Stochastic factorizations of birth-death chains and Darboux transformations*

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MAIN IDEA OF THE TALK

Let $\{X_n : n = 0, 1, \ldots\}$ be an irreducible discrete-time birth-death chain with space state $\mathcal{S} \subseteq \mathbb{Z}$ and P its one-step transition probability tridiagonal matrix.

We will study stochastic factorizations of P of the form

$$P = P_1 P_2$$

where P_1 and P_2 are also stochastic matrices.

The natural choice is looking for P_1 and P_2 as upper or lower bidiagonal matrices or viceversa (UL or LU factorizations) but we will see that we can consider different factorizations if $\mathcal{S} = \mathbb{Z}$.

Main motivation: divide the probabilistic model for P into two simpler probabilistic models and consider the combination of both as a model for P.

We will be focused on answering the following two questions:

1. Under what conditions can we guarantee a stochastic factorization of the form $P = P_1 P_2$?

2. If ψ is the spectral measure (scalar or matrix-valued) associated with P, what is the spectral measure of the discrete Darboux transformation $\widetilde{P} = P_2 P_1$?

OUTLINE

1. State space $\mathcal{S} = \{0, 1, \ldots, N\}$.

2. State space $\mathcal{S} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

3. State space $\mathcal{S} = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$.

1. State space $\mathcal{S} = \{0, 1, \ldots, N\}$.

UL FACTORIZATION

Let $\{X_n : n = 0, 1, \ldots\}$ be an irreducible discrete-time birth-death chain with space state $\{0, 1, \ldots, N\}$ and P its one-step transition probability matrix. Consider a UL factorization of P in the following way

$$\underbrace{\begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{N-1} & b_{N-1} & a_{N-1} \\ & & & c_N & b_N \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} y_0 & x_0 & & & \\ 0 & y_1 & x_1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & y_{N-1} & x_{N-1} \\ & & & 0 & y_N \end{pmatrix}}_{P_U} \underbrace{\begin{pmatrix} s_0 & 0 & & & \\ r_1 & s_1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & r_{N-1} & s_{N-1} & 0 \\ & & & r_N & s_N \end{pmatrix}}_{P_L}$$

with the condition that P_U and P_L are also stochastic matrices.

We obviously must have $s_0 = y_N = 1$. A direct computation shows that

$$\begin{aligned} &a_n = x_n s_{n+1}, \quad n = 0, \ldots, N-1, \\ &b_n = x_n r_{n+1} + y_n s_n, \quad n = 0, \ldots, N-1, \quad b_N = s_N \\ &c_n = y_n r_n, \quad n = 1, \ldots, N. \end{aligned}$$

Since P, P_U and P_L are stochastic we must have $a_n + b_n + c_n = 1$, $y_n + x_n = 1$ and $s_n + r_n = 1$ and the only relevant relations are the first and the third one.

EXAMPLES

For instance, the symmetric random walk with transition probability matrix

$$P = \begin{pmatrix} 0 & 1 & & & \\ q & 0 & p & & \\ & \ddots & \ddots & \ddots & \\ & & q & 0 & p \\ & & & 1 & 0 \end{pmatrix}, \quad p+q=1$$

does not have a UL stochastic factorization since the eigenvalues are not inside the interval $[0,1]$.

But nevertheless we have

$$\begin{pmatrix} 3/4 & 1/4 & & & \\ 1/4 & 1/2 & 1/4 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/4 & 1/2 & 1/4 \\ & & & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & & & \\ 0 & 1/2 & 1/2 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1/2 & 1/2 \\ & & & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & & & \\ 1/2 & 1/2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/2 & 1/2 & 0 \\ & & & 1/2 & 1/2 \end{pmatrix}$$

Darboux transformation: if $Pv = \lambda v$ then $\widetilde{P}\widetilde{v} = \lambda \widetilde{v}$, where $\widetilde{P} = P_L P_U$ and $\widetilde{v} = P_L v$, i.e. the eigenvalues of P and $\widetilde{P}$ are the same (then only thing that changes are the eigenvectors).

LU FACTORIZATION

If we consider now a LU factorization of P

$$\underbrace{\begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{N-1} & b_{N-1} & a_{N-1} \\ & & & c_N & b_N \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} \tilde{s}_0 & 0 & & & \\ \tilde{r}_1 & \tilde{s}_1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{r}_{N-1} & \tilde{s}_{N-1} & 0 \\ & & & \tilde{r}_N & \tilde{s}_N \end{pmatrix}}_{\widetilde{P}_L} \underbrace{\begin{pmatrix} \tilde{y}_0 & \tilde{x}_0 & & & \\ 0 & \tilde{y}_1 & \tilde{x}_1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & \tilde{y}_{N-1} & \tilde{x}_{N-1} \\ & & & 0 & \tilde{y}_N \end{pmatrix}}_{\widetilde{P}_U}$$

with $\widetilde{P}_U$ and $\widetilde{P}_L$ stochastic matrices, then we have again $\tilde{s}_0 = \tilde{y}_N = 1$ but now

$$\begin{aligned} a_n &= \tilde{s}_n \tilde{x}_n, \quad n =, \dots, N-1, \\ b_n &= \tilde{r}_n \tilde{x}_{n-1} + \tilde{s}_n \tilde{y}_n, \quad n = 0, \dots, N, \\ c_n &= \tilde{r}_n \tilde{y}_n, \quad n = 1, \dots, N. \end{aligned}$$

Now it is possible to see, performing the same computation, that the LU stochastic factorization is always possible as long as the eigenvalues of P are contained in the interval $[0, 1]$. Same considerations for the Darboux transformation.

2. State space $\mathcal{S} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

UL AND LU FACTORIZATIONS

Now the space state is $\mathcal{S} = \mathbb{N}_0$ and P is a semi-infinite matrix. We can consider as before

$$\underbrace{\begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} y_0 & x_0 & & & \\ 0 & y_1 & x_1 & & \\ & 0 & y_2 & x_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{P_U} \underbrace{\begin{pmatrix} s_0 & 0 & & & \\ r_1 & s_1 & 0 & & \\ & r_2 & s_2 & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{P_L}$$

with the condition that P_U and P_L are also stochastic matrices.

A direct computation shows that

$$\begin{aligned} a_n &= x_n s_{n+1}, \quad n \geq 0 \\ b_n &= x_n r_{n+1} + y_n s_n, \quad n \geq 0 \\ c_n &= y_n r_n, \quad n \geq 1 \end{aligned}$$

As before, it is possible to compute all the coefficients x_n, y_n, r_n, s_n of P_U and P_L in terms of a free parameter y_0 using the previous relations.

The same can be applied for the LU stochastic factorization but now there is no free parameter.

CONDITIONS FOR STOCHASTICITY

For that we will need that the following continued fraction

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}} \doteq 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots}$$

is convergent and $0 < H < 1$.

Theorem. *Let H the continued fraction given before and the corresponding convergents $h_n = A_n/B_n$. Assume that*

$$0 < A_n < B_n, \quad n \geq 1$$

Then H is convergent. Moreover, if $P = P_U P_L$, then both P_U and P_L are stochastic matrices if and only if we choose y_0 in the following range

$$0 < y_0 \leq H = 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots}$$

The stochastic LU factorization is possible if and only if H is convergent and $0 < H < 1$.

Moreover, let ψ be the spectral measure associated with P. Then $0 < H < 1$ if and only if $supp(\psi) \subset [0,1]$.

DARBOUX TRANSFORMATIONS

UL case: Let ψ be the spectral measure associated with P. The Darboux transformation gives a family of discrete-time birth-death chains $\widetilde{P} = P_L P_U$. Let us call $\widetilde{\psi}$ the spectral matrix associated with $\widetilde{P}$. If $\mu_{-1} = \int_{-1}^{1} d\psi(x)/x$ is well defined, then the family of spectral measures is given by

$$\widetilde{\psi}(x) = y_0 \frac{\psi(x)}{x} + M\delta_0(x), \quad M = 1 - y_0\mu_{-1}$$

where $\delta_0(x)$ is the Dirac delta located at $x = 0$ and y_0 is the free parameter from the UL factorization. This transformation of the spectral measure ψ is also known as a Geronimus transformation.

LU case: the corresponding Darboux transformation $\widehat{P} = \widetilde{P}_U \widetilde{P}_L$ gives rise to a tridiagonal stochastic matrix and a spectral measure $\widehat{\psi}$. In this case, it is possible to see that this new spectral measure is given by

$$\widehat{\psi}(x) = x\psi(x)$$

or, in other words, a Christoffel transformation of ψ.

EXAMPLE: JACOBI POLYNOMIALS

Consider the family of Jacobi polynomials $Q_n^{(\alpha,\beta)}(x)$, which are orthogonal with respect to the weight

$$w(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^\alpha (1-x)^\beta, \quad x \in [0,1], \quad \alpha, \beta > -1$$

normalized by the condition

$$Q_n^{(\alpha,\beta)}(1) = 1$$

Then the Jacobi polynomials satisfy the three-term recursion relation

$$x Q_n^{(\alpha,\beta)}(x) = a_n Q_{n+1}^{(\alpha,\beta)}(x) + b_n Q_n^{(\alpha,\beta)}(x) + c_n Q_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 0$$

where the coefficients a_n, b_n, c_n are defined by

$$a_n = \frac{(n+\beta+1)(n+1+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)}, \quad n \geq 0$$

$$b_n = \frac{(n+\beta+1)(n+1)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)} + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad n \geq 0$$

$$c_n = \frac{n(n+\alpha)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad n \geq 1$$

UL FACTORIZATION

Let us call $\alpha_n, n \geq 1$, the sequence of alternating coefficients $a_0, c_1, a_1, c_2, \ldots$, respectively. Then the sequence α_n is a chain sequence, i.e. $\alpha_n = (1 - m_{n-1})m_n$ where $0 \leq m_0 < 1$ and $0 < m_n < 1$ for $n \geq 1$. In this case we have

$$m_{2n} = \frac{n}{2n + \alpha + \beta + 1}, \quad m_{2n+1} = \frac{n + \beta + 1}{2n + \alpha + \beta + 2}, \quad n \geq 0$$

Then the continued fraction H with partial numerators given by α_n is convergent to $(1+L)^{-1}$ where L is given by (see book of Chihara)

$$L = \sum_{n=1}^{\infty} \frac{m_1 m_2 \cdots m_n}{(1 - m_1)(1 - m_2) \cdots (1 - m_n)}$$

It is possible to see (hypergeometric series) that

$$L = \frac{\beta + 1}{\alpha}$$

Using the main theorem we have that the stochastic UL factorization is always possible if we choose the free parameter y_0 in the range

$$0 < y_0 \leq \frac{\alpha}{\alpha + \beta + 1}, \quad \alpha > 0, \beta > -1$$

So for every y_0 in that range we can always have a stochastic UL factorization.

Since $0 < H < 1$ the stochastic LU factorization is always possible.

DARBOUX TRANSFORMATIONS

The spectral measure associated with the Darboux transformation $\widetilde{P} = P_L P_U$ is the Geronimus transformation of the Jacobi weight w. In this case it is easy to see that

$$\mu_{-1} = \int_0^1 \frac{w(x)}{x} dx = \frac{\alpha + \beta + 1}{\alpha}$$

Therefore we have

$$\widetilde{w}(x) = y_0 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha-1}(1-x)^{\beta} + \left(1 - y_0 \frac{\alpha+\beta+1}{\alpha}\right) \delta_0(x), \quad x \in [0,1]$$

This measure is integrable as long as $\alpha > 0$ and $\beta > -1$. We see that if y_0 is in the range of a stochastic UL factorization, then the mass at 0 is always nonnegative, and vanishes if

$$y_0 = \frac{\alpha}{\alpha + \beta + 1}$$

For the LU decomposition, the spectral measure associated with the Darboux transformation $\widehat{P} = \widetilde{P}_U \widetilde{P}_U$ is the Christoffel transformation of the Jacobi weight, i.e.

$$\widehat{w}(x) = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)} x^{\alpha+1}(1-x)^{\beta}, \quad x \in [0,1]$$

3. State space $\mathcal{S} = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2\ldots\}$.

UL AND LU FACTORIZATIONS

Now the space state is $\mathcal{S} = \mathbb{Z}$ and P is the doubly infinite stochastic matrix

$$P = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ & c_{-1} & b_{-1} & a_{-1} & & \\ \hline & & c_0 & b_0 & a_0 & \\ & & & c_1 & b_1 & a_1 \\ & & & & \ddots & \ddots & \ddots \end{array}\right)$$

Again, we look for stochastic UL (or LU) factorizations of P of the form

$$P = \underbrace{\left(\begin{array}{cc|cccc} \ddots & \ddots & & & & \\ 0 & y_{-1} & x_{-1} & & & \\ \hline & 0 & y_0 & x_0 & & \\ & & 0 & y_1 & x_1 & \\ & & & & \ddots & \ddots \end{array}\right)}_{P_U} \underbrace{\left(\begin{array}{ccc|ccc} \ddots & \ddots & & & & \\ & r_{-1} & s_{-1} & 0 & & \\ \hline & & r_0 & s_0 & 0 & \\ & & & r_1 & s_1 & 0 \\ & & & & \ddots & \ddots \end{array}\right)}_{P_L}$$

A direct computation shows that

$$\begin{aligned} a_n &= x_n s_{n+1}, \quad n \in \mathbb{Z} \\ b_n &= x_n r_{n+1} + y_n s_n, \quad n \in \mathbb{Z} \\ c_n &= y_n r_n, \quad n \in \mathbb{Z} \end{aligned}$$

CONDITIONS FOR STOCHASTICITY

Now we will have to consider two continued fractions, the same H as before and a new one corresponding to the negative states

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}}, \quad H' = \cfrac{c_0}{1 - \cfrac{a_{-1}}{1 - \cfrac{c_{-1}}{1 - \cfrac{a_{-2}}{1 - \cdots}}}}.$$

We assume that both H, H' are convergent and $0 < H' \leq H < 1$.

Theorem.

- ***UL case:*** *The stochastic UL factorization is possible if and only if the free parameter y_0 satisfies the following condition*

$$H' \leq y_0 \leq H$$

- ***LU case:*** *The stochastic LU factorization is possible if and only if the free parameter $\tilde{r}_0$ satisfies the following condition*

$$H' \leq \tilde{r}_0 \leq H$$

SPECTRAL ANALYSIS OF P

(Nikishin, 1986 or Masson-Repka, 1991). If we consider the eigenvalue equation $xq^{\alpha}(x) = Pq^{\alpha}(x)$ where $q^{\alpha}(x) =$ $(\cdots, Q_{-1}^{\alpha}(x), Q_0^{\alpha}(x), Q_1^{\alpha}(x), \cdots)^T, \alpha = 1, 2$, then, for each x real or complex there exist two polynomial familie of linearly independent solutions $Q_n^{\alpha}(x), \alpha = 1, 2, n \in \mathbb{Z}$, given by

$$
\begin{aligned}
Q_0^1(x) &= 1, \quad Q_0^2(x) = 0, \\
Q_{-1}^1(x) &= 0, \quad Q_{-1}^2(x) = 1, \\
xQ_n^{\alpha}(x) &= a_n Q_{n+1}^{\alpha}(x) + b_n Q_n^{\alpha}(x) + c_n Q_{n-1}^{\alpha}(x), \quad n \in \mathbb{Z}, \quad \alpha = 1, 2.
\end{aligned}
$$

P is self-adjoint in $\ell_{\pi}^2(\mathbb{Z})$ where $(\pi_n)_{n \in \mathbb{Z}}$ is given by

$$
\pi_0 = 1, \quad \pi_n = \frac{a_0 a_1 \cdots a_{n-1}}{c_1 c_2 \cdots c_n}, \quad \pi_{-n} = \frac{c_0 c_{-1} \cdots c_{-n+1}}{a_{-1} a_{-2} \cdots a_{-n}}, \quad n \geq 1
$$

It is possible to see that $Q_i^1(P)e^{(0)} + Q_i^2(P)e^{(-1)} = e^{(i)}, i \in \mathbb{Z}$, where $e^{(i)} = (e_j^{(i)})_{j \in \mathbb{Z}}$ is defined by $e_j^{(i)} = \delta_{i,j}/\pi_i$

Therefore we need to apply the Spectral Theorem 4 times and obtain 3 measures $\psi_{\alpha,\beta}, \alpha, \beta = 1, 2$ (sinc $\psi_{12} = \psi_{21}$ by the symmetry) such that

$$
\sum_{\alpha,\beta=1}^{2} \int_{-1}^{1} Q_i^{\alpha}(x) Q_j^{\beta}(x) d\psi_{\alpha\beta}(x) = \frac{\delta_{i,j}}{\pi_j}, \quad i, j \in \mathbb{Z}
$$

These 3 measures can be written in matrix form as the 2×2 matrix

$$
\Psi(x) = \begin{pmatrix} \psi_{11}(x) & \psi_{12}(x) \\ \psi_{12}(x) & \psi_{22}(x) \end{pmatrix}
$$

which it is called the spectral matrix associated with P.

RELATION WITH MATRIX-VALUED OPS

If we define the matrix-valued polynomials

$$\boldsymbol{Q}_n(x) = \begin{pmatrix} Q_n^1(x) & Q_n^2(x) \\ Q_{-n-1}^1(x) & Q_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0$$

then we have

$$x\boldsymbol{Q}_0(x) = A_0\boldsymbol{Q}_1(x) + B_0\boldsymbol{Q}_0(x), \quad \boldsymbol{Q}_0(x) = I_{2\times 2},$$
$$x\boldsymbol{Q}_n(x) = A_n\boldsymbol{Q}_{n+1}(x) + B_n\boldsymbol{Q}_n(x) + C_n\boldsymbol{Q}_{n-1}(x), \quad n \geq 1,$$

where $I_{2\times 2}$ denotes the 2×2 identity matrix and

$$B_0 = \begin{pmatrix} b_0 & c_0 \\ a_{-1} & b_{-1} \end{pmatrix}, \quad B_n = \begin{pmatrix} b_n & 0 \\ 0 & b_{-n-1} \end{pmatrix}, \quad n \geq 1,$$
$$A_n = \begin{pmatrix} a_n & 0 \\ 0 & c_{-n-1} \end{pmatrix}, \quad n \geq 0, \quad C_n = \begin{pmatrix} c_n & 0 \\ 0 & a_{-n-1} \end{pmatrix}, \quad n \geq 1.$$

The matrix orthogonality is defined in terms of the (matrix-valued) inner product

$$\int_{-1}^{1} \boldsymbol{Q}_n(x)\Psi(x)\boldsymbol{Q}_m^T(x)dx = \begin{pmatrix} 1/\pi_n & 0 \\ 0 & 1/\pi_{-n-1} \end{pmatrix} \delta_{nm}$$

The matrix P "folds" into a semi-infinite 2×2 block tridiagonal matrix $\boldsymbol{P}$.

UL DARBOUX TRANSFORMATION

Theorem.
UL case: *Let* $\Psi(x)$ *the spectral matrix associated with* $P = P_U P_L$ *and consider the Darboux transformation* $\widetilde{P} = P_L P_U$. *Assume that* $M_{-1} = \int_{-1}^{1} \Psi(x) x^{-1} dx$ *is well-defined. Then the spectral matrix associated with* $\widetilde{P}$ *is given by*

$$\widetilde{\Psi}(x) = \boldsymbol{S}_0(x) \Psi_S(x) \boldsymbol{S}_0^*(x)$$

where $\boldsymbol{S}_0(x)$ *is defined by*

$$\boldsymbol{S}_0(x) = \begin{pmatrix} s_0 & r_0 \\ -\dfrac{x_{-1}s_0}{y_{-1}} & \dfrac{x - x_{-1}r_0}{y_{-1}} \end{pmatrix}$$

and

$$\Psi_S(x) = \frac{y_0}{s_0} \frac{\Psi(x)}{x} + \left[\begin{pmatrix} 1/s_0 & 0 \\ 0 & 1/r_0 \end{pmatrix} - \frac{y_0}{s_0} M_{-1} \right] \delta_0(x)$$

LU DARBOUX TRANSFORMATION

Theorem.
LU case: *Let* $\Psi(x)$ *the spectral matrix associated with* $P = \widetilde{P}_L \widetilde{P}_U$ *and consider the Darboux transformation* $\widehat{P} = \widetilde{P}_U \widetilde{P}_L$. *Assume that* $M_{-1} = \int_{-1}^{1} \Psi(x) x^{-1} dx$ *is well-defined. Then the spectral matrix associated with* $\widehat{P}$ *is given by*

$$\widehat{\Psi}(x) = \boldsymbol{T}_0(x) \Psi_T(x) \boldsymbol{T}_0^*(x)$$

where $\boldsymbol{T}_0(x)$ *is defined by*

$$\boldsymbol{T}_0(x) = \begin{pmatrix} \dfrac{x - \tilde{r}_0 \tilde{x}_{-1}}{\tilde{s}_0} & -\dfrac{\tilde{r}_0 \tilde{y}_{-1}}{\tilde{s}_0} \\ \tilde{x}_{-1} & \tilde{y}_{-1} \end{pmatrix}$$

and

$$\Psi_T(x) = \frac{\tilde{s}_0}{\tilde{y}_0} \frac{\Psi(x)}{x} + \left[\frac{\hat{a}_{-1}}{\hat{c}_0} \begin{pmatrix} 1/\tilde{x}_{-1} & 0 \\ 0 & 1/\tilde{y}_{-1} \end{pmatrix} - \frac{\tilde{s}_0}{\tilde{y}_0} M_{-1} \right] \delta_0(x)$$

AN EXAMPLE: RANDOM WALK ON Z

We have that the transition probabilities are constant. Then, if $a \leq (1-\sqrt{c})^2$, we have

$$H = \frac{1}{2}\left(1+c-a+\sqrt{(1+c-a)^2-4c}\right), \quad H' = \frac{1}{2}\left(1+c-a-\sqrt{(1+c-a)^2-4c}\right)$$

Therefore for every parameter y_0 such that $H' \leq y_0 \leq H$ we have a stochastic UL factorization.

The spectral matrix associated with P is given by

$$\Psi(x) = \frac{1}{\pi\sqrt{(x-\sigma_-)(\sigma_+-x)}}\begin{pmatrix} 1 & \dfrac{x-b}{2c} \\ \dfrac{x-b}{2c} & a/c \end{pmatrix}, \quad x \in [\sigma_-, \sigma_+], \quad \sigma_\pm = 1 - \left(\sqrt{a} \mp \sqrt{c}\right)^2.$$

A straightforward computation gives that the spectral matrix for the Darboux transformation is given by

$$\widetilde{\Psi}(x) = \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}\left[\widetilde{A} + \widetilde{B}x + \widetilde{C}x^2\right] + \widetilde{\boldsymbol{M}}\delta_0(x),$$

where

$$\widetilde{A} = \frac{(H'-y_0)(H-y_0)}{s_0 y_0}\begin{pmatrix} 1 & -x_{-1}/y_{-1} \\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2 \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} 1 & -\dfrac{by_0}{2cy_{-1}} \\ -\dfrac{by_0}{2cy_{-1}} & \dfrac{(y_0 b - c(1-c))x_{-1}^2}{acy_{-1}^2} \end{pmatrix}, \quad \widetilde{C} = \frac{y_0}{2cy_{-1}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

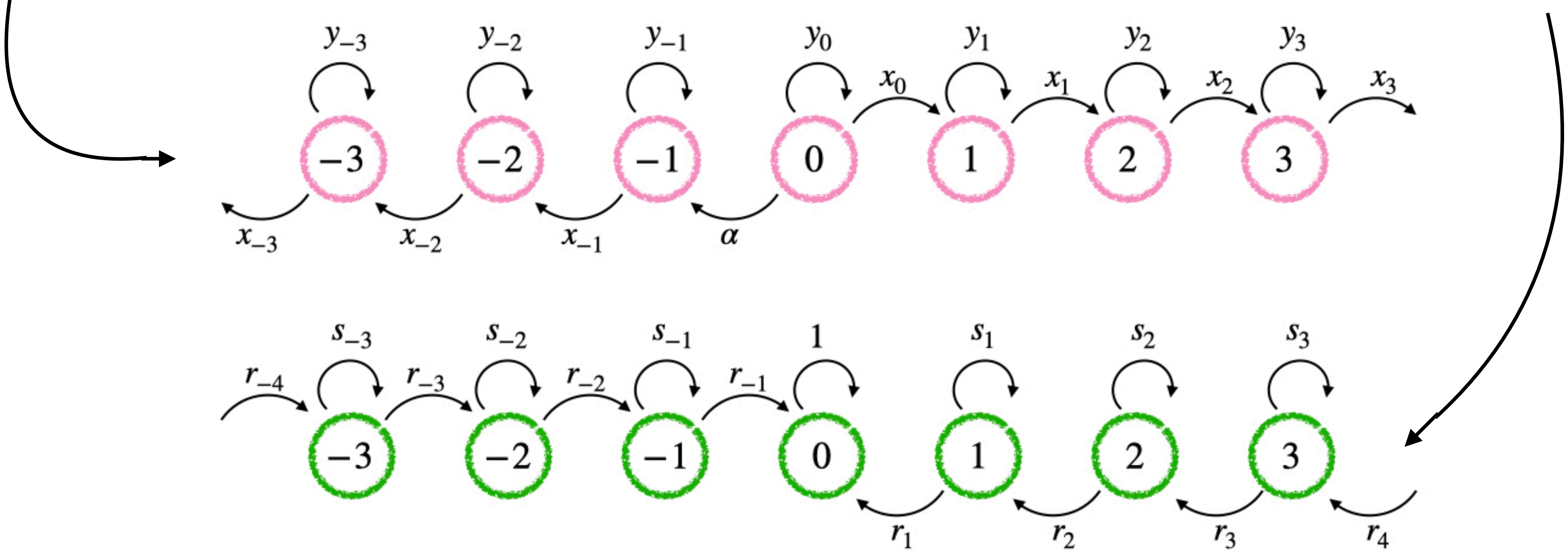
$$\widetilde{\boldsymbol{M}} = \frac{(y_0-H')(H-y_0)}{s_0 y_0\sqrt{\sigma_-\sigma_+}}\begin{pmatrix} 1 & -x_{-1}/y_{-1} \\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2 \end{pmatrix}.$$

REFLECTING-ABSORBING FACTORIZATION

Now, instead of considering a UL or LU factorization (where the upper bidiagonal matrix represents a pure-birth chain and the lower bidiagonal matrix represents a pure-death chain) we consider a different factorization, given by a *reflecting* birth-death chain from the state 0 and an *absorbing* birth-death chain to the state 0.

Then, consider $P = P_R P_A$ where

$$P_R = \left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & x_{-2} & y_{-2} & 0 & & & & \\ & & x_{-1} & y_{-1} & 0 & & & \\ \hline & & & \alpha & y_0 & x_0 & & \\ & & & & 0 & y_1 & x_1 & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right), \quad P_A = \left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & 0 & s_{-2} & r_{-2} & & & & \\ & & 0 & s_{-1} & r_{-1} & & & \\ \hline & & & 0 & 1 & 0 & & \\ & & & & r_1 & s_1 & 0 & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right).$$



REFLECTING-ABSORBING FACTORIZATION

Remarks.

1. If the state space is $\mathcal{S} = \mathbb{N}_0$ then the reflecting-absorbing factorization is just an UL stochastic factorization.

2. We need to introduce a new parameter α to connect the reflecting birth-death chain from the state 0 to the state -1.

3. $P = P_R P_A$ does not preserve the UL or LU structure, but, after the "folding trick", if we write $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$, then we have

$$\boldsymbol{P}_R = \left(\begin{array}{cc|cc|cc} y_0 & \alpha & x_0 & 0 & & \\ 0 & y_{-1} & 0 & x_{-1} & & \\ \hline & & y_1 & 0 & x_1 & 0 \\ & & 0 & y_{-2} & 0 & x_{-2} \\ \hline & & & & \ddots & & \ddots \end{array}\right), \quad \boldsymbol{P}_A = \left(\begin{array}{cc|cc|c} 1 & 0 & & & \\ r_{-1} & s_{-1} & & & \\ \hline r_1 & 0 & s_1 & 0 & \\ 0 & r_{-2} & 0 & s_{-2} & \\ \hline & & \ddots & & \ddots \end{array}\right)$$

which *preserves the block UL structure.*

Theorem. *We have now TWO free parameters α and x_0. Assuming that the previous continued fractions H, H' are convergent and $H + H' \leq 1$, then the reflecting-absorbing factorization is possible if and only if*

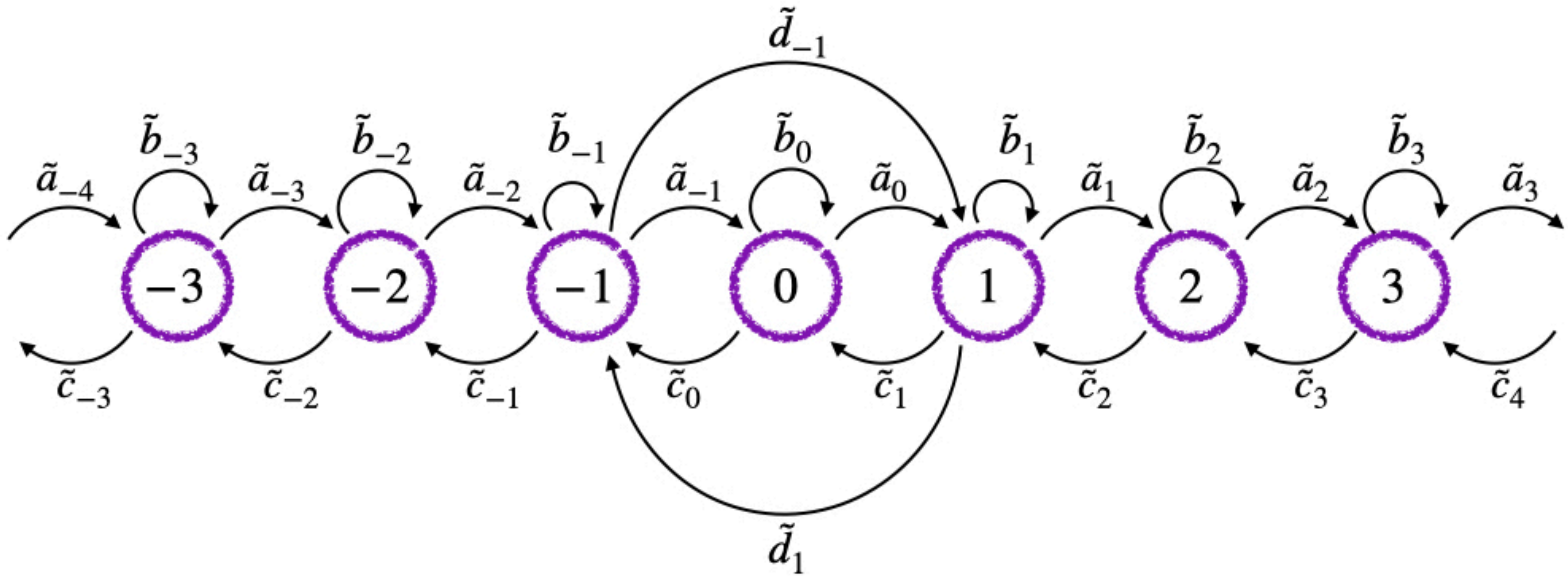
$$\alpha \geq H', \quad \text{and} \quad x_0 \geq H$$

DARBOUX TRANSFORMATION

The Darboux transformation $\widetilde{P} = P_A P_R$ is *not* a birth-death chain since now we have extra transitions between states -1 and 1, given by

$$\tilde{d}_1 = \mathbb{P}(\widetilde{X}_1 = -1 | \widetilde{X}_0 = 1) = r_1 \alpha, \quad \tilde{d}_{-1} = \mathbb{P}(\widetilde{X}_1 = 1 | \widetilde{X}_0 = -1) = r_{-1} x_0.$$

The diagram of $\widetilde{P}$ is given by



DARBOUX TRANSFORMATION

Using the UL block factorization of $\widetilde{\boldsymbol{P}}$ after the "folding", we can compute the spectral matrix associated with $\widetilde{P}$.

Theorem. *Let $\Psi(x)$ the spectral matrix associated with $P = P_R P_A$ and consider the Darboux transformation $\widehat{P} = P_A P_R$. Assume that $M_{-1} = \int_{-1}^{1} \Psi(x) x^{-1} dx$ is well-defined. Then the spectral matrix associated with $\widehat{P}$ is given by*

$$\widetilde{\Psi}(x) = S_0 \Psi_U(x) S_0^T$$

where the constant matrix S_0 is defined by

$$S_0 = \begin{pmatrix} 1 & 0 \\ r_{-1} & s_{-1} \end{pmatrix}$$

and

$$\Psi_U(x) = \frac{\Psi(x)}{x} + \left[\frac{1}{y_0} \begin{pmatrix} 1 & -r_{-1}/s_{-1} \\ -r_{-1}/s_{-1} & (r_{-1}/s_{-1})^2 + y_0 r_{-1}/\alpha s_{-1}^2 \end{pmatrix} - M_{-1} \right] \delta_0(x)$$

Similar considerations can be obtained if we consider absorbing-reflecting factorizations, but now we have to start with an "almost" birth-death chain with extra transitions between states -1 and 1.

AN EXAMPLE: RANDOM WALK ON Z

Again, if $a \leq (1-\sqrt{c})^2$ then we have

$$H = \frac{1}{2}\left(1+c-a+\sqrt{(1+c-a)^2-4c}\right), \quad H' = \frac{1}{2}\left(1+c-a-\sqrt{(1+c-a)^2-4c}\right)$$

and the reflecting-absorbing factorization will be possible if and only if $\alpha \geq H'$ and $x_0 \geq H$.

Recall that the spectral matrix associated with P is given by

$$\Psi(x) = \frac{1}{\pi\sqrt{(x-\sigma_-)(\sigma_+-x)}}\begin{pmatrix} 1 & \dfrac{x-b}{2c} \\ \dfrac{x-b}{2c} & a/c \end{pmatrix}, \quad x \in [\sigma_-, \sigma_+], \quad \sigma_\pm = 1-\left(\sqrt{a}\mp\sqrt{c}\right)^2.$$

A straightforward computation gives that the spectral matrix for the Darboux transformation $\widetilde{P}$ (with transitions between the states -1 and 1) is given by

$$\widetilde{\Psi}(x) = \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}[\widetilde{A}+\widetilde{B}x] + \widetilde{M}_{-1}\delta_0$$

where

$$\widetilde{A} = \begin{pmatrix} 1 & \dfrac{2\alpha+H-H'-1}{2\alpha} \\ \dfrac{2\alpha+H-H'-1}{2\alpha} & \dfrac{(\alpha-H')(H+\alpha-1)}{\alpha^2} \end{pmatrix}, \quad \widetilde{B} = \frac{1}{2\alpha}\begin{pmatrix} 0 & 1 \\ 1 & \dfrac{2(\alpha-c)}{\alpha} \end{pmatrix}$$

$$\widetilde{M}_{-1} = \begin{pmatrix} \dfrac{x_0-H+\alpha-H'}{y_0(1-H-H')} & \dfrac{H'-\alpha}{\alpha(1-H-H')} \\ \dfrac{H'-\alpha}{\alpha(1-H-H')} & -\dfrac{(\alpha-H')(1-H-\alpha)}{\alpha^2(1-H-H')} \end{pmatrix}$$

THANK YOU