Non-commutative Painlevé equations and Hermite-type matrix orthogonal polynomials*

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*Joint work with Mattia Cafasso, Université d'Angers

OUTLINE

1. Motivation and preliminaries

2. Integral representations

3. Non-commutative Painlevé IV equation

MOTIVATION

Let $(H_n)_n$ be the classical Hermite polynomials such that

$$\int_{\mathbb{R}} H_n(x) H_m(x) \mathrm{e}^{-x^2} dx = \delta_{nm}$$

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$$\boxed{K_n(x,y) = \sum_{k=0}^{n-1} H_k(x) H_k(y) \mathrm{e}^{-\frac{x^2+y^2}{2}}}$$

describes the statistical properties of the eigenvalues of a random matrix $\boldsymbol{M}$ in the space of $(n \times n)$ Hermitian matrices with the measure $\mu(\boldsymbol{M}) = \mathrm{e}^{-\mathrm{Tr}(\boldsymbol{M}^2)} d\boldsymbol{M}$ (GUE, Mehta).

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The last particle distribution is given by the Fredholm determinant

$$F(s) = \mathbb{P}\left[\lambda_{\max} \leq s\right] = \det(\mathrm{Id} - \chi_s \mathbb{K}_n)$$

where χ_s is the indicator function of the interval $[s, \infty)$ and $\mathbb{K}_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the integral operator

$$[\mathbb{K}_n f](x) = \int_{\mathbb{R}} K_n(x,y) f(y) dy \quad \forall\, f \in L^2(\mathbb{R})$$

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THEOREM (Tracy-Widom, 1994)
The log derivative of the Fredholm determinant

$$R(s) = \partial_s \log(\det(\mathrm{Id} - \chi_s \mathbb{K}_n))$$

solves the sigma-form of the Painlevé IV equation

$$(R'')^2 + 4(R')^2(R' + 2n) - 4(sR' - R)^2 = 0$$

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Extend these results to CD kernels associated to Hermite-type matrix-valued orthogonal polynomials (MOP).

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- Double integral representations of some Hermite-type MOP

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- Non-commutative integrable systems

  P. Etingof, I.Gelfand, V. Retakh, V. Rubtsov, V. Sokolov...

PRELIMINARIES

Let $\boldsymbol{W}$ be a weight matrix (positive definite and finite moments). Consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ the weighted space with the inner product

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{F}(x)\boldsymbol{W}(x)\boldsymbol{G}^*(x)dx$$

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A sequence $(\boldsymbol{P}_n)_n$ of matrix orthonormal polynomials (MOP) with respect to $\boldsymbol{W}$ is a sequence satisfying (Krein, 1949)

$$\deg \boldsymbol{P}_n = n, \quad \langle \boldsymbol{P}_n, \boldsymbol{P}_m \rangle_{\boldsymbol{W}} = \boldsymbol{I}_N \delta_{nm}$$

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If $(\boldsymbol{P}_n)_n$ is complete, the Christoffel-Darboux (CD) kernel is

$$\boldsymbol{K}_n(x,y) = \sum_{k=0}^{n-1} \boldsymbol{P}_k^*(y)\boldsymbol{P}_k(x), \quad x, y \in \mathbb{R}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

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$$z^{\boldsymbol{J}} = \begin{pmatrix} z^{N-1} & & & \\ & \ddots & & \\ & & z & \\ & & & 1 \end{pmatrix}, \quad z^{-\boldsymbol{J}} = \begin{pmatrix} \frac{1}{z^{N-1}} & & & \\ & \ddots & & \\ & & \frac{1}{z} & \\ & & & 1 \end{pmatrix}$$

THE EXAMPLE

Let us consider the weight matrix (Durán-Grünbaum, 2004)

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\boldsymbol{A}x}\mathrm{e}^{\boldsymbol{A}^*x}, \quad x \in \mathbb{R}$$

and the family of MOP $(\boldsymbol{P}_n)_n$
satisfying the second order differential equation

$$\boldsymbol{P}_n''(x) + \boldsymbol{P}_n'(x)(-2x\boldsymbol{I} + 2\boldsymbol{A}) + \boldsymbol{P}_n(x)(\boldsymbol{A}^2 - 2\boldsymbol{J}) = (-2n\boldsymbol{I} - 2\boldsymbol{J})\boldsymbol{P}_n(x)$$

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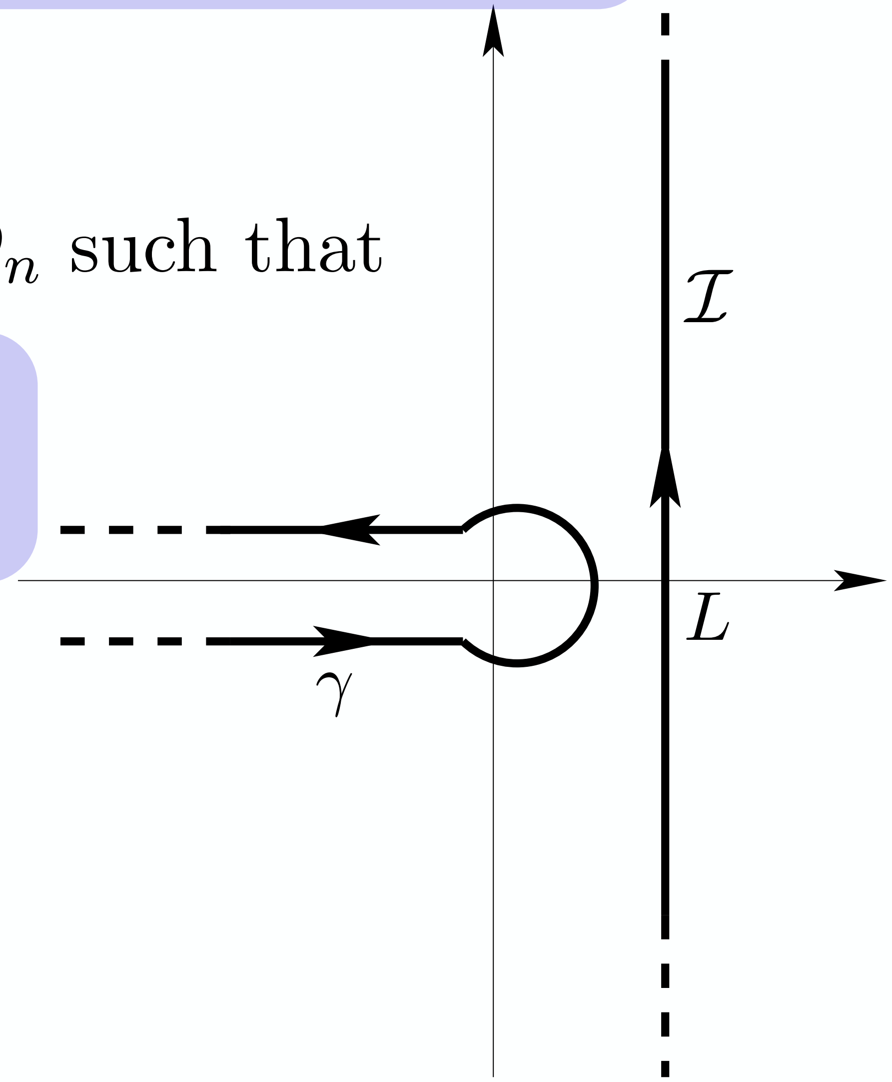
THEOREM (Cafasso-MdI, 2013)

There exist suitable constant matrices $\boldsymbol{C}_n$ and $\boldsymbol{D}_n$ such that

$$\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x} = \oint_\gamma z^{-\boldsymbol{J}}\boldsymbol{C}_n z^{\boldsymbol{J}}\mathrm{e}^{-z^2+2zx}\frac{dz}{z^{n+1}}$$



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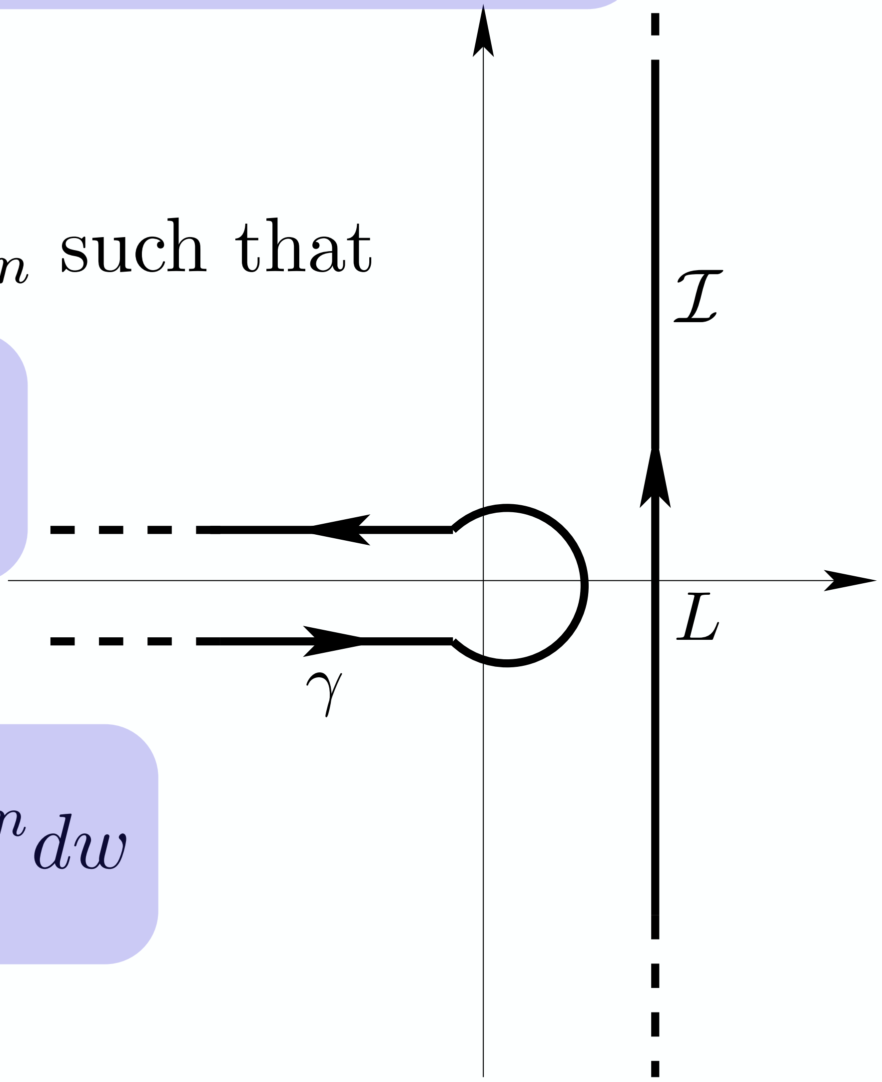

CASE N=2

For the family of MOP $(\boldsymbol{P}_n)_n$ with respect to $(N=2)$

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2} \begin{pmatrix} 1 + x^2\nu^2 & \nu x \\ \nu x & 1 \end{pmatrix}, \quad x \in \mathbb{R}$$

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$$\boldsymbol{P}_n(x)\begin{pmatrix} 1 & \nu x \\ 0 & 1 \end{pmatrix} = \frac{n!}{2^{n+1}\pi i}\oint_\gamma \begin{pmatrix} 1 & \dfrac{(n+1)\nu}{2z} \\ -\dfrac{z\nu}{\gamma_n^2} & \dfrac{1}{\gamma_n^2} \end{pmatrix} \mathrm{e}^{-z^2+2zx}\frac{dz}{z^{n+1}}$$

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CHRISTOFFEL-DARBOUX KERNEL (N=2)

The CD kernel

$$\boldsymbol{K}_n(x,y)=\sum_{k=0}^{n-1}\boldsymbol{\Phi}_k^*(y)\boldsymbol{\Phi}_k(x)$$

where $\boldsymbol{\Phi}_n(x)=\mathrm{e}^{-x^2/2}\|\boldsymbol{P}_n\|_{\boldsymbol{W}}^{-1}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ can be written as

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In other words

$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}} dw \oint_{\gamma} dz \begin{pmatrix} \frac{z(\gamma_n^2-1)+w}{w\gamma_n^2} & \frac{\nu(w-z)}{\gamma_n^2} \\ \frac{n\nu(z-w)}{2\gamma_n^2} & \frac{w(\gamma_n^2-1)+z}{z\gamma_n^2} \end{pmatrix} \frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

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$\boldsymbol{G}(\lambda)$ depends on $\lambda^{\boldsymbol{J}_2}\boldsymbol{B}_n\lambda^{-\boldsymbol{J}_2}$

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Lax pair

$$\partial_\lambda \boldsymbol{\Psi} = \mathcal{A}\boldsymbol{\Psi}$$

$$\partial_s \boldsymbol{\Psi} = \mathcal{U}\boldsymbol{\Psi}$$

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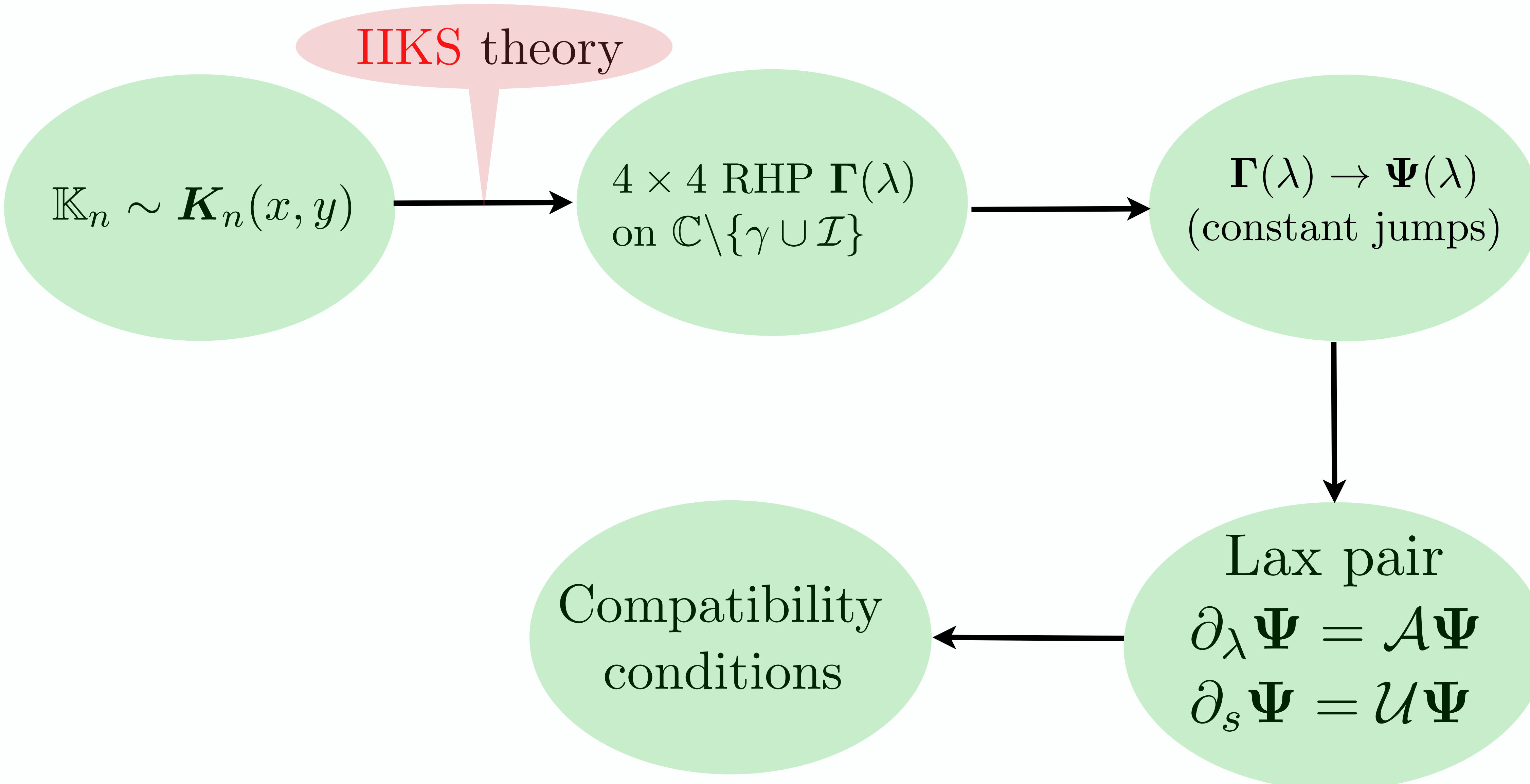

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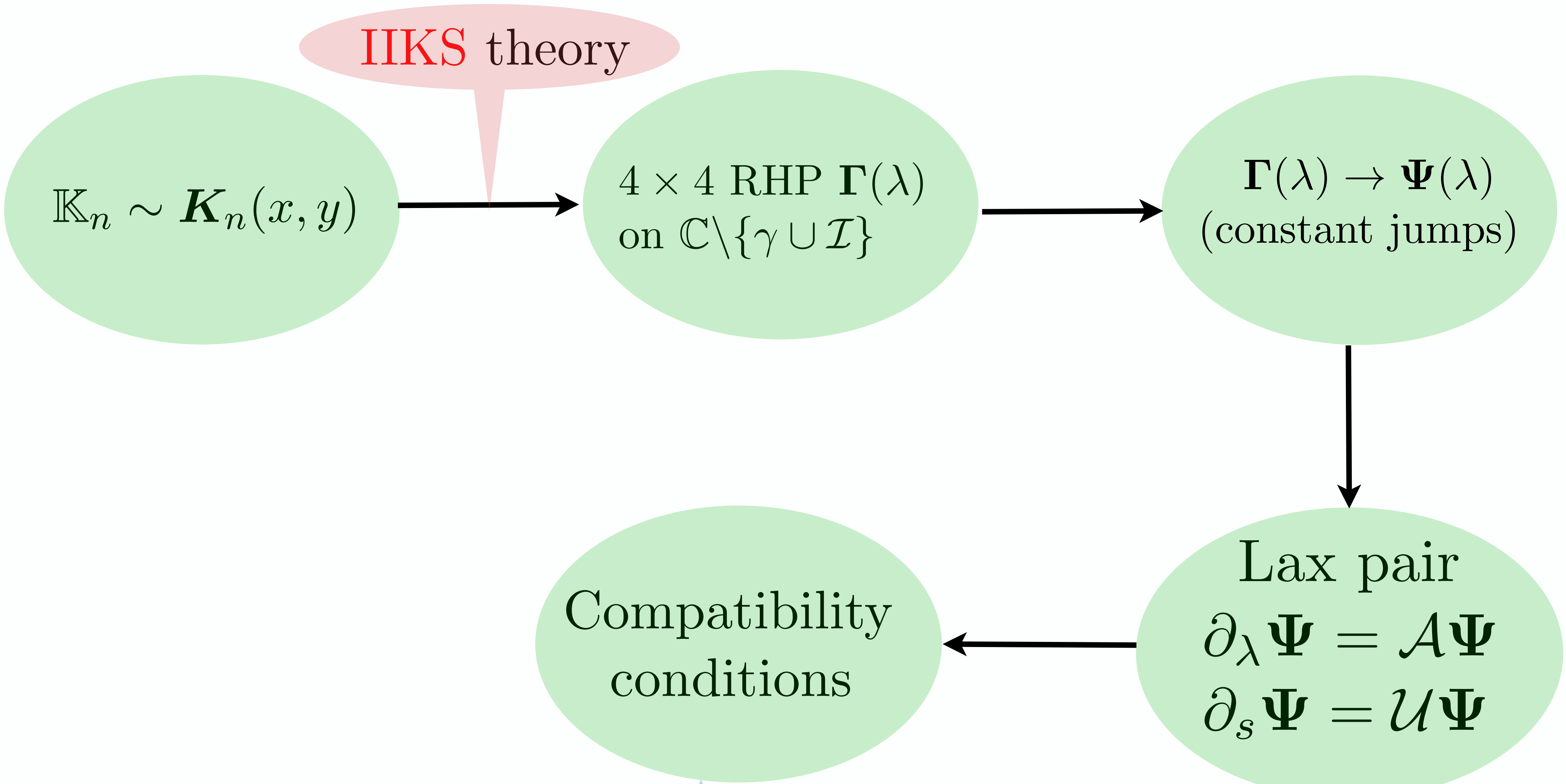

$$\begin{cases} \boldsymbol{u}' &= -\boldsymbol{u}^2 + 2s\boldsymbol{u} + 4\boldsymbol{z} - 2n\boldsymbol{I}_2 + \boldsymbol{V_A} \\ \boldsymbol{z}'' &= 2\boldsymbol{u}'\boldsymbol{z} + 2\boldsymbol{u}\boldsymbol{z}' - 2s\boldsymbol{z}' \end{cases}$$

where $\boldsymbol{V_A} = 2[\boldsymbol{J}_2, \boldsymbol{y}]\boldsymbol{y}^{-1}$ $([\boldsymbol{x}, \boldsymbol{y}] = \boldsymbol{x}\boldsymbol{y} - \boldsymbol{y}\boldsymbol{x})$ and

$$\boldsymbol{z} = -(\boldsymbol{\Gamma}_1)'_{11}, \quad \boldsymbol{y} = -2(\boldsymbol{\Gamma}_1)_{12}, \quad \boldsymbol{u} = (\boldsymbol{\Gamma}_1)'_{12}(\boldsymbol{\Gamma}_1)^{-1}_{12} + 2s\boldsymbol{I}_2$$

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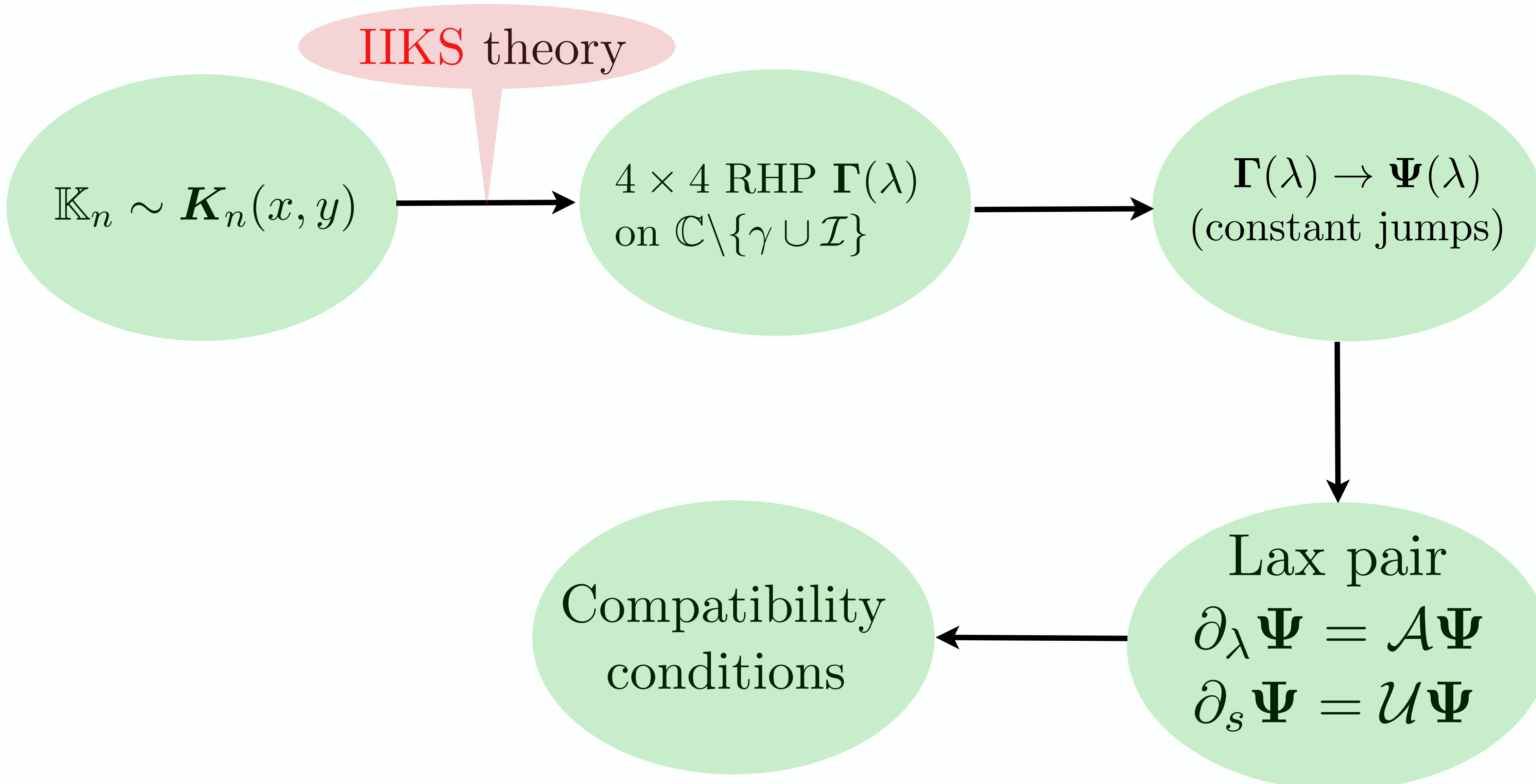

Non-commutative version of the derived Painlevé IV equation

$$\boldsymbol{u}''' + [\boldsymbol{u}'', \boldsymbol{u}] - 4(n+1+s^2)\boldsymbol{u}' - 2\left(\{\boldsymbol{u}', \boldsymbol{u}^2\} + \boldsymbol{u}\boldsymbol{u}'\boldsymbol{u}\right)$$
$$+6s\{\boldsymbol{u}', \boldsymbol{u}\} + 4\boldsymbol{u}(\boldsymbol{u} - s\boldsymbol{I}_2) + (\boldsymbol{V}_{\boldsymbol{A}}' - 2(\boldsymbol{u}\boldsymbol{V}_{\boldsymbol{A}}))' + 2s\boldsymbol{V}_{\boldsymbol{A}}' = \boldsymbol{0}$$

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If we assume that all the variables commute, we get the equation

$$u''' - 4u' - 6u^2u' + 12u'u - 4nu' + 4u^2 - 4su - 4s^2u' = 0$$

and this equation is the derivative of the Painlevé IV equation

$$u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 - 4su^2 + 2(s^2+1+n)u - \frac{2n^2}{u}$$

FINAL REMARKS

- It is well known that

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} K_n^{\mathrm{Hermite}}\left(\sqrt{2n}+\frac{x}{\sqrt{2}n^{1/6}}, \sqrt{2n}+\frac{y}{\sqrt{2}n^{1/6}}\right) = K_{\mathrm{Ai}}(x,y),$$

In our case we have that

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- M. D. de la Iglesia and M. Cafasso, *Non-commutative Painlevé equations and Hermite-type matrix orthogonal polynomials*, accepted in Communications in Mathematical Physics.


THANK YOU