Bivariate Markov processes and matrix orthogonality

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Outline

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1-D Markov processes

Let $(\Omega, \mathcal{F}, \Pr)$ a probability space, a (1-D) Markov process with state space $\mathcal{S} \subset \mathbb{R}$ is a collection of $\mathcal{S}$-valued random variables $\{X_t : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) such that

$$\Pr(X_{t_0+t_1} \leq y | X_{t_0} = x, X_\tau, 0 \leq \tau < t_0) = \Pr(X_{t_0+t_1} \leq y | X_{t_0} = x)$$

for all $t_0, t_1 > 0$. This is what is called the Markov property.

The main goal is to find a description of the transition probabilities

$$P(t; x, y) \equiv \Pr(X_t = y | X_0 = x), \quad x, y \in \mathcal{S}$$

if $\mathcal{S}$ is discrete or the transition density

$$p(t; x, y) \equiv \frac{\partial}{\partial y} \Pr(X_t \leq y | X_0 = x), \quad x, y \in \mathcal{S}$$

if $\mathcal{S}$ is continuous.

1-D Markov processes

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1-D Markov processes

On the set $\mathfrak{B}(\mathcal{S})$ of all real-valued, bounded, Borel measurable functions define the transition operator

$$(T_t f)(x) = E[f(X_t)|X_0 = x], \quad t \geq 0$$

The family $\{T_t, t > 0\}$ has the *semigroup property* $T_{s+t} = T_s T_t$
The infinitesimal operator $\mathcal{A}$ of the family $\{T_t, t > 0\}$ is

$$(\mathcal{A}f)(x) = \lim_{s \downarrow 0} \frac{(T_s f)(x) - f(x)}{s}$$

and $\mathcal{A}$ determines all $\{T_t, t > 0\}$.
There are 3 important cases related to orthogonal polynomials

1. Random walks: $\mathcal{S} = \{0, 1, 2, \dots\}$, $\mathcal{T} = \{0, 1, 2, \dots\}$.
2. Birth and death processes: $\mathcal{S} = \{0, 1, 2, \dots\}$, $\mathcal{T} = [0, \infty)$.
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Random walks

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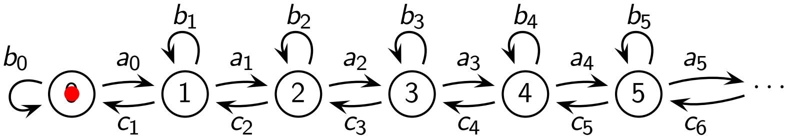
$$\Pr(X_{n+1} = j | X_n = i) = 0 \quad \text{for} \quad |i - j| > 1$$

i.e. a tridiagonal transition probability matrix (stochastic)

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_i \geq 0, a_i, c_i > 0, \quad a_i + b_i + c_i = 1$$

$$\Rightarrow \mathcal{A}f(i) = a_i f(i+1) + b_i f(i) + c_i f(i-1), \; f \in \mathfrak{B}(\mathcal{S})$$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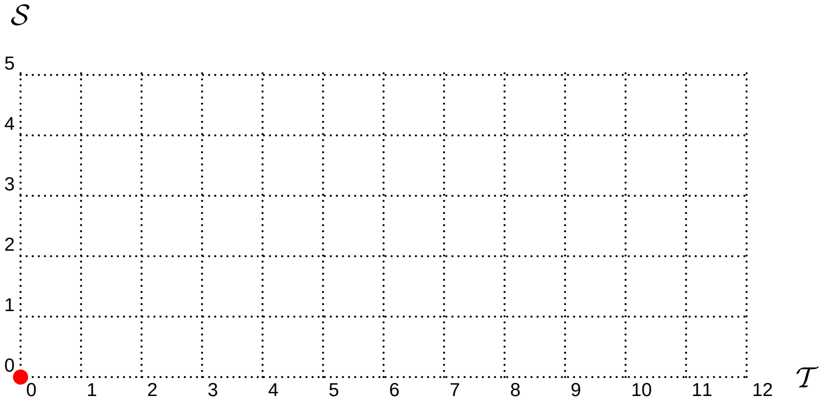
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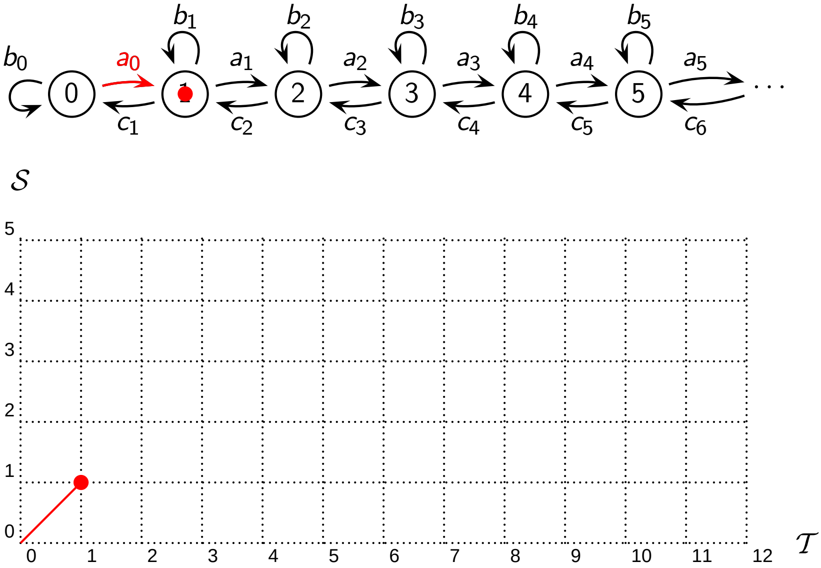
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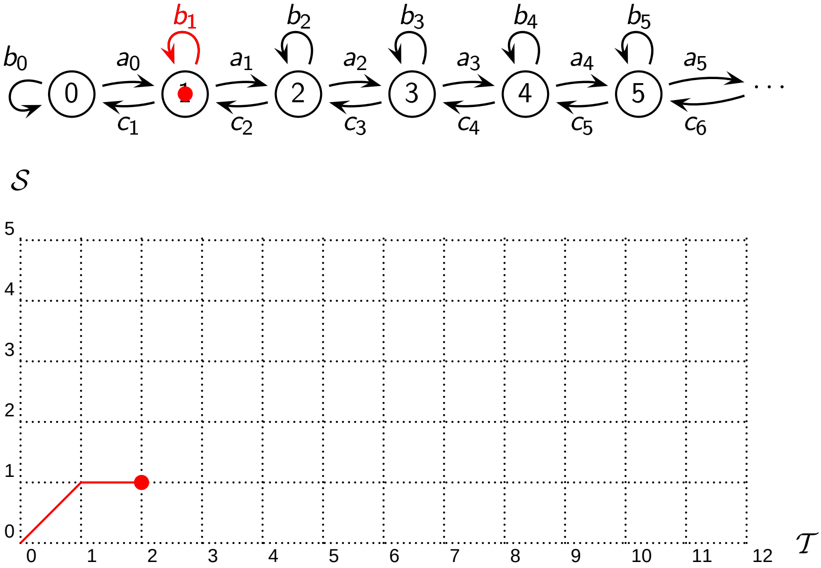
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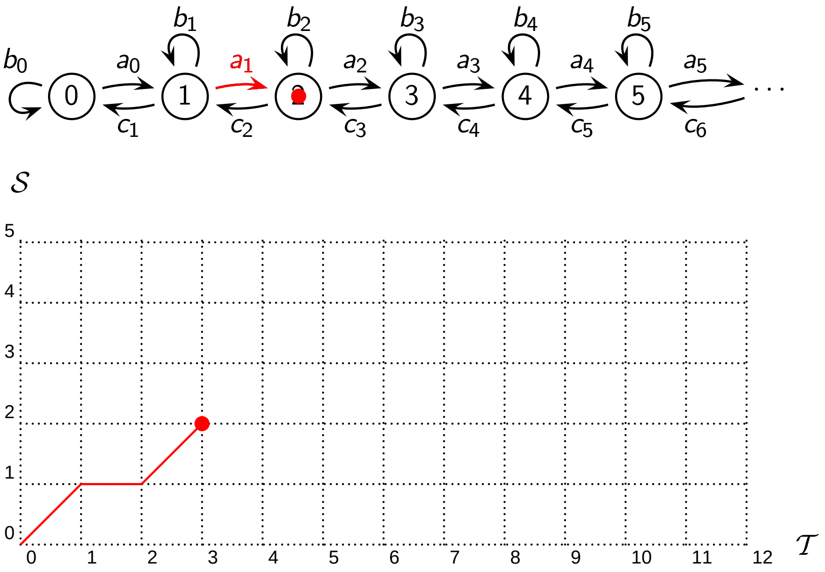
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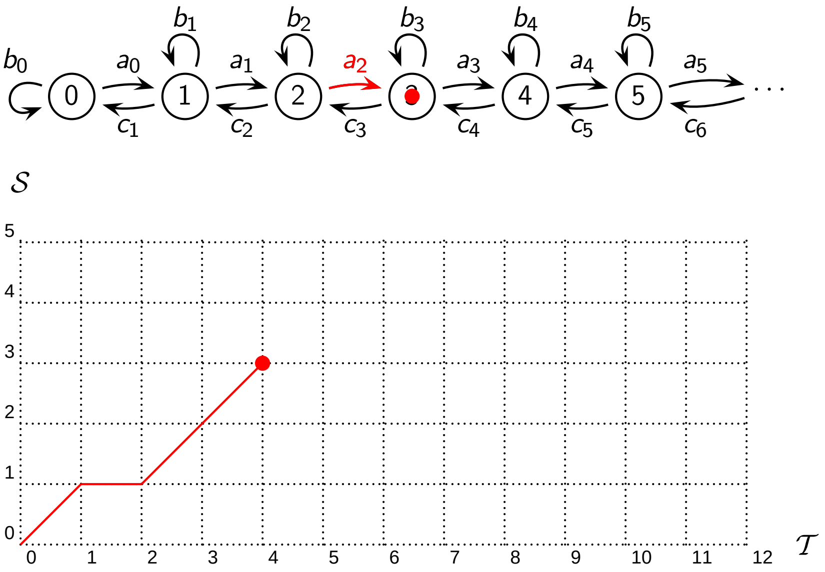
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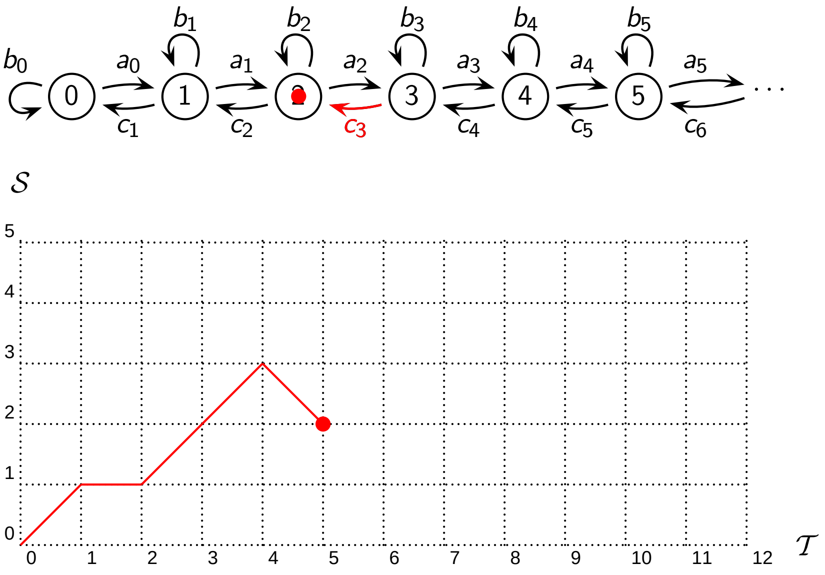
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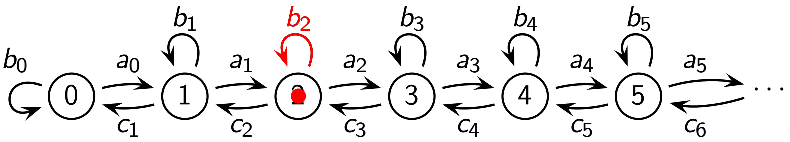
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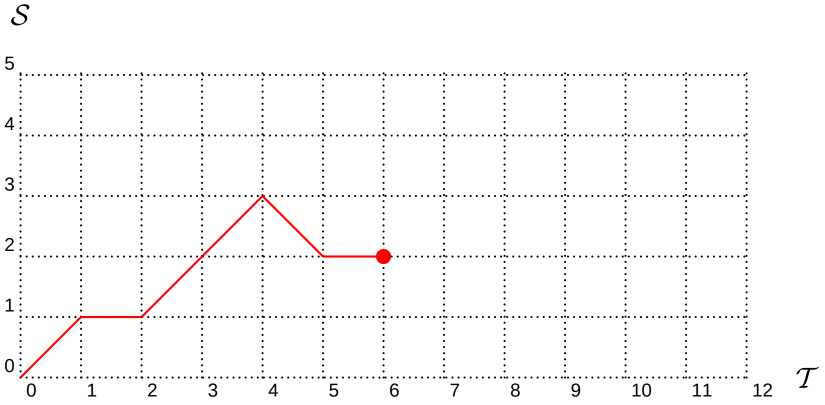
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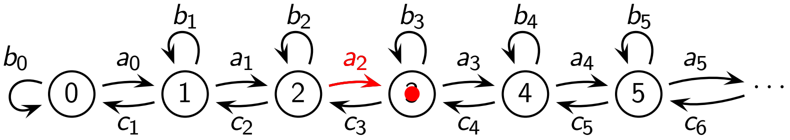
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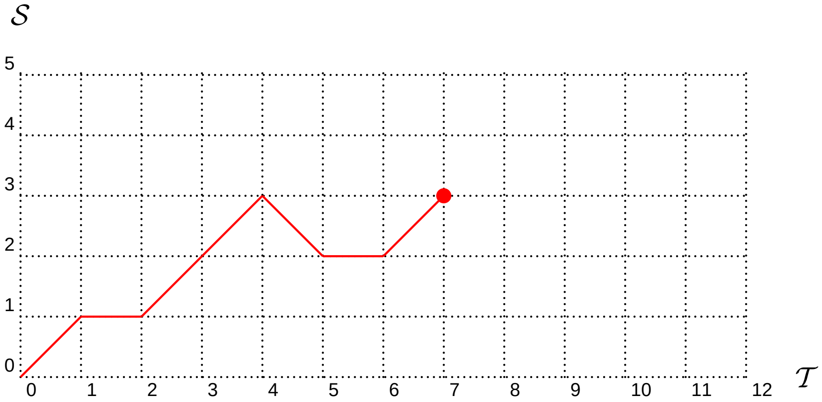
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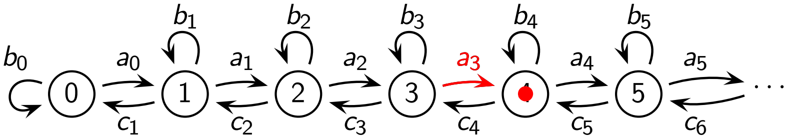
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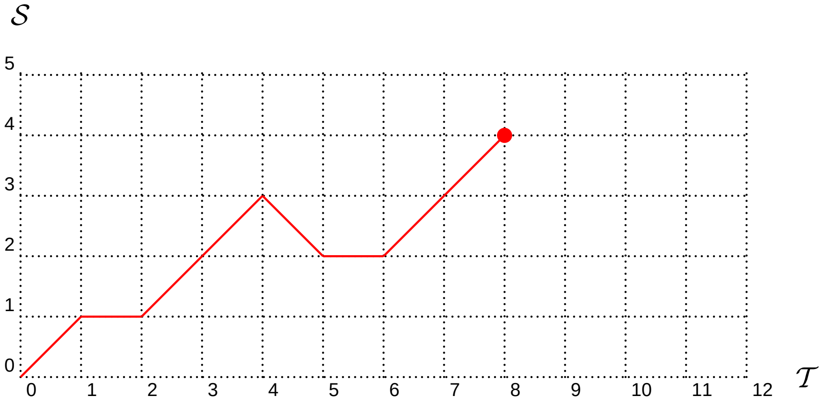
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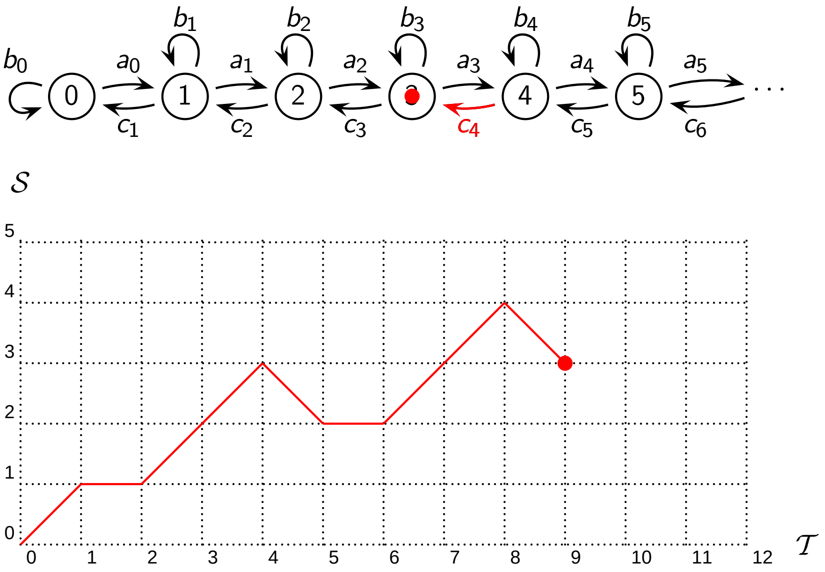

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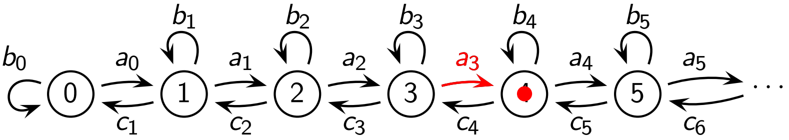
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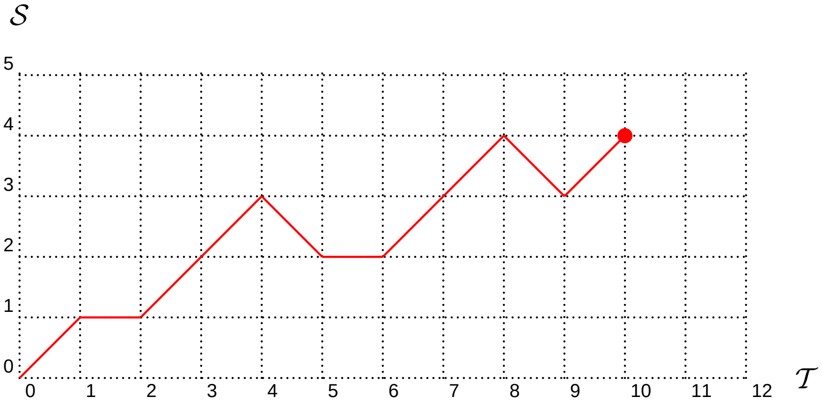
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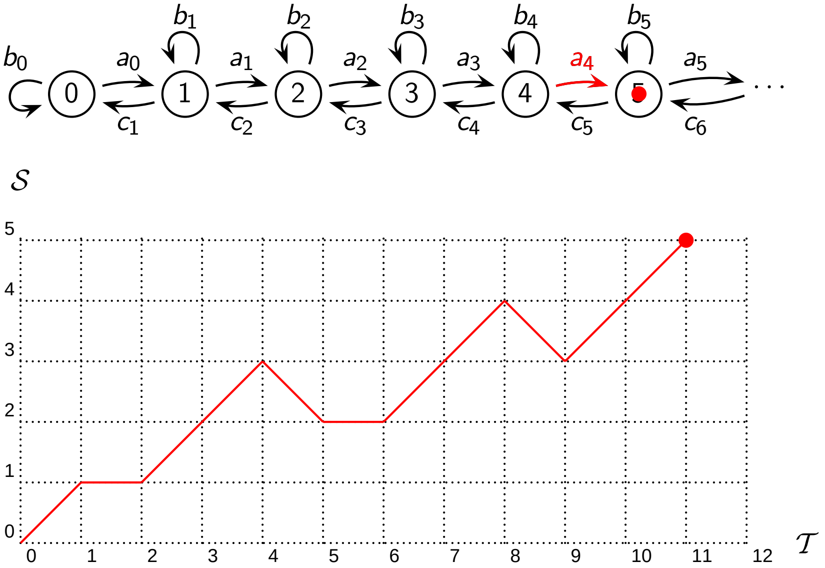
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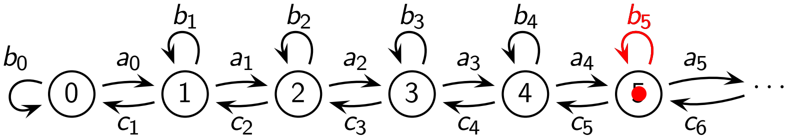
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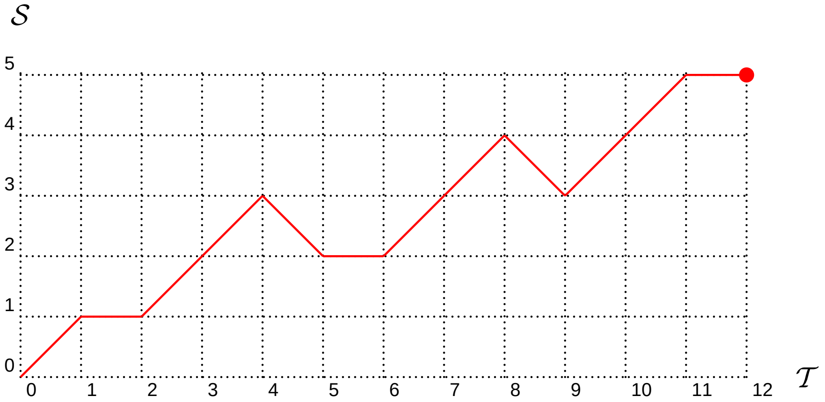
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Birth and death processes

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- $P_{i,i+1}(h) = \lambda_i h + o(h)$, as $h \downarrow 0, \lambda_i > 0, i \in \mathcal{S}$;
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$P(t)$ satisfies the backward and forward equation $P'(t) = \mathcal{A}P(t)$ and $P'(t) = P(t)\mathcal{A}$ with initial condition $P(0) = I$ where $\mathcal{A}$ is the tridiagonal infinitesimal operator matrix

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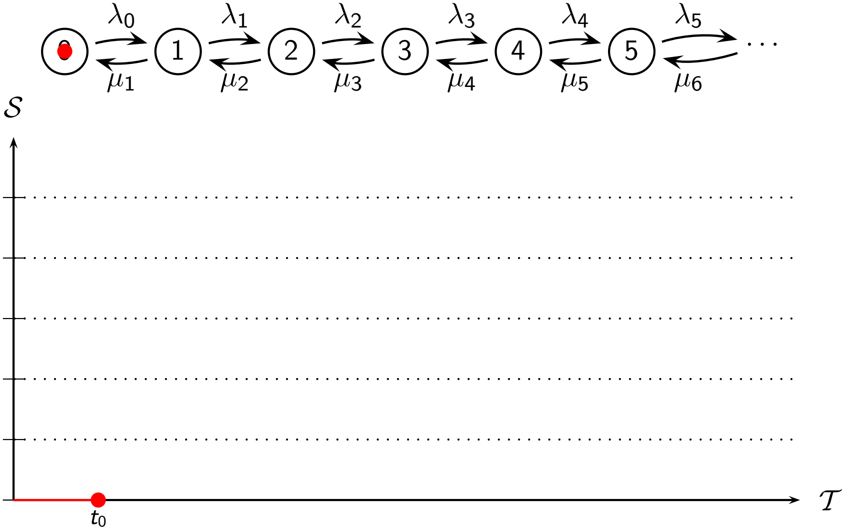
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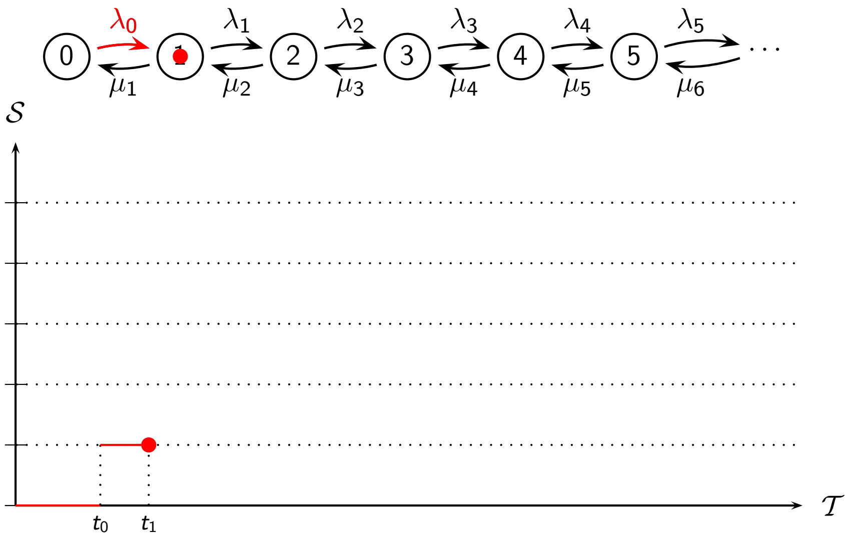
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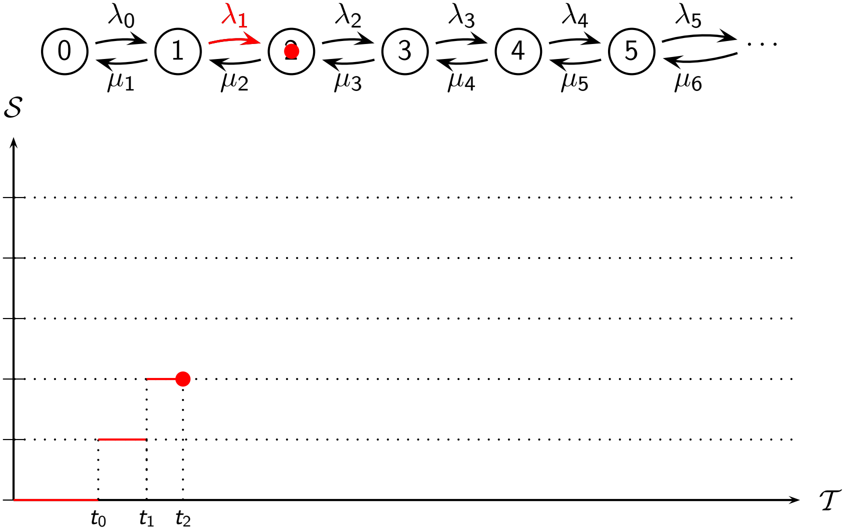
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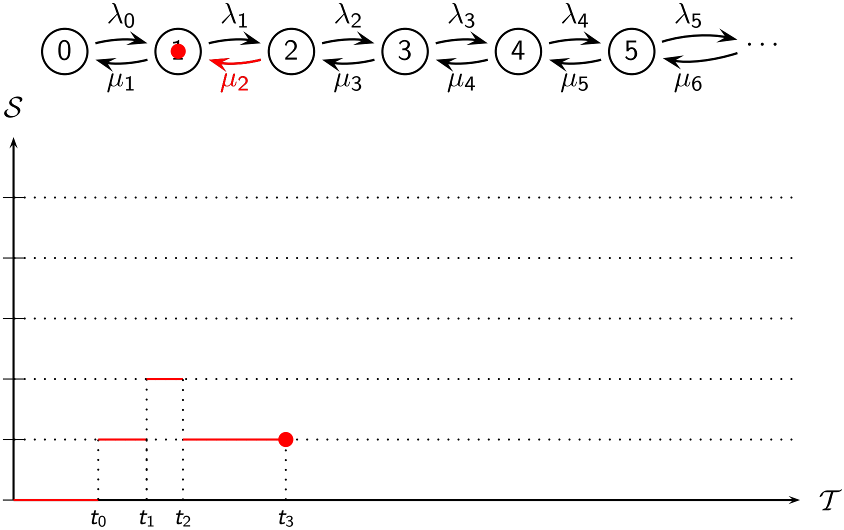
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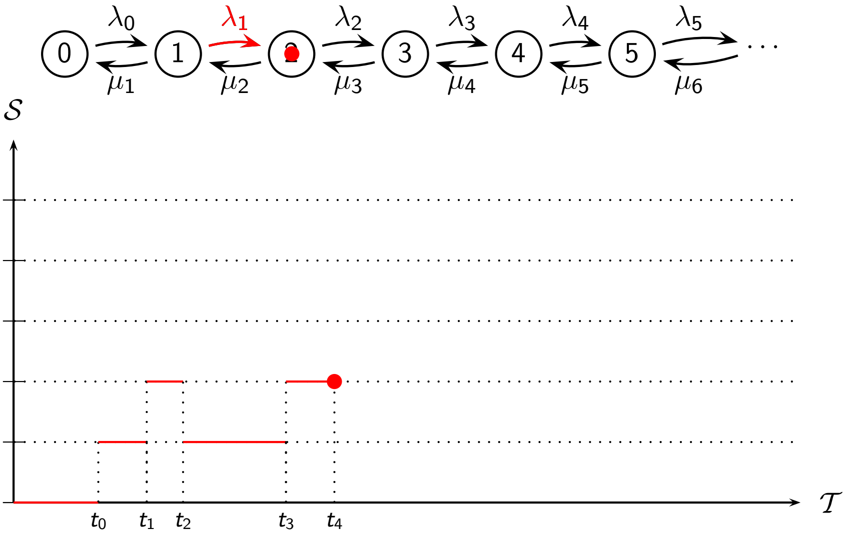
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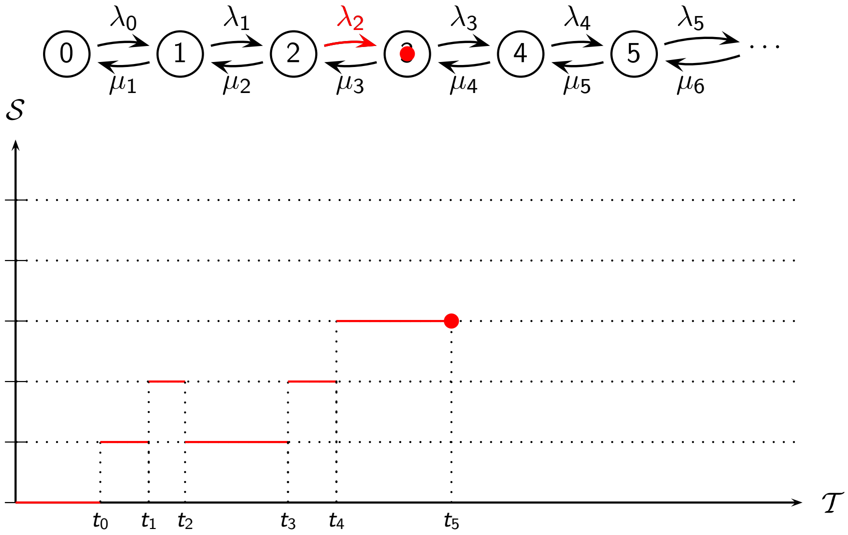
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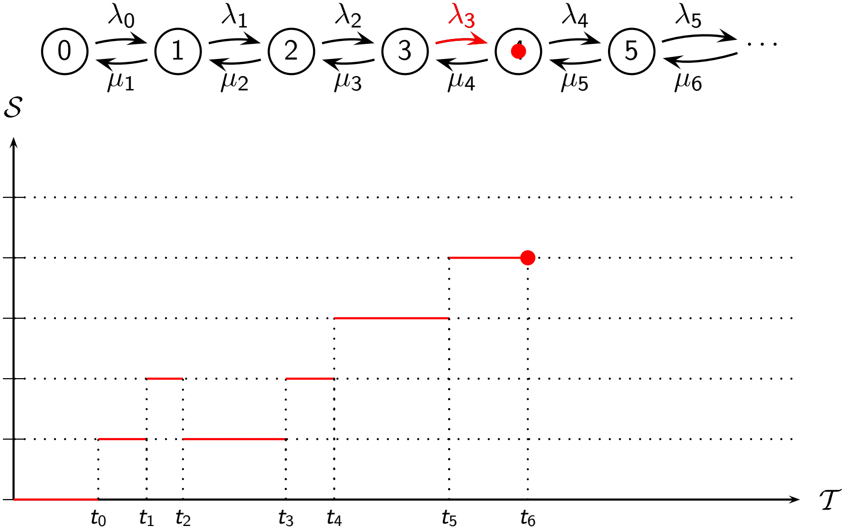
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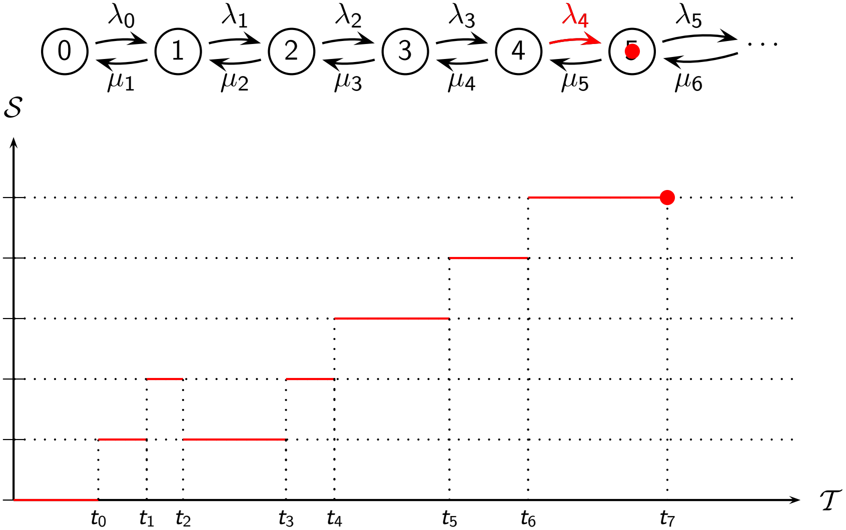
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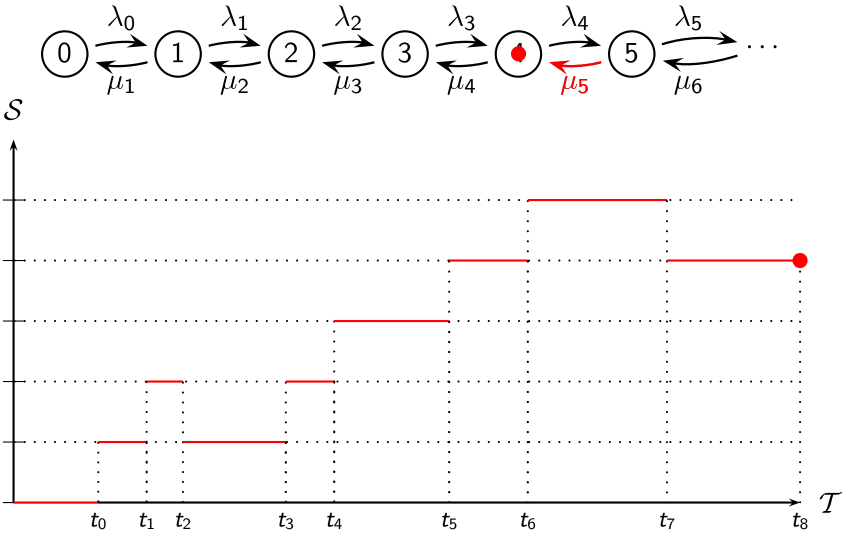
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Diffusion processes

We have $\mathcal{S} = (a, b), -\infty \leq a < b \leq \infty$, $\mathcal{T} = [0, \infty)$.

Suppose that as $t \downarrow 0$

- $E[X_{s+t} - X_s | X_s = x] = t\tau(x) + o(t)$;
- $E[(X_{s+t} - X_s)^2 | X_s = x] = t\sigma^2(x) + o(t)$;
- $E[|X_{s+t} - X_s|^3 | X_s = x] = o(t)$.

$\mu(x)$ is the drift coefficient and $\sigma^2(x) > 0$ the diffusion coefficient.

Infinitesimal generator

$$\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx}$$

The transition density satisfies the backward differential equation

$$\frac{\partial}{\partial t}p(t; x, y) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 p(t; x, y)}{\partial x^2} + \tau(x)\frac{\partial p(t; x, y)}{\partial x}$$

and the forward or evolution differential equation

$$\frac{\partial}{\partial t}p(t; x, y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y)p(t; x, y)] - \frac{\partial}{\partial y}[\tau(y)p(t; x, y)]$$

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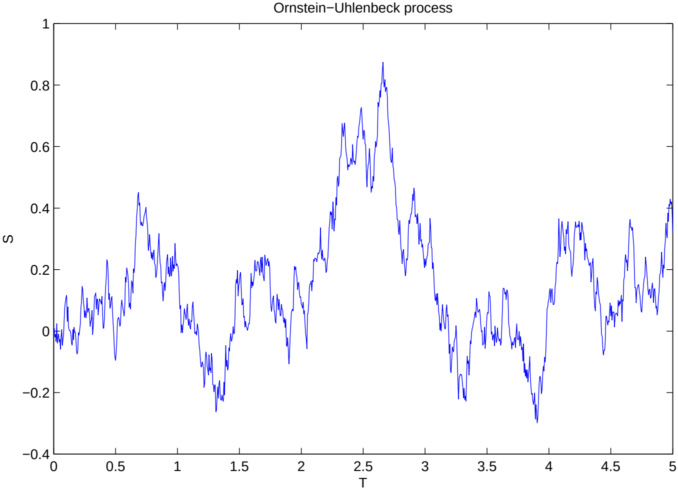
Ornstein-Uhlenbeck diffusion process with $\mathcal{S} = \mathbb{R}$ and $\sigma^2(x) = 1$, $\tau(x) = -x$.



SPECTRAL METHODS

Given a infinitesimal operator $\mathcal{A}$, if we can find a measure $\omega(x)$ associated with $\mathcal{A}$, and a set of orthogonal eigenfunctions $f(i,x)$ such that

$$\mathcal{A}f(i,x) = \lambda(i,x)f(i,x),$$

then it is possible to find spectral representations of

- Transition probabilities $P_{ij}(t)$ (discrete case) or densities $p(t;x,y)$ (continuous case).
- Invariant measure or distribution $\boldsymbol{\pi} = (\pi_j)$ (discrete case) with

$$\pi_j = \lim_{t\to\infty} P_{ij}(t)$$

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Transition probabilities

1. Random walks: $\mathcal{S} = \mathcal{T} = \{0, 1, 2, \ldots\}$

$$f(i,x) = q_i(x), \quad \lambda(i,x) = x, \quad i \in \mathcal{S}, \quad x \in [-1,1].$$

$$\Pr(X_n = j | X_0 = i) = P_{ij}^n = \frac{1}{\|q_i\|^2} \int_{-1}^{1} x^n q_i(x) q_j(x) \mathrm{d}\omega(x)$$

2. Birth and death processes: $\mathcal{S} = \{0, 1, 2, \ldots\}$, $\mathcal{T} = [0, \infty)$

$$f(i,x) = q_i(x), \quad \lambda(i,x) = -x, \quad i \in \mathcal{S}, \quad x \in [0,\infty].$$

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3. Diffusion processes: $\mathcal{S} = (a,b) \subseteq \mathbb{R}$, $\mathcal{T} = [0, \infty)$

$$f(i,x) = \phi_i(x), \quad \lambda(i,x) = \alpha_i, \quad i \in \{0, 1, 2, \ldots\}, \quad x \in S.$$

$$p(t; x, y) = \sum_{n=0}^{\infty} e^{\alpha_n t} \phi_n(x) \phi_n(y) \omega(y)$$

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2. Birth-death: a non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots) \geq 0$

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Outline

2-D Markov processes

Now we have a bivariate or 2-component Markov process of the form $\{(X_t, Y_t) : t \in \mathcal{T}\}$ indexed by a parameter set $\mathcal{T}$ (time) and with state space $\mathcal{C} = \mathcal{S} \times \{1, 2, \dots, N\}$, where $\mathcal{S} \subset \mathbb{R}$. The first component is the level while the second component is the phase.
Now the transition probabilities can be written in terms of a matrix-valued function $\mathbf{P}(t; x, A)$, defined for every $t \in \mathcal{T}, x \in \mathcal{S}$, and any Borel set A of $\mathcal{S}$, whose entry (i, j) gives

$$\mathbf{P}_{ij}(t; x, A) = \Pr\{X_t \in A, Y_t = j | X_0 = x, Y_0 = i\}.$$

Every entry must be nonnegative and

$$\mathbf{P}(t; x, A)\mathbf{e}_N \leq \mathbf{e}_N, \quad \mathbf{e}_N = (1, 1, \dots, 1)^T$$

The infinitesimal operator $\mathcal{A}$ is now matrix-valued.

Ideas behind: *random evolutions*
(Griego-Hersh-Papanicolaou-Pinsky-Kurtz...60's and 70's).

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Discrete time quasi-birth-and-death processes

Now we have $\mathcal{C} = \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}$, $\mathcal{T} = \{0, 1, 2, \ldots\}$ and

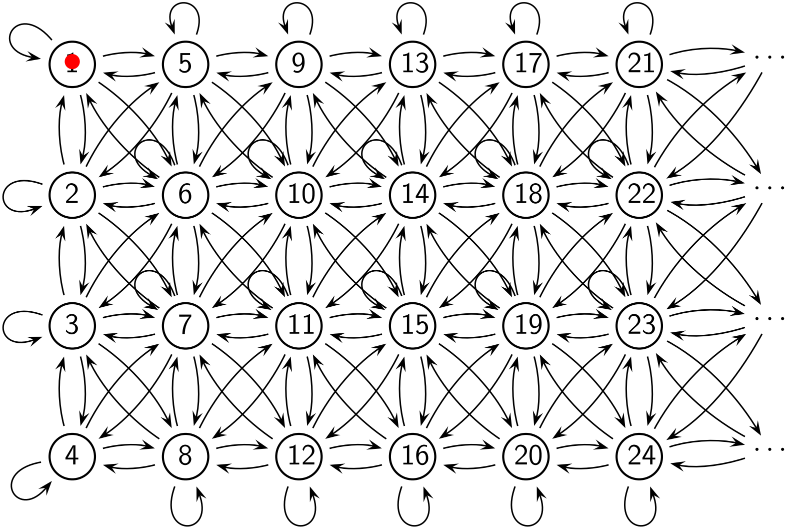
$$(\mathbf{P}_{ii'})_{jj'} = \Pr(X_{n+1} = i, Y_{n+1} = j | X_n = i', Y_n = j') = 0 \quad \text{for} \quad |i - i'| > 1$$

i.e. a $N \times N$ block tridiagonal transition probability matrix

$$\mathbf{P} = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$(A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geq 0, \ \det(A_n), \det(C_n) \neq 0$$
$$\sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1, \ i = 1, \ldots, N$$

$N = 4$ PHASES



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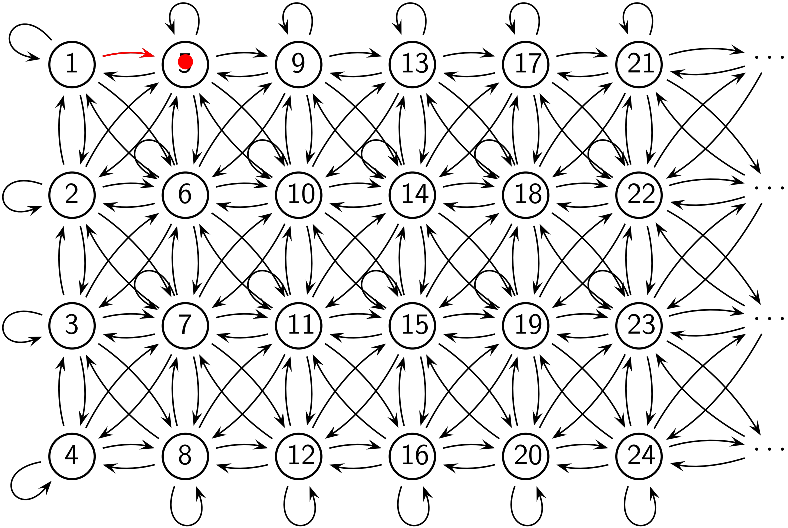

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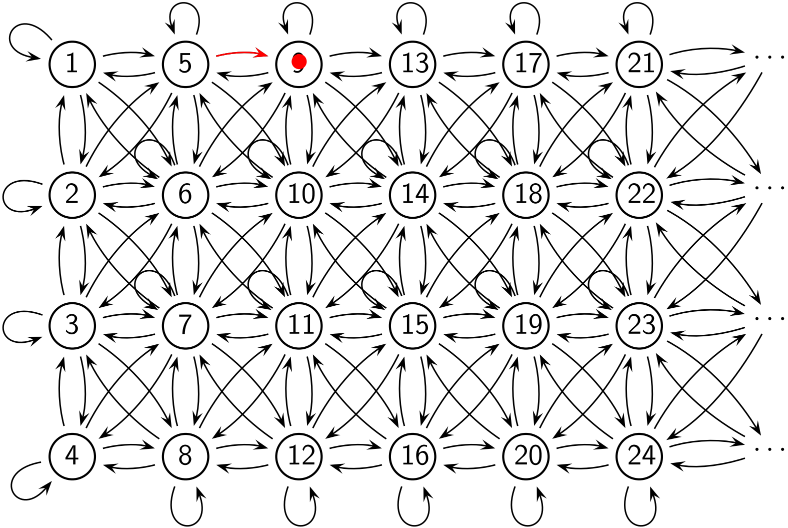

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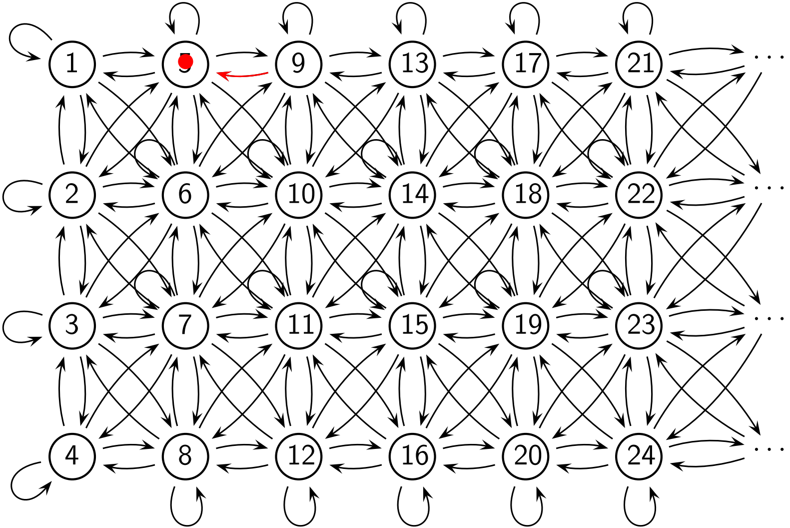

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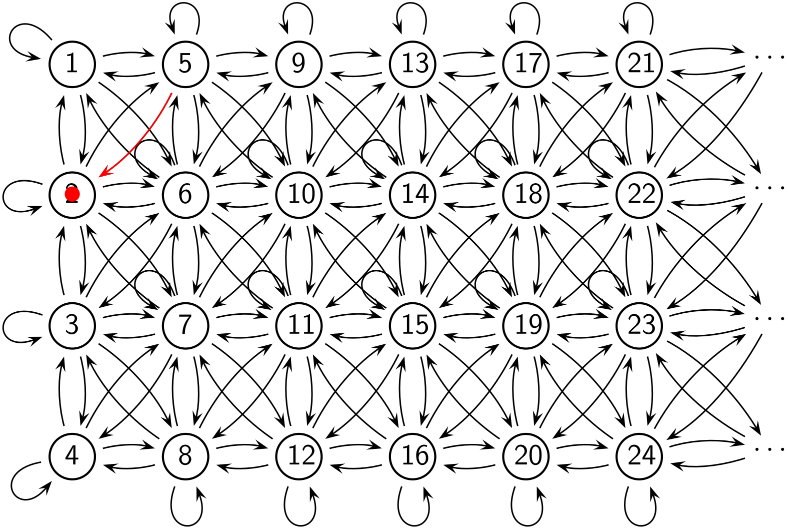

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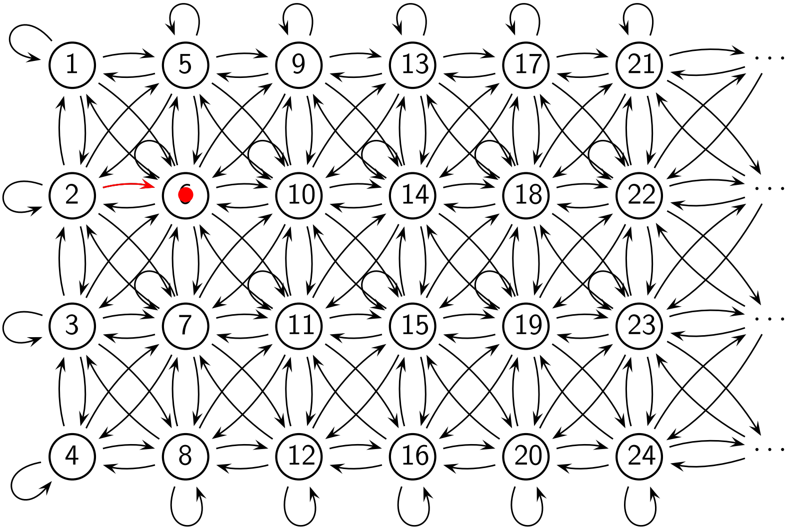

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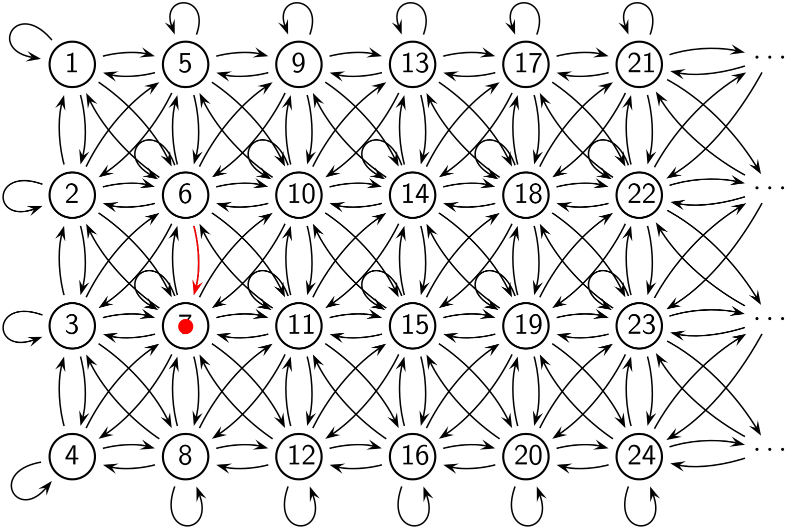

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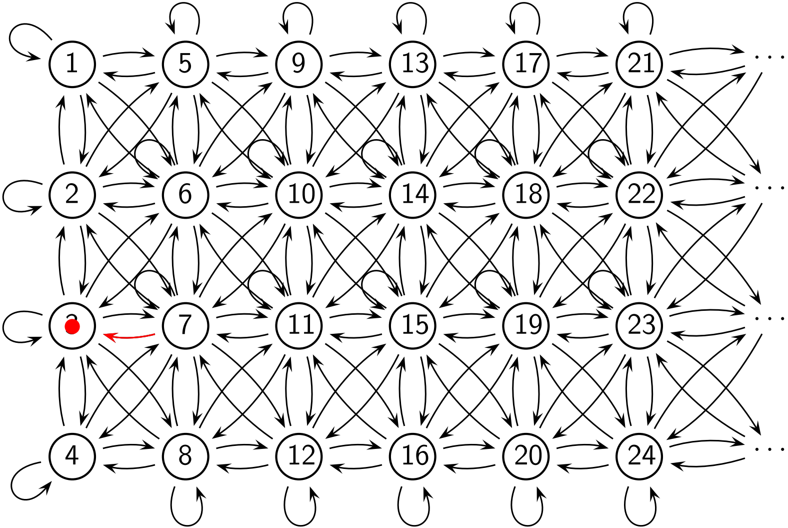

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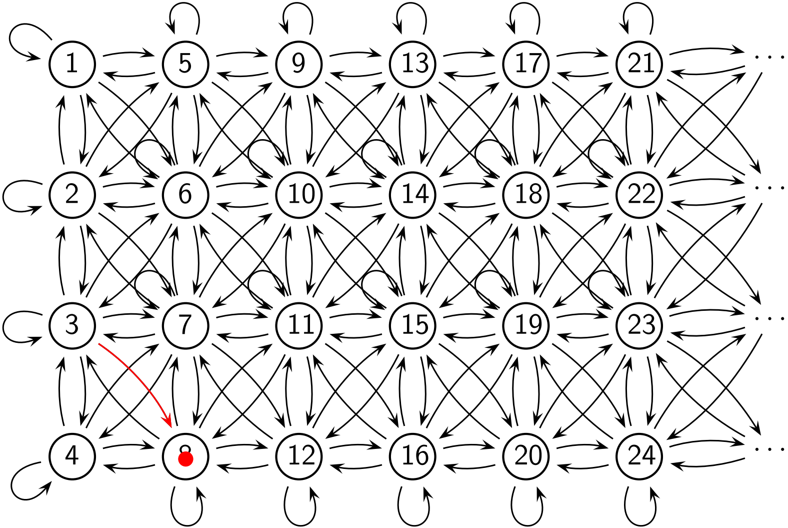

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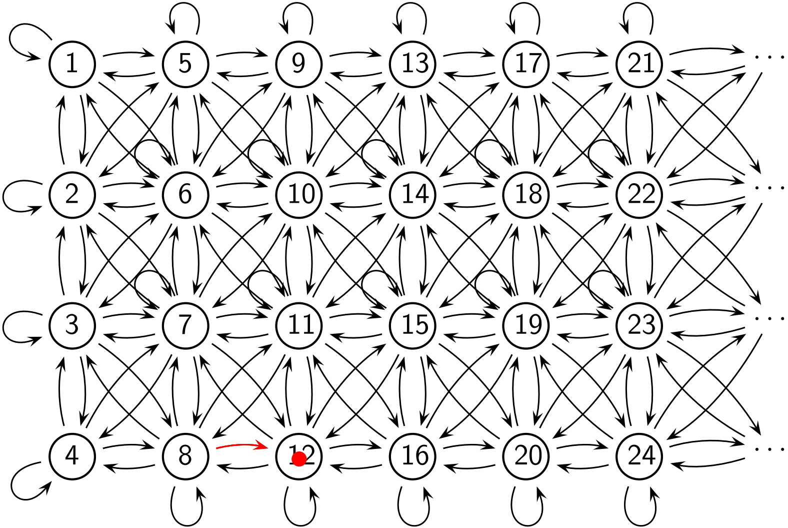

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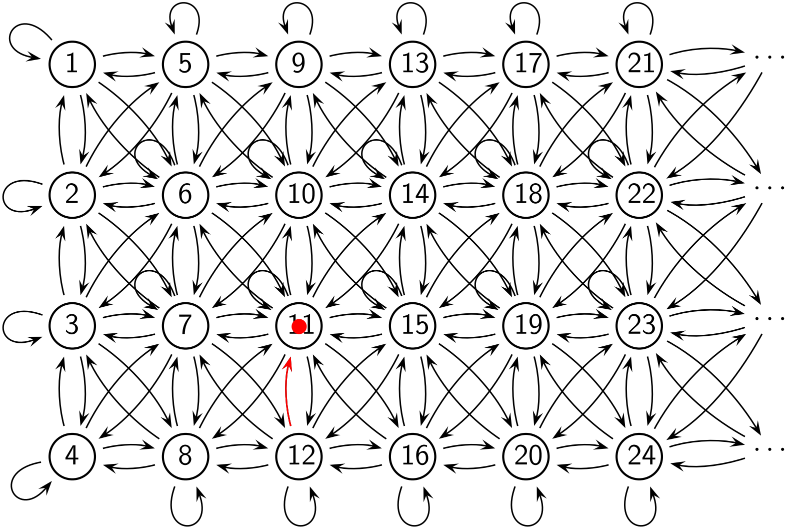

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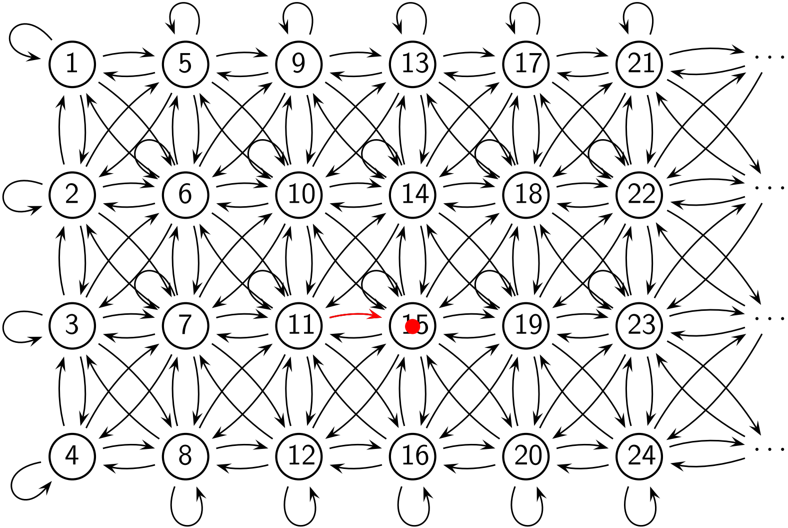

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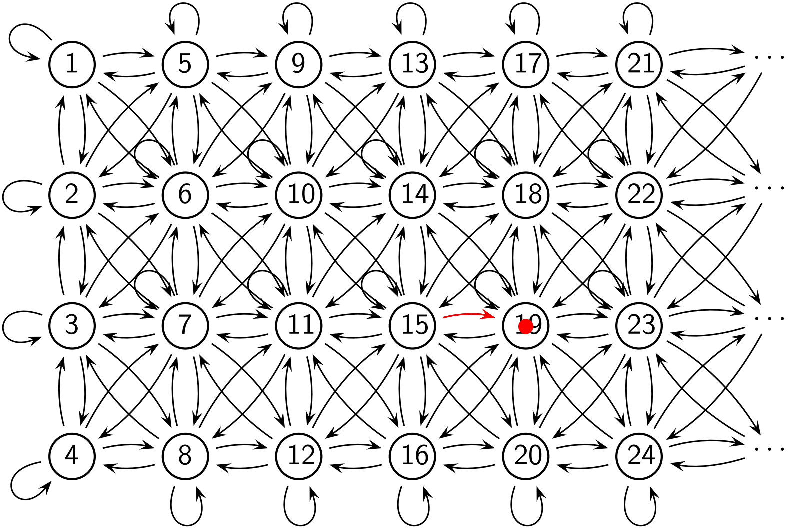

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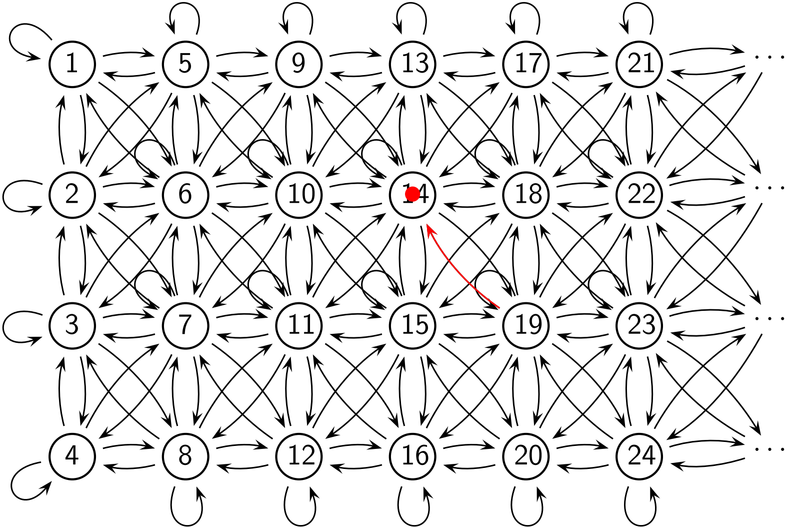

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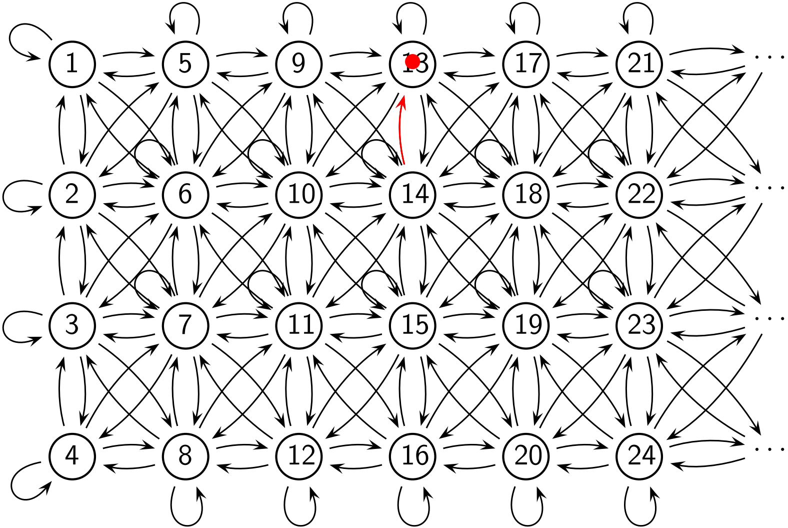

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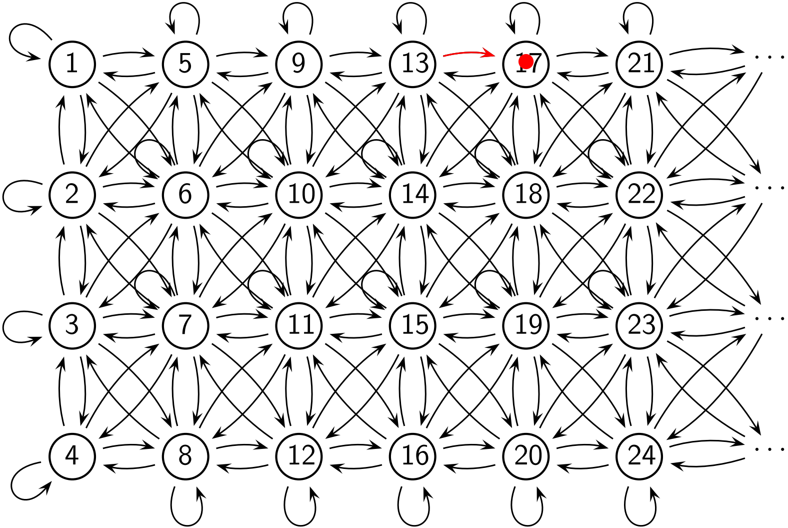

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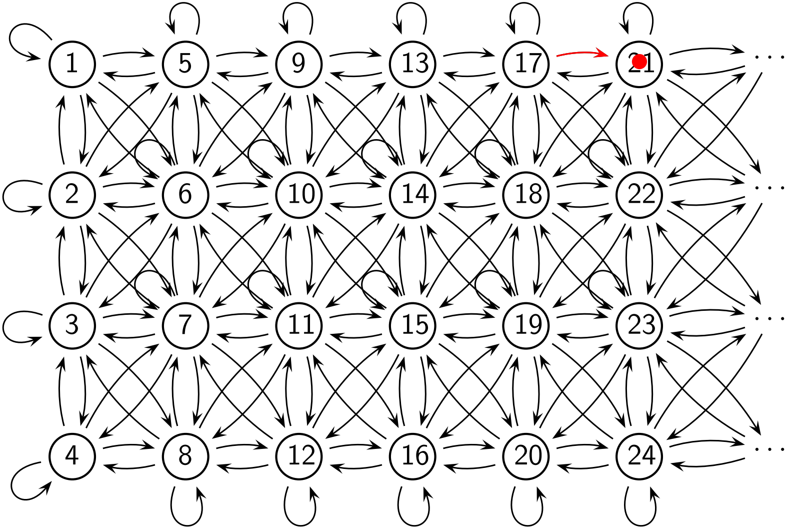

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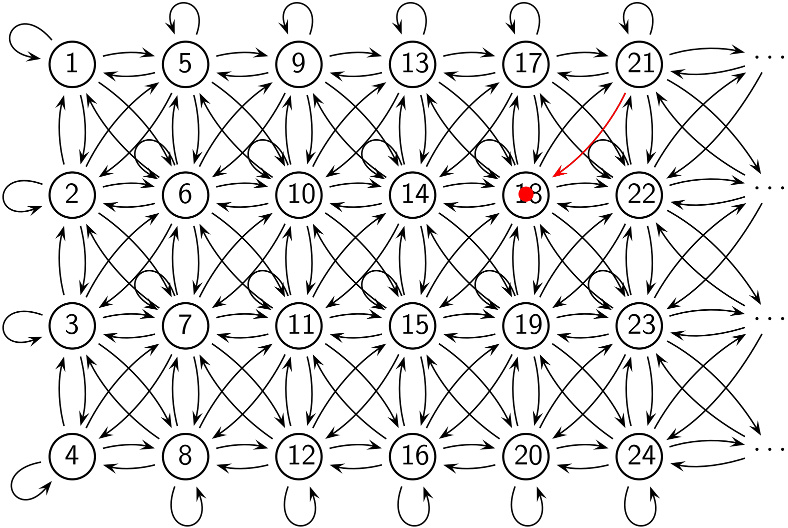

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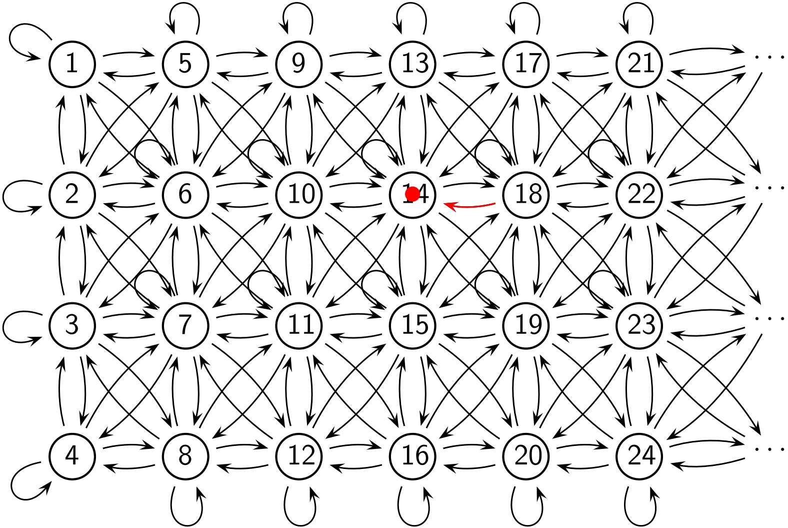

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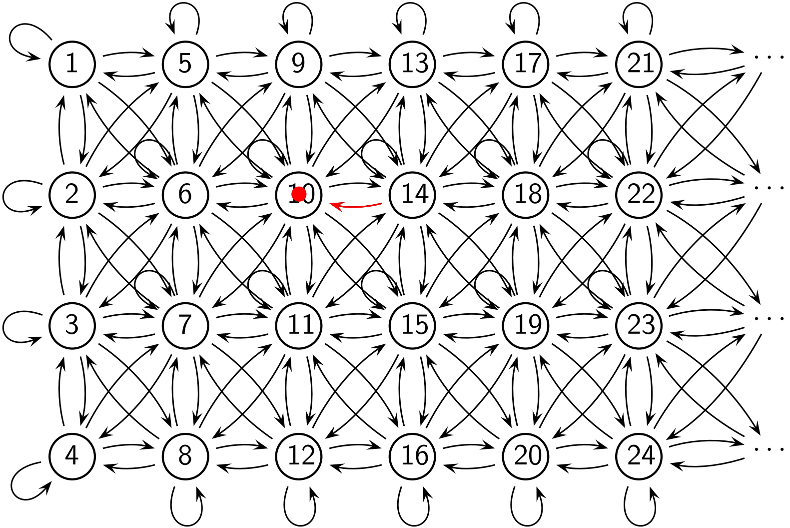

Switching diffusion processes

We have $\mathcal{C} = (a, b) \times \{1, 2, \ldots, N\}$, $\mathcal{T} = [0, \infty)$. The transition probability density is now a matrix which entry (i, j) gives

$$\mathbf{P}_{ij}(t; x, A) = \Pr(X_t \in A, Y_t = j | X_0 = x, Y_0 = i)$$

for any $t > 0$, $x \in (a, b)$ and A any Borel set.

The infinitesimal operator $\mathcal{A}$ is now a matrix-valued differential operator (Berman, 1994)

$$\mathcal{A} = \frac{1}{2}\mathbf{A}(x)\frac{d^2}{dx^2} + \mathbf{B}(x)\frac{d^1}{dx^1} + \mathbf{Q}(x)\frac{d^0}{dx^0}$$

We have that $\mathbf{A}(x)$ and $\mathbf{B}(x)$ are diagonal matrices and $\mathbf{Q}(x)$ is the infinitesimal operator of a continuous time Markov chain, i.e.

$$\mathbf{Q}_{ii}(x) \leq 0, \quad \mathbf{Q}_{ij}(x) \geq 0, i \neq j, \quad \mathbf{Q}(x)\mathbf{e}_N = \mathbf{0}$$

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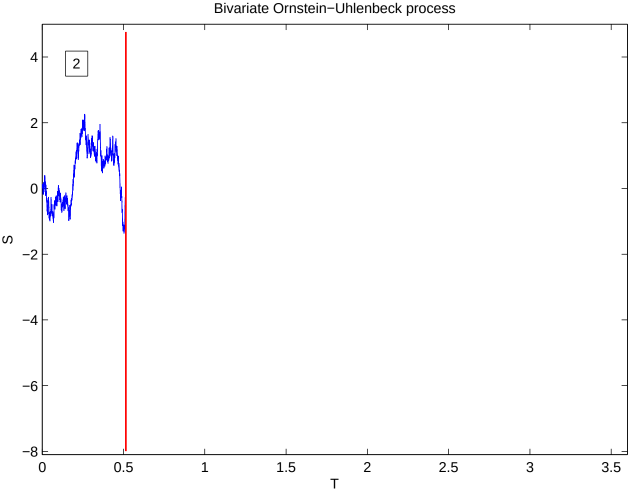
An illustrative example

$N = 3$ phases and $\mathcal{S} = \mathbb{R}$ with

$$\mathbf{A}_{ii}(x) = i^2, \quad \mathbf{B}_{ii}(x) = -ix, \quad i = 1, 2, 3.$$



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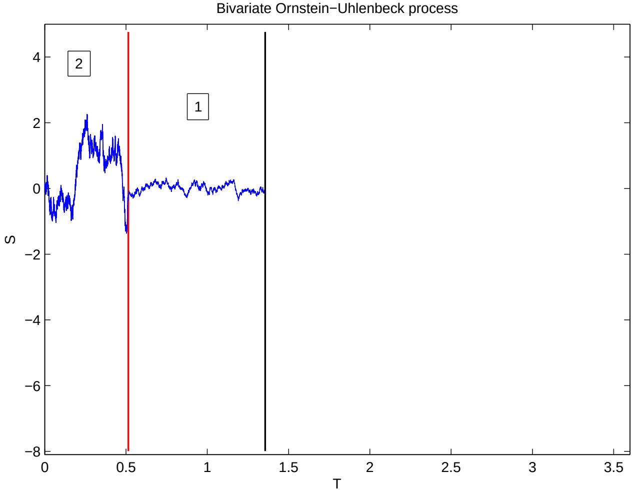

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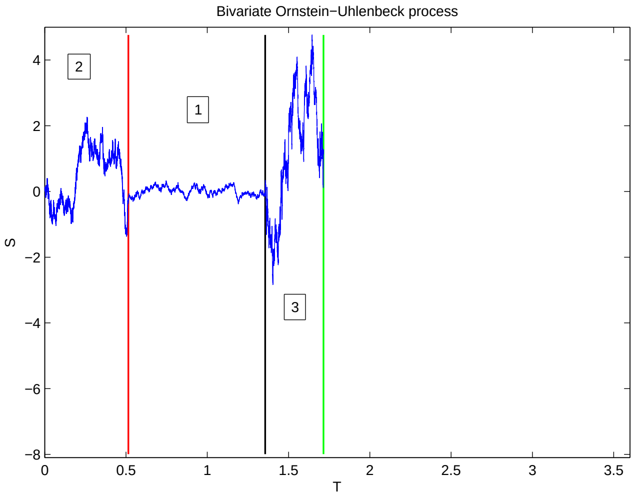

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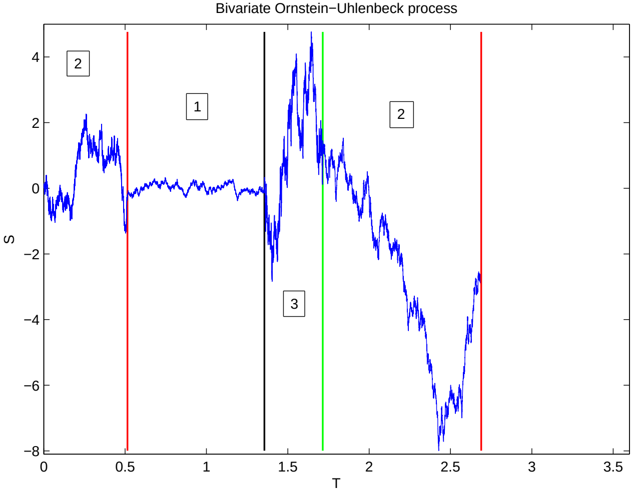

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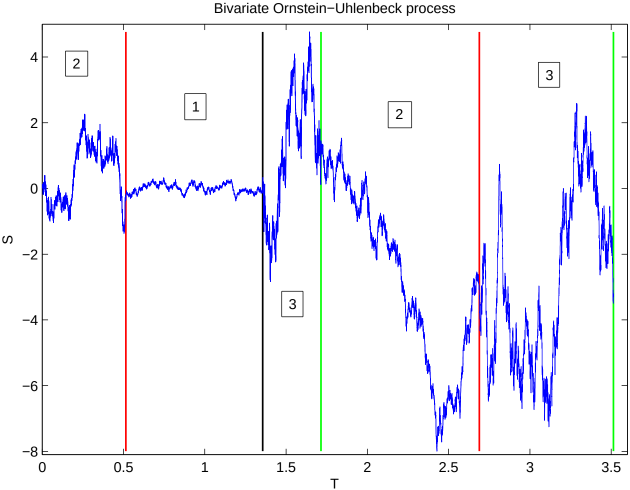

Spectral methods

Now, given a matrix-valued infinitesimal operator $\mathcal{A}$, if we can find a weight matrix $\mathbf{W}(x)$ associated with $\mathcal{A}$, and a set of orthogonal matrix eigenfunctions $\mathbf{F}(i,x)$ such that

$$\mathcal{A}\mathbf{F}(i,x) = \mathbf{\Lambda}(i,x)\mathbf{F}(i,x),$$

then it is possible to find spectral representations of

- Transition probabilities $\mathbf{P}(t;x,y)$.
- Invariant measure or distribution $\pi = (\pi_j)$ (discrete case) with

$$\pi_j = \lim_{t\to\infty} \mathbf{P}_{\cdot j}(t) \in \mathbb{R}^N$$

or $\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y))$ (continuous case) with

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Transition probabilities

1. Discrete time quasi-birth-and-death processes:
   $\mathcal{C} = \{0,1,2,\dots\} \times \{1,2,\dots,N\}$, $\mathcal{T} = \{0,1,2,\dots\}$
   (Grünbaum and Dette-Reuther-Studden-Zygmunt, 2007)

$$\mathbf{F}(i,x) = \mathbf{\Phi}_i(x), \quad \mathbf{\Lambda}(i,x) = x\mathbf{I}, \quad i \in \{0,1,2,\dots\}, \quad x \in [-1,1].$$

$$\mathbf{P}_{ij}^n = \left(\int_{-1}^1 x^n \mathbf{\Phi}_i(x) d\mathbf{W}(x) \mathbf{\Phi}_j^*(x)\right)\left(\int_{-1}^1 \mathbf{\Phi}_j(x) d\mathbf{W}(x) \mathbf{\Phi}_j^*(x)\right)^{-1}$$

2. Switching diffusion processes: $\mathcal{C} = (a,b) \times \{1,2,\dots,N\}$,
   $\mathcal{T} = [0,\infty)$ (MdI, 2011)

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Invariant measure

1. Discrete time quasi-birth-and-death processes (MdI, 2011)

Non-null vector with non-negative components

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0; \boldsymbol{\pi}_1; \cdots) \equiv (\Pi_0 \mathbf{e}_N; \Pi_1 \mathbf{e}_N; \cdots)$$

such that $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ where $\mathbf{e}_N = (1, \ldots, 1)^T$ and

$$\Pi_n = (C_1^T \cdots C_n^T)^{-1} \Pi_0 (A_0 \cdots A_{n-1}) = \left(\int_{-1}^{1} \boldsymbol{\Phi}_n(x) d\mathbf{W}(x) \boldsymbol{\Phi}_n^*(x) \right)^{-1}$$

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$$\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y))$$

$$\mathcal{A}^* \psi(y) = \mathbf{0} \Leftrightarrow \frac{1}{2} \frac{\partial^2}{\partial y^2} [\psi(y) \mathbf{A}(y)] - \frac{\partial}{\partial y} [\psi(y) \mathbf{B}(y)] + \psi(y) \mathbf{Q}(y) = \mathbf{0}$$

$$\Rightarrow \psi(y) = \left(\int_a^b \mathbf{e}_N^T \mathbf{W}(x) \mathbf{e}_N dx \right)^{-1} \mathbf{e}_N^T \mathbf{W}(y)$$

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An example coming from group representation

Let $N \in \{1,2,\ldots\}$, $\alpha,\beta > -1$, $0 < k < \beta+1$ and $\mathbf{E}_{ij}$ will denote the matrix with 1 at entry (i,j) and 0 otherwise.
For $x \in (0,1)$, we have a symmetric pair $\{\mathbf{W},\mathcal{A}\}$ (Grünbaum-Pacharoni-Tirao, 2002) where

$$\mathbf{W}(x) = x^{\alpha}(1-x)^{\beta}\sum_{i=1}^{N}\binom{\beta-k+i-1}{i-1}\binom{N+k-i-1}{N-i}x^{N-i}\mathbf{E}_{ii}$$

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$$\mathbf{Q}(x) = \sum_{i=2}^{N} \mu_i(x)\mathbf{E}_{i,i-1} - \sum_{i=1}^{N}(\lambda_i(x) + \mu_i(x))\mathbf{E}_{ii} + \sum_{i=1}^{N-1} \lambda_i(x)\mathbf{E}_{i,i+1},$$

$$\lambda_i(x) = \frac{1}{1-x}(N-i)(i+\beta-k), \quad \mu_i(x) = \frac{x}{1-x}(i-1)(N-i+k).$$

- The orthogonal eigenfunctions $\Phi_i(x)$ of $\mathcal{A}$ are called matrix-valued spherical functions associated with the complex projective space. There are many structural formulas available studied in the last years (Grünbaum-Pacharoni-Tirao-Román-MdI).
- Bispectrality: $\Phi_i(x)$ satisfy a three-term recurrence relation

$$x\Phi_i(x) = A_i\Phi_{i+1}(x) + B_i\Phi_i(x) + C_i\Phi_{i-1}(x), \quad i = 0, 1, \ldots$$

  whose Jacobi matrix describes a discrete-time quasi-birth-and-death process (Grünbaum-MdI, 2008). It was recently connected with urn and Young diagram models (Grünbaum-Pacharoni-Tirao, 2011).
- The infinitesimal operator $\mathcal{A}$ describes a nontrivial switching diffusion process from which we can give a description of the matrix-valued probability density $\mathbf{P}(t; x, y)$ and invariant distribution $\psi(y)$ in terms of the eigenfunctions $\Phi_i(x)$, among other properties (MdI, 2011).

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A variant of the Wright-Fisher model

The Wright-Fisher diffusion model involving only mutation effects considers a big population of constant size M composed of two types A and B.

$$A \xrightarrow{\frac{1+\beta}{2}} B, \quad B \xrightarrow{\frac{1+\alpha}{2}} A, \quad \alpha, \beta > -1$$

As $M \to \infty$, this model can be described by a diffusion process whose state space is $S = [0,1]$ with drift and diffusion coefficient

$$\tau(x) = \alpha + 1 - x(\alpha + \beta + 2), \quad \sigma^2(x) = 2x(1-x), \quad \alpha, \beta > -1$$

The N phases of our bivariate Markov process are variations of the Wright-Fisher model in the drift coefficients:

$$\mathbf{B}_{ii}(x) = \alpha + 1 + N - i - x(\alpha + \beta + 2 + N - i), \quad \mathbf{A}_{ii}(x) = 2x(1-x)$$

Now there is an extra parameter $k \in (0, \beta + 1)$ in $\mathbf{Q}(x)$, which measures how the process moves through all the phases.

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The parameters α and β generally describe if the boundaries 0 and 1 are either *absorbing* or *reflecting*.

The matrix $\mathbf{Q}(x)$ controls the *waiting times* at each phase and the *tendency* of moving forward or backward in phases. These also depend on the position x of the particle.

Meaning of k (MdI, 2011)

- If $k \lesssim \beta + 1 \Rightarrow$ phase 1 is absorbing $\Rightarrow$ Backward tendency
  Meaning: The parameter k helps the population of A's to survive against the population of B's.
- If $k \gtrsim 0 \Rightarrow$ phase N is absorbing $\Rightarrow$ Forward tendency
  Meaning: Both populations A and B 'fight' in the same conditions.
- Middle values of k: Forward/backward tendency.

More information: arXiv:1107.3733.

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Feliz cumpleaños, Paco!