x) + \partial^1 F_1(x) + \partial^1 F_0, \quad \partial = \frac{d}{dx}$$

Scalar case: Bochner (1929): Hermite, Laguerre and Jacobi

New matrix examples (2003): Durán, Grünbaum, Pacharoni and Tirao. Typically the weight matrices are of the form $W = \omega TT^*$

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- Matrix spherical functions associated with $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$ Grünbaum-Pacharoni-Tirao (2003)
- Durán-Grünbaum (2004): Symmetry equations

New phenomena

- For a fixed family of OMP there exist several linearly independent second-order differential operators having them as eigenfunctions
- OMP satisfying odd-order differential equations
- For a fixed second-order differential operator, there can be more than one family of lin. ind. OMP having them as eigenfunctions

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The Riemann-Hilbert problem

The Riemann-Hilbert problem (RHP) for orthogonal polynomials was introduced by Fokas-Its-Kitaev (1990)

For a given ω with $x^i\omega, x^j\omega' \in L^1(\mathbb{R})$ we try to find $Y^n : \mathbb{C} \to \mathbb{C}^{2\times 2}$ s.t.

1. Y^n is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $Y^n_+(x) = Y^n_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$
3. $Y^n(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

Advantages

1. Algebraic properties: three term recurrence relation, ladder operators, second order differential equation
2. Uniform asymptotics: *steepest descent analysis for RHP* (Deift-Zhou,1993). Very useful for functions which do not have an integral representation form

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The RHP for OMP

The unique solution of the RHP for OMP is given by

$$Y^n(z) = \begin{pmatrix} \widehat{P}_n(z) & C(\widehat{P}_n W)(z) \\ -2\pi i \gamma_{n-1} \widehat{P}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\widehat{P}_{n-1} W)(z) \end{pmatrix}, \quad n \geq 1$$

where $\gamma_n = \kappa_n^* \kappa_n$ and $C(F)(z) = \frac{1}{2\pi i} \int_a^b \frac{F(t)}{t-z} dt$

$Y^n(z)$ satisfies the following pair of first-order difference-differential relations (also known as Lax pair)

$$Y^{n+1}(z) = E_n(z) Y^n(z), \quad \frac{d}{dz} Y^n(z) = F_n(z) Y^n(z)$$

Cross-differentiation gives compatibility conditions (or string equations)

$$E_n'(z) + E_n(z) F_n(z) = F_{n+1}(z) E_n(z)$$

Problem: get explicit expression of $F_n(z)$

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Transformation of the RHP

Let $W(x) = T(x)T^*(x), \quad x \in \mathbb{R}$ and consider

$$X^n(z) = Y^n(z) \begin{pmatrix} T(z) & 0 \\ 0 & T^{-*}(\bar{z}) \end{pmatrix}$$

Therefore we have a class of Lax pairs

$$X^{n+1}(z) = E_n^S(z) X^n(z), \quad \frac{d}{dz} X^n(z) = F_n^S(z) X^n(z)$$

And a class of compatibility conditions

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Example

Let us consider ($S = I$)

$$W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R}$$

for any $A \in \mathbb{C}^{N\times N}$ (Durán-Grünbaum, 2004)

$$X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2} e^{Az} & 0 \\ 0 & e^{z^2/2} e^{-A^* z} \end{pmatrix}$$

Lax pair

$$X^{n+1}(z) = \begin{pmatrix} zI - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} X^n(z), \quad \frac{d}{dz} X^n(z) = \begin{pmatrix} -zI + A & -\frac{1}{\pi i}\gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & zI - A^* \end{pmatrix} X^n(z)$$

Compatibility conditions

$$\alpha_n = (A + \gamma_n^{-1} A^* \gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I$$

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$$X^{n+1}(z) = \begin{pmatrix} zI - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} X^n(z), \quad \frac{d}{dz} X^n(z) = \begin{pmatrix} -zI + A & -\frac{1}{\pi i}\gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & zI - A^* \end{pmatrix} X^n(z)$$

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Example

Let us consider ($S = I$)

$$W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R}$$

for any $A \in \mathbb{C}^{N\times N}$ (Durán-Grünbaum, 2004)

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Ladder operators

$$\widehat{P}_n'(z) + \widehat{P}_n(z)A - A\widehat{P}_n(z) = 2\beta_n\widehat{P}_{n-1}(z)$$
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Combining them we get a second order differential equation

Second order differential equation

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$$J\alpha_n - \alpha_n J + \alpha_n = \mathcal{A} + \frac{1}{2}(\mathcal{A}^2\alpha_n - \alpha_n\mathcal{A}^2)$$
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New ladder operators (0-th order)

$$\widehat{P}_n J - J\widehat{P}_n - x(\widehat{P}_n\mathcal{A} - \mathcal{A}\widehat{P}_n) + 2\beta_n\widehat{P}_n - n\widehat{P}_n = 2(\mathcal{A} - \alpha_n)\beta_n\widehat{P}_{n-1}$$
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Finally, something remarkable happens. Combining the second and the first order differential equation will give surprisingly

Sturm-Liouville type differential equation

$$\widehat{P}_n''(x) + \widehat{P}_n'(x)(2\mathcal{A} - 2xI) + \widehat{P}_n(x)(\mathcal{A}^2 - 2J) = (-2nI + \mathcal{A}^2 - 2J)\widehat{P}_n(x)$$

This is a second-order differential equation of Sturm-Liouville type satisfied by the OMP, already given by Durán-Grünbaum (2004)

Conclusions

1. The ladder operators method gives more insight about the differential properties of OMP and new phenomena
2. This method works for every weight matrix W. The corresponding OMP satisfy differential equations, but not necessarily of Sturm-Liouville type

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Outline

1. Preliminaries
2. Methods and new phenomena
3. Applications

New applications

Quantum mechanics

[Durán–Grünbaum] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006)

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