Differential properties of orthogonal matrix polynomials

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Outline

1. Scalar orthogonality
   - Difference and differential equations
   - Random walks and OP's
2. Matrix orthogonality
   - Difference and differential equations
   - Quasi-birth-and-death processes and OMP's
   - An example
   - Other applications

Scalar orthogonality

Let ω be a positive measure on $\mathbb{R}$ with finite moments. We can construct a family of orthonormal polynomials $(p_n)_n$

$$\langle p_n, p_m \rangle = \int_{\mathbb{R}} p_n(x) p_m(x) \mathrm{d}\omega(x) = \delta_{nm}, \quad n, m \geq 0$$

This is equivalent to a three term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad a_{n+1} \neq 0, \quad b_n \in \mathbb{R}$$

Jacobi operator (tridiagonal):

$$x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}$$

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$$xp_n(x) = a_{n+1}p_{n+1}(x)+b_np_n(x)+a_np_{n-1}(x), \quad a_{n+1} \neq 0, \quad b_n \in \mathbb{R}$$

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Bochner problem

Bochner (1929): characterize $(p_n)_n$ satisfying

$$\mathcal{A}p_n \equiv \underbrace{(\alpha_2 x^2 + \alpha_1 x + \alpha_0)}_{\sigma(x)} p_n''(x) + \underbrace{(\beta_1 x + \beta_0)}_{\tau(x)} p_n'(x) = \lambda_n p_n(x)$$

This is equivalent to the symmetry of $\mathcal{A}$ with respect to $\langle \cdot, \cdot \rangle$,i.e.

$$\langle \mathcal{A}p_n, p_m \rangle = \langle p_n, \mathcal{A}p_m \rangle$$

- Hermite: $\sigma(x) = 1, \omega(x) = e^{-x^2}$, $x \in (-\infty, \infty)$
- Laguerre: $\sigma(x) = x, \omega(x) = x^{\alpha} e^{-x}$, $\alpha > -1$, $x \in (0, \infty)$
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Random walks

A random walk is a Markov chain $\{X_n : n = 0, 1, 2, \dots\}$ with state space $\mathcal{S} = \{0, 1, 2, \dots\}$ where

$$\Pr(X_{n+1} = j | X_n = i) = 0 \quad \text{for} \quad |i - j| > 1, \quad i, j \in \mathcal{S}$$

i.e. a tridiagonal transition probability matrix (stochastic)

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_n \geq 0, a_n, c_n > 0, \quad a_n + b_n + c_n = 1$$

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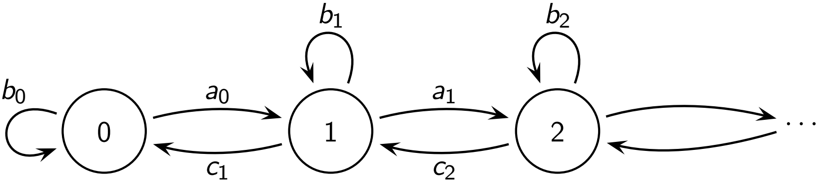

Introducing the polynomials $(q_n)_n$ by the conditions $q_{-1}(x) = 0$, $q_0(x) = 1$ and the recursion relation

$$xq_n(x) = a_n q_{n+1}(x) + b_n q_n(x) + c_n q_{n-1}(x), \quad n = 0, 1, \ldots$$

there exists a unique measure $d\omega(x)$ supported in $[-1, 1]$ such that $(q_n)_n$ are orthogonal w.r.t $d\omega(x)$.

Karlin-McGregor formula (1959)

$$\Pr(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

Invariant measure or distribution

A non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \geq 0$ such that

$$\boldsymbol{\pi} P = \boldsymbol{\pi}$$

$$\Rightarrow \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\|q_i\|^2}$$

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Matrix case

Krein (1949): orthogonal matrix polynomials on $\mathbb{R}$ (OMP)

Orthogonality: weight matrix W. Matrix valued inner product:

$$\langle P, Q\rangle_W = \int_{\mathbb{R}} P(x) dW(x) Q^*(x) \in \mathbb{C}^{N\times N}, \quad P, Q \in \mathbb{C}^{N\times N}[x]$$

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$$P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n P_n(x), \quad n \geq 0$$

grad $F_i \leq i$, $\quad \Lambda_n$ Hermitian

Equivalent to the symmetry of

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Methods and new phenomena

Methods

- Matrix spherical functions associated with $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$ Grünbaum-Pacharoni-Tirao (2003)
- Durán-Grünbaum (2004): Symmetry equations

New phenomena

- For a fixed family of OMP there exist several linearly independent second-order differential operators having them as eigenfunctions
- OMP satisfying odd-order differential equations
- For a fixed second-order differential operator, there can be more than one family of lin. ind. OMP having them as eigenfunctions

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QUASI-BIRTH-AND-DEATH PROCESSES

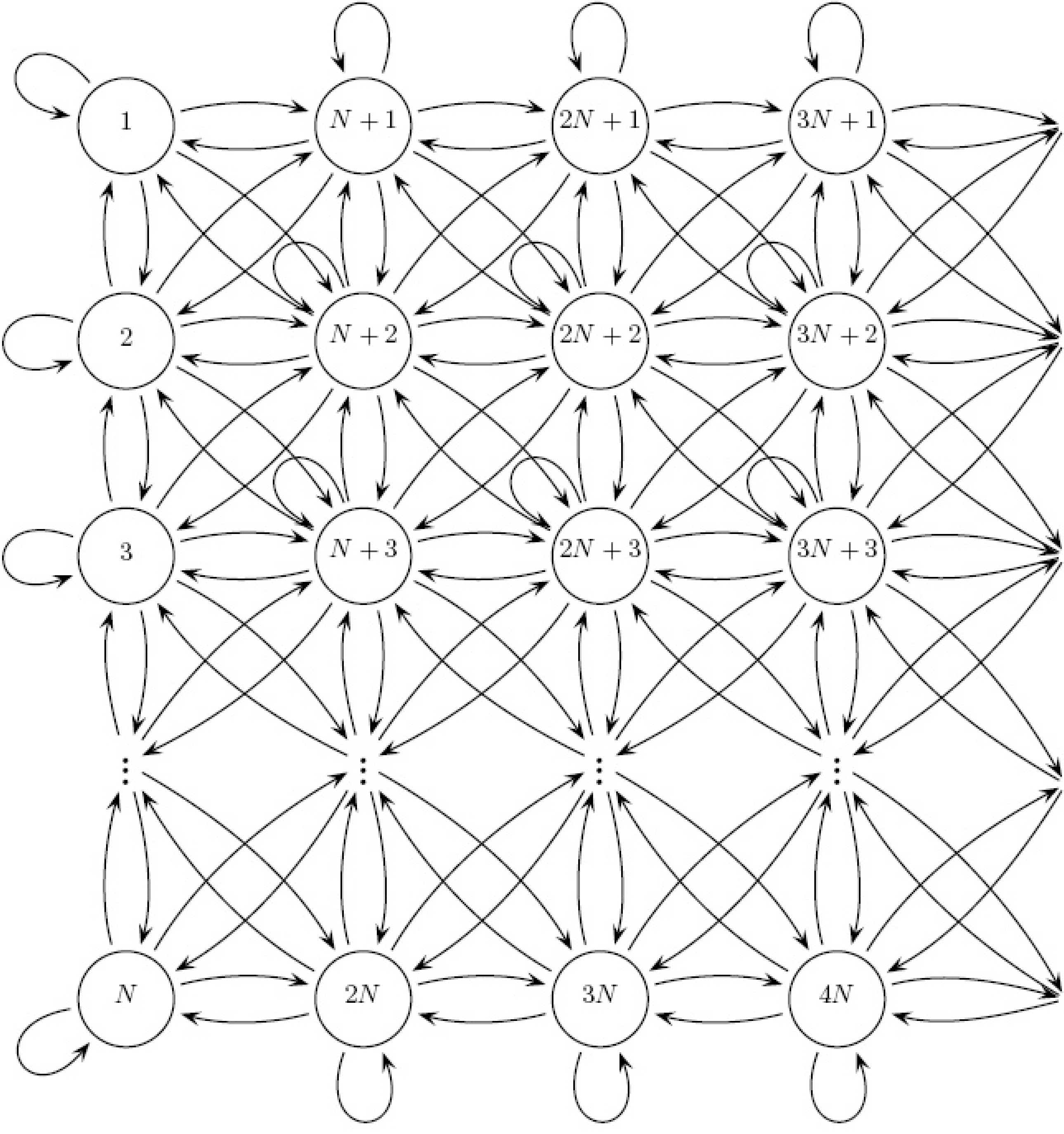
A discrete time quasi-birth-and-death process is a 2-dimensional Markov chain $\{Z_n = (X_n, Y_n) : n = 0, 1, 2, \dots\}$ with state space $\mathcal{C} = \{0, 1, 2, \dots\} \times \{1, 2, \dots, N\}$ where

$$(P_{ii'})_{jj'} = \Pr(X_{n+1} = i, Y_{n+1} = j | X_n = i', Y_n = j') = 0 \quad \text{for} \quad |i - i'| > 1$$

i.e. a $N \times N$ block tridiagonal transition probability matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \begin{array}{l} (A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geq 0, \ |A_n|, |C_n| \neq 0 \\ \sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1 \end{array}$$

The first component is called the level while the second component is the phase.



1
$N+1$
$2N+1$
$3N+1$
2
$N+2$
$2N+2$
$3N+2$
3
$N+3$
$2N+3$
$3N+3$
N
$2N$
$3N$
$4N$

OMP: Grünbaum and Dette-Reuther-Studden-Zygmunt (2007):
Introducing the matrix polynomials $(Q_n)_n$ by the conditions $Q_{-1}(x) = 0$, $Q_0(x) = I$ and the recursion relation

$$xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x), \quad n = 0, 1, \ldots$$

and under certain technical conditions over A_n, B_n, C_n, there exists an unique weight matrix $dW(x)$ supported in $[-1, 1]$ such that $(Q_n)_n$ are orthogonal w.r.t $dW(x)$.

Karlin-McGregor formula

$$P_{ij}^n = \left(\int_{-1}^1 x^n Q_i(x) dW(x) Q_j^*(x)\right)\left(\int_{-1}^1 Q_j(x) dW(x) Q_j^*(x)\right)^{-1}$$

Invariant measure or distribution (MdI, 2010)

Non-null vector with non-negative components

$$\pi = (\pi^0; \pi^1; \cdots) \equiv (\Pi_0 e_N; \Pi_1 e_N; \cdots)$$

such that $\pi P = \pi$ where $e_N = (1, \ldots, 1)^T$ and

$$\Pi_n = (C_1^T \cdots C_n^T)^{-1} \Pi_0 (A_0 \cdots A_{n-1}) = \left(\int_{-1}^1 Q_n(x) dW(x) Q_n^*(x)\right)^{-1}$$

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An example

Conjugation

$$W(x) = T^* \widetilde{W}(x) T$$

where

$$T = \begin{pmatrix} 1 & \dfrac{1}{\alpha+\beta-k+2} \\ 0 & -\dfrac{\alpha+\beta-k+2}{\beta-k+1} \end{pmatrix}$$

Grünbaum-MdI (2008)

$$\widetilde{W}(x) = x^\alpha(1-x)^\beta \begin{pmatrix} kx+\beta-k+1 & (1-x)(\beta-k+1) \\ (1-x)(\beta-k+1) & (1-x)^2(\beta-k+1) \end{pmatrix}$$

$x \in (0,1)$, $\alpha, \beta > -1$, $0 < k < \beta + 1$
Pacharoni-Tirao (2006)

We consider the family of OMP $(Q_n(x))_n$ such that

- Three term recurrence relation

$$xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x), \quad n = 0, 1, \ldots$$

where the Jacobi matrix is stochastic

- Choosing $Q_0(x) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)} \begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)} \times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

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Pentadiagonal Jacobi matrix

Particular case $\alpha = \beta = 0$, $k = 1/2$:

$$P = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{50} & & & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & & & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} & & \\ & & & & \frac{81}{392} & \frac{3}{196} & \frac{1955}{3969} & \frac{5}{324} & \frac{175}{648} & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$\Rightarrow$ Discrete time quasi-birth-and-death process with 2 phases

Invariant measure

Invariant measure

The row vector

$$\pi = (\pi^0; \pi^1; \cdots)$$

$$\pi^n = \left(\frac{1}{\left(\|Q_n\|_W^2\right)_{1,1}}, \frac{1}{\left(\|Q_n\|_W^2\right)_{2,2}}, \cdots, \frac{1}{\left(\|Q_n\|_W^2\right)_{N,N}} \right), \quad n \geq 0$$

is an invariant measure of P

Particular case $N = 2$, $\alpha = \beta = 0$, $k = 1/2$:

$$\pi^n = \left(\frac{2(n+1)^3}{(2n+3)(2n+1)}, \frac{(n+1)(n+2)}{2n+3} \right), \quad n \geq 0$$

$$\pi = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \cdots \right)$$

Invariant measure

Invariant measure

The row vector

$$\pi = (\pi^0; \pi^1; \cdots)$$

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Other applications

Quantum mechanics

[Durán–Grünbaum] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006)

Time-and-band limiting

[Durán–Grünbaum] *A survey on orthogonal matrix polynomials satisfying second order differential equations*, J. Comput. Appl. Math. (2005)

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