Factorization of stochastic Jacobi matrices into two stochastic factors and Darboux transformations*

Manuel Domínguez de la Iglesia

Instituto de Matemáticas, UNAM, México

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*Joint work with F. Alberto Grünbaum

MAIN MOTIVATION

Let $\{X_n : n = 0, 1, \ldots\}$ be an irreducible random walk with space state $\mathbb{Z}_{\geq 0}$ and P its one-step transition probability matrix given by

$$P = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \\ c_1 & b_1 & a_1 & 0 & \\ 0 & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

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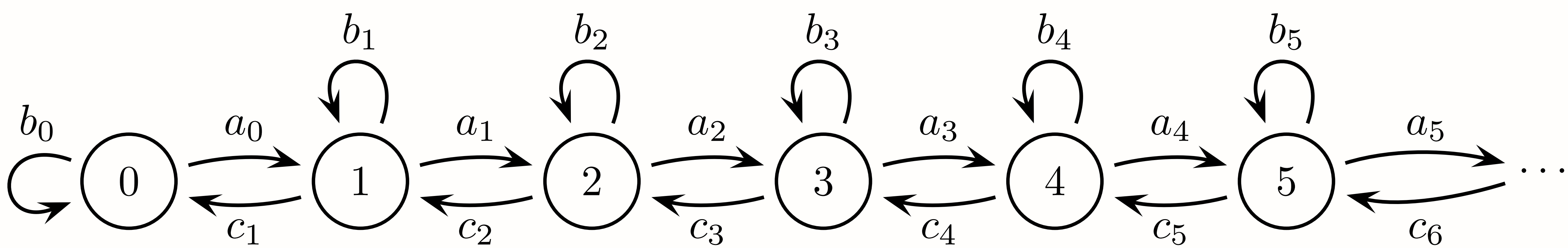

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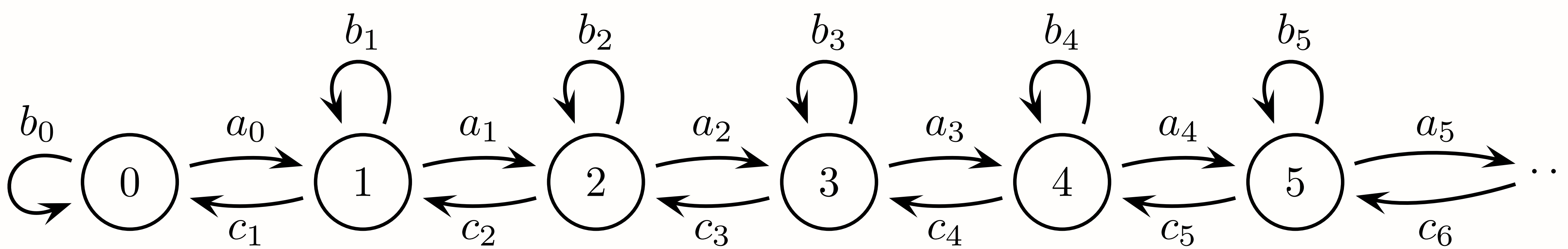

MAIN GOAL: find simple ways to describe certain random walks in terms of urn models, where the coefficients a_n, b_n, c_n are complicated rational functions.

HOW?: we will try to describe the urn model as the composition of two easier urn models factorizing the matrix P.

OUTLINE

1. Stochastic LU and UL factorizations

2. Stochastic Darboux transformations

3. An urn model for the Jacobi polynomials

1. Stochastic LU and UL factorizations

UL FACTORIZATION

Let $\{X_n : n = 0, 1, \ldots\}$ be an irreducible random walk with space state $\mathbb{Z}_{\geq 0}$ and P its one-step transition probability matrix. We would like to perform a UL decomposition of the matrix P in the following way

$$P = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} y_0 & x_0 & & \\ 0 & y_1 & x_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} s_0 & 0 & & \\ r_1 & s_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = P_U P_L$$

with the condition that P_U and P_L are also stochastic matrices.

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A direct computation shows that

$$\begin{aligned} a_n &= x_n s_{n+1}, \quad n \geq 0 \\ b_n &= x_n r_{n+1} + y_n s_n, \quad n \geq 0 \\ c_n &= y_n r_n, \quad n \geq 1 \end{aligned}$$

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Since P is stochastic we have that all entries of P are nonnegative, $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1, n \geq 1$. We also want that P_U and P_L are stochastic, i.e. all entries nonnegative and

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$$x_n + y_n = 1, \quad n \geq 0, \quad s_0 = 1, \quad r_n + s_n = 1, \quad n \geq 1.$$

The only relevant equations in these relations are going to be the first and third ones, i.e.

$$a_n = (1 - y_n)s_{n+1}, \quad c_{n+1} = y_{n+1}(1 - s_{n+1}), \quad n \geq 0$$

LU FACTORIZATION

One could have performed the factorization the other way around like

$$P = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \tilde{s}_0 & 0 & & \\ \tilde{r}_1 & \tilde{s}_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{y}_0 & \tilde{x}_0 & & \\ 0 & \tilde{y}_1 & \tilde{x}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \widetilde{P}_L \widetilde{P}_U$$

in which case we have a LU factorization with relations

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The important difference between both cases is that in the UL factorization case there will be a free parameter y_0 while in the LU factorization case the decomposition will be unique. The computation in both cases is similar so we will focus on the UL factorization case.

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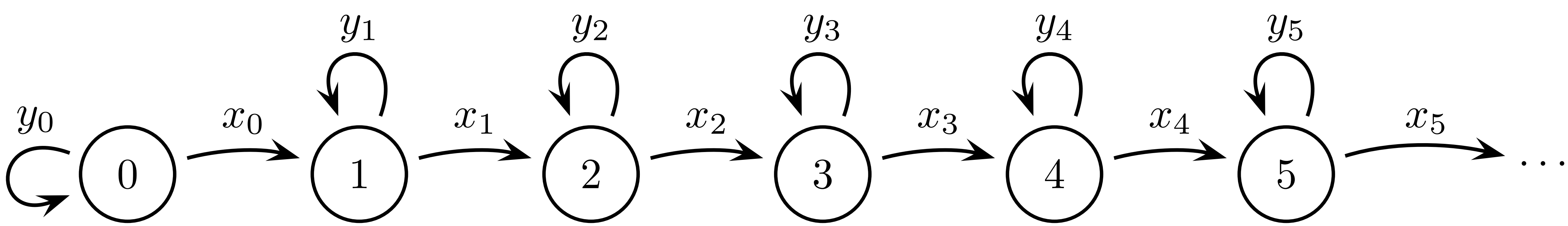
UL and LU decompositions of stochastic matrices have been considered earlier in the literature (W.K. Grassmann, D.P. Heyman, V. Vigon, etc.) in a different context related with censored Markov chains and Wiener-Hopf factorizations.

PROBABILISTIC INTERPRETATION

From the factorization $P = P_U P_L$, we observe that P_U is a pure birth random walk on $\mathbb{Z}_{\geq 0}$ with diagram



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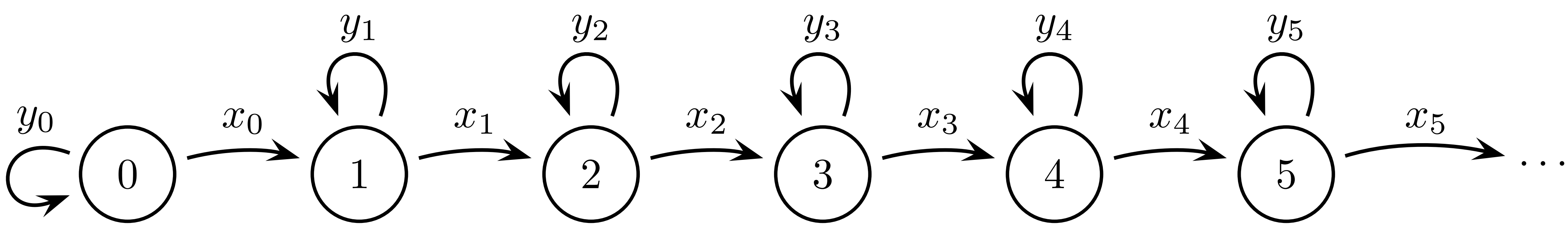
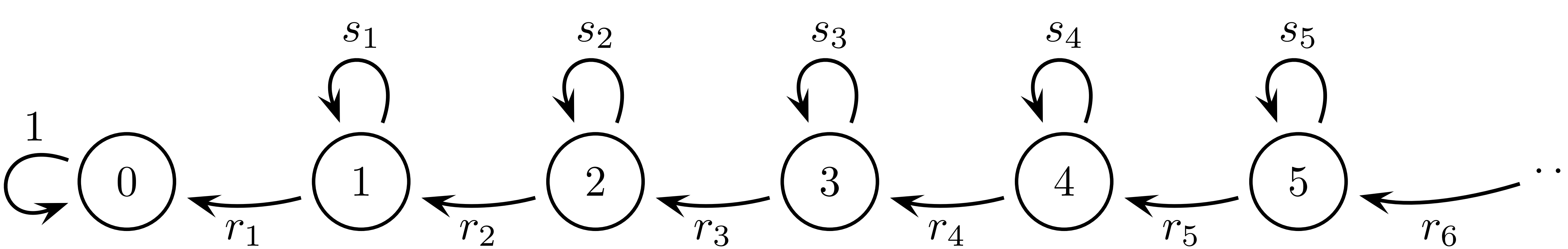


while P_L is a pure death random walk on $\mathbb{Z}_{\geq 0}$ with absorbing state at 0 with diagram



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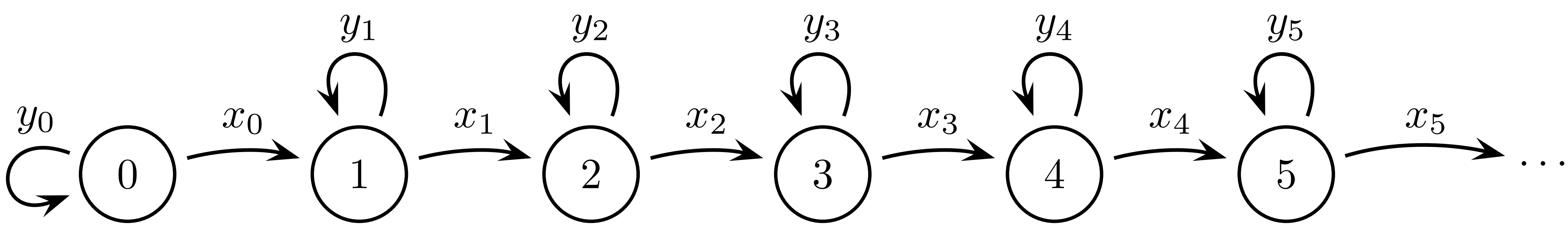

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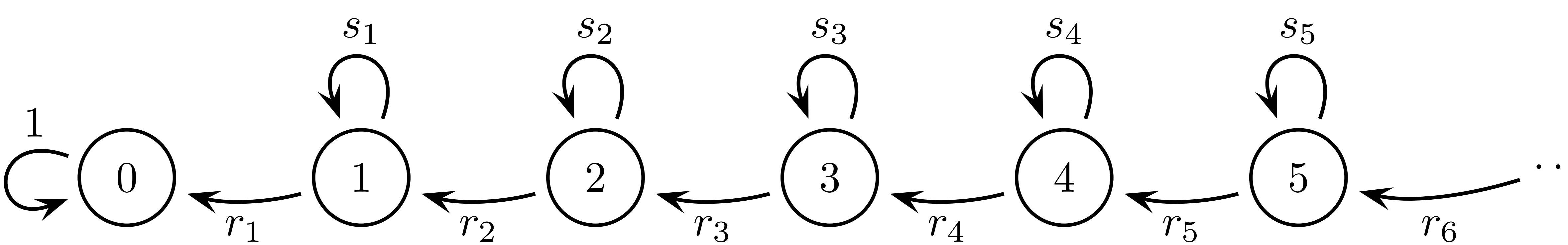

The random walk for P will be the combination of performing first the pure birth random walk and immediately after the pure death random walk.

STOCHASTIC FACTORS (UL)

From the relations

$$a_n = (1 - y_n)s_{n+1}, \quad c_{n+1} = y_{n+1}(1 - s_{n+1}), \quad n \geq 0$$

it is possible to compute all the coefficients x_n, y_n, r_n, s_n of P_U and P_L in terms of y_0.

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Indeed, if y_0 is fixed and $s_0 = 1$, then

$$s_1 = \frac{a_0}{1 - y_0}, \quad y_1 = \frac{c_1}{1 - s_1}, \quad s_2 = \frac{a_1}{1 - y_1}, \quad y_2 = \frac{c_2}{1 - s_2}, \quad \text{etc.}$$

and for each y_n and s_n we have $x_n = 1 - y_n$ and $r_n = 1 - s_n$.

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This will give that the sum of each row of P_U and P_L is exactly 1, but this does not mean that both factors are stochastic matrices, since all entries must be nonnegative.

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For that we will need that the following continued fraction

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1 - \cdots}}}} \doteq 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots}$$

is convergent and $0 < H < 1$.

MAIN THEOREM

Theorem. *Let H the continued fraction given before and the corresponding convergents $h_n = A_n/B_n$. Assume that*

$$0 < A_n < B_n, \quad n \geq 1$$

Then H is convergent. Moreover, if $P = P_U P_L$, then both P_U and P_L are stochastic matrices if and only if we choose y_0 in the following range

$$0 < y_0 \leq H = 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots}$$

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Sketch of the proof. Assume that both P_U and P_L are stochastic. That means that $0 < y_0 < 1$, $0 < x_n, s_{n+1} < 1, n \geq 0$ and $0 < y_n, r_n < 1, n \geq 1$. Consider the sequence of numbers $\alpha_n, n \geq 0$, given by $\alpha_{2n} = y_n, \alpha_{2n+1} = s_{n+1}, n \geq 0$. Then the condition $\alpha_n < 1$ implies that $y_0 < h_n$ for every n.

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For $\alpha_0 = y_0$ it is clear that $0 \leq y_0 < 1 = h_0$. For $\alpha_1 = s_1$, we have, by definition,

$$\alpha_1 = s_1 = \frac{a_0}{1 - y_0} < 1 \Leftrightarrow y_0 < 1 - a_0 = h_1$$

For $\alpha_2 = y_1$ we have, using the previous bounds, that

$$\alpha_2 = y_1 = \frac{c_1}{1 - s_1} < 1 \Leftrightarrow s_1 < 1 - c_1 \Leftrightarrow \frac{a_0}{1 - y_0} < 1 - c_1 \Leftrightarrow y_0 < 1 - \frac{a_0}{1 - c_1} = h_2$$

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Sketch of the proof. In general, for an even index $2n$ we have, using all the previous bounds, that

$$\alpha_{2n} = y_n = \frac{c_n}{1 - s_n} < 1 \Leftrightarrow s_n < 1 - c_n \Leftrightarrow \frac{a_{n-1}}{1 - y_{n-1}} < 1 - c_n \Leftrightarrow y_{n-1} < 1 - \frac{a_{n-1}}{1 - c_n}$$

$$\Leftrightarrow s_{n-1} < 1 - \frac{c_{n-1}}{1 - \frac{a_{n-1}}{1 - c_n}} \Leftrightarrow \cdots \Leftrightarrow y_0 < 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \cdots - \frac{c_n|}{|1} = h_{2n}$$

Similarly, for an odd index $2n+1$ we have, using all the previous bounds, that

$$\alpha_{2n+1} = s_{n+1} = \frac{a_n}{1 - y_n} < 1 \Leftrightarrow y_n < 1 - a_n \Leftrightarrow \frac{c_n}{1 - s_n} < 1 - a_n \Leftrightarrow s_n < 1 - \frac{c_n}{1 - a_n}$$

$$\Leftrightarrow y_{n-1} < 1 - \frac{a_{n-1}}{1 - \frac{c_n}{1 - a_n}} \Leftrightarrow \cdots \Leftrightarrow y_0 < 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \cdots - \frac{a_n|}{|1} = h_{2n+1}$$

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Sketch of the proof. Therefore $y_0 < h_n$ for every n and since h_n is a positive bounded and decreasing sequence less than 1 (by properties of continued fractions) we get the result.

On the contrary, if we have the bound, in particular we have that $0 \leq y_0 \leq H < h_n$ for every $n \geq 0$. Following the same steps as before, using an argument of strong induction will lead us to the fact that both P_U and P_L are stochastic matrices. $\square$

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Then H is convergent. Moreover, if $P = P_U P_L$, then both P_U and P_L are stochastic matrices if and only if we choose y_0 in the following range

$$0 < y_0 \leq H = 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots}$$

MAIN THEOREM

Theorem. *Let H the continued fraction given before and the corresponding convergents $h_n = A_n/B_n$. Assume that*

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For the LU decomposition there is no free parameter, so the positivity condition comes in terms of an upper bound of the coefficient $\tilde{y}_0 = a_0$. Indeed, one must have

$$0 < a_0 \leq \widetilde{H}$$

where

$$\widetilde{H} = 1 - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|1} - \frac{a_2|}{|\cdots}$$

as long as we have $0 < \widetilde{A}_n < \widetilde{B}_n, n \geq 1$, where $\tilde{h}_n = \widetilde{A}_n/\widetilde{B}_n$ are the convergents of $\widetilde{H}$.

2. Stochastic Darboux transformations

DARBOUX TRANSFORMATIONS

If $P = P_U P_L$, then by inverting the order of multiplication we obtain another tridiagonal matrix of the form

$$\widetilde{P} = P_L P_U = \begin{pmatrix} s_0 & 0 & & \\ r_1 & s_1 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} y_0 & x_0 & & \\ 0 & y_1 & x_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_0 & & \\ \tilde{c}_1 & \tilde{b}_1 & \tilde{a}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

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This is called a discrete Darboux transformation. It appeared for the first time in [Matveev-Salle] in connection with Toda lattices. Later, many other authors (Grünbaum, Haine, Horozov, Iliev, etc.) have used this transformation in the description of some families of Krall polynomials.

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Now the new coefficients are given by

$$\begin{aligned} \tilde{a}_n &= s_n x_n, \quad n \geq 0 \\ \tilde{b}_n &= r_n x_{n-1} + s_n y_n, \quad n \geq 0 \\ \tilde{c}_n &= r_n y_{n-1}, \quad n \geq 1 \end{aligned}$$

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The matrix $\widetilde{P}$ is actually stochastic, since the multiplication of two stochastic matrices is again a stochastic matrix. Therefore it gives a *family* of new random walks with coefficients $(\tilde{a}_n)_n$, $(\tilde{b}_n)_n$ and $(\tilde{c}_n)_n$ and depending on a free parameter y_0.

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Probabilistic interpretation. In terms of a model driven by urn experiments the factorization $P = P_U P_L$ may be thought as two urn experiments, Experiment 1 and Experiment 2, respectively. We first perform the Experiment 1 and with the result we immediately perform the Experiment 2. The urn model for $\widetilde{P} = P_L P_U$ will proceed in the reversed order, first the Experiment 2 and with the result the Experiment 1. The same can be done for the LU decomposition.

SPECTRAL MEASURES

One important property of the Darboux transformation is how to transform the spectral measure associated P. It is very well known that for every tridiagonal stochastic matrix P (or Jacobi matrix) there exists an unique positive measure ω supported on the interval $-1 \leq x \leq 1$ (Spectral or Favard's Theorem).

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The Darboux transformation gives a family of random walks $\widetilde{P}$ which is also a tridiagonal stochastic matrix. If the moment $\mu_{-1} = \int_{-1}^{1} d\omega(x)/x$ is well defined, then a candidate for the family of spectral measures is then

$$\widetilde{\omega}(x) = y_0 \frac{\omega(x)}{x} + M\delta_0(x), \quad M = 1 - y_0\mu_{-1}$$

where $\delta_0(x)$ is the Dirac delta located at $x = 0$ and y_0 is the free parameter from the UL factorization. This transformation of the spectral measure ω is also known as a Geronimus transformation.

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Similarly, for the LU decomposition, the corresponding Darboux transformation $\widehat{P}$ gives rise to a tridiagonal stochastic matrix and a spectral measure $\widehat{\omega}$. In this case, it is possible to see that this new spectral measure is given by

$$\widehat{\omega}(x) = x\omega(x)$$

or, in other words, a Christoffel transformation of ω.

3. An urn model for the Jacobi polynomials

JACOBI POLYNOMIALS

Consider the family of Jacobi polynomials $Q_n^{(\alpha,\beta)}(x)$, which are orthogonal with respect to the weight

$$w(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}, \quad x \in [0,1], \quad \alpha, \beta > -1$$

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Then the Jacobi polynomials satisfy the three-term recursion relation

$$xQ_n^{(\alpha,\beta)}(x) = a_n Q_{n+1}^{(\alpha,\beta)}(x) + b_n Q_n^{(\alpha,\beta)}(x) + c_n Q_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 0$$

where the coefficients a_n, b_n, c_n are defined by

$$a_n = \frac{(n+\beta+1)(n+1+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)}, \quad n \geq 0$$

$$b_n = \frac{(n+\beta+1)(n+1)}{(2n+\alpha+\beta+1)(2n+2+\alpha+\beta)} + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad n \geq 0$$

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c_n &= \frac{n(n+\alpha)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad n \geq 1
\end{aligned}$$

We remark that all these coefficients are nonnegative, $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1, n \geq 1$, so they are the coefficients of a discrete time random walk on the nonnegative integers and depend on the state of the system. The corresponding Jacobi matrix is then stochastic.

UL FACTORIZATION

Let us call $\alpha_n, n \geq 1$, the sequence of alternating coefficients $a_0, c_1, a_1, c_2, \ldots$, respectively. Then the sequence α_n is a chain sequence, i.e. $\alpha_n = (1 - m_{n-1})m_n$ where $0 \leq m_0 < 1$ and $0 < m_n < 1$ for $n \geq 1$. In this case we have

$$m_{2n} = \frac{n}{2n + \alpha + \beta + 1}, \quad m_{2n+1} = \frac{n + \beta + 1}{2n + \alpha + \beta + 2}, \quad n \geq 0$$

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Then the continued fraction H with partial numerators given by α_n is convergent to $(1+L)^{-1}$ where L is given by (see book of Chihara)

$$L = \sum_{n=1}^{\infty} \frac{m_1 m_2 \cdots m_n}{(1 - m_1)(1 - m_2) \cdots (1 - m_n)}$$

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Using the main theorem we have that the stochastic UL factorization is always possible if we choose the free parameter y_0 in the range

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So for every y_0 in that range we can always have a stochastic UL factorization.

UL FACTORIZATION II

If we make the substitution $y_0 = \frac{\alpha}{\alpha+\beta+1} h_0$ (in which case $0 < h_0 \leq 1$), then we have

$$x_n = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} \cdot \frac{h_0 n!(\beta+1)_n(n+\beta+1) + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_{n+1}}{h_0 n!(\beta+1)_n(n+\alpha+\beta+1) + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_{n+1}}$$

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$$r_n = \frac{n+\alpha}{2n+\alpha+\beta} \cdot \frac{h_0 n!(\beta+1)_n + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_n}{h_0(n-1)!(n+\alpha)(\beta+1)_n + (1-h_0)(\alpha+1)_n(\alpha+\beta+1)_n}$$

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From these formulas we clearly see that each coefficient x_n, y_n, s_n, r_n is the multiplication of two positive numbers less than 1. The simplest simplification is taking $h_0 = 1$, in which case we have

$$x_n = \frac{n+\beta+1}{2n+\alpha+\beta+1}, \quad y_n = \frac{n+\alpha}{2n+\alpha+\beta+1}, \quad n \geq 0$$
$$s_n = \frac{n+\alpha+\beta}{2n+\alpha+\beta}, \quad r_n = \frac{n}{2n+\alpha+\beta}, \quad n \geq 1, \quad s_0 = 1$$

We will use these coefficients later to give a simplified urn model for the Jacobi polynomials.

LU FACTORIZATION

Now there is no free parameter and we need to have

$$a_0 = \frac{\beta+1}{\alpha+\beta+2} \leq \widetilde{H}$$

where $\widetilde{H}$ is the continued fraction with partial numerators given by $c_1, a_1, c_2, a_2, \ldots$. Again this is a chain sequence with coefficients

$$m_{2n} = \frac{n+\beta+1}{2n+\alpha+\beta+2}, \quad m_{2n+1} = \frac{n+1}{2n+\alpha+\beta+3}, \quad n \geq 0$$

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$$\widetilde{H} = m_0 + \frac{1-m_0}{1+L}$$

where L is given earlier (observe now that $m_0 \neq 0$). In this case we have $L = 1/\alpha$ and

$$\widetilde{H} = \frac{\beta+1}{\alpha+\beta+2} + \frac{\frac{\alpha+1}{\alpha+\beta+2}}{1+1/\alpha} = \frac{\alpha+\beta+1}{\alpha+\beta+2}$$

We clearly see that $a_0 \leq \widetilde{H}$ and therefore we can always perform a stochastic LU factorization, but now without a free parameter.

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Finally, it is possible to see that

$$\tilde{x}_n = \frac{n+\beta+1}{2n+\alpha+\beta+2}, \quad \tilde{y}_n = \frac{n+\alpha+1}{2n+\alpha+\beta+2}, \quad n \geq 0$$
$$\tilde{s}_n = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}, \quad \tilde{r}_n = \frac{n}{2n+\alpha+\beta+1}, \quad n \geq 1, \quad \tilde{s}_0 = 1$$

SPECTRAL MEASURES

The spectral measure associated with the Darboux transformation $\widetilde{P} = P_L P_U$ is the Geronimus transformation of the Jacobi weight w. In this case it is easy to see that

$$\mu_{-1} = \int_0^1 \frac{w(x)}{x} dx = \frac{\alpha+\beta+1}{\alpha}$$

Therefore we have

$$\widetilde{w}(x) = y_0 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha-1}(1-x)^{\beta} + \left(1 - y_0 \frac{\alpha+\beta+1}{\alpha}\right) \delta_0(x), \quad x \in [0,1]$$

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The spectral measure associated with the Darboux transformation $\widetilde{P} = P_L P_U$ is the Geronimus transformation of the Jacobi weight w. In this case it is easy to see that

$$\mu_{-1} = \int_0^1 \frac{w(x)}{x} dx = \frac{\alpha+\beta+1}{\alpha}$$

Therefore we have

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This measure is integrable as long as $\alpha > 0$ and $\beta > -1$. We see that if y_0 is in the range of a stochastic UL factorization, then the mass at 0 is always nonnegative, and vanishes if

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For the LU decomposition, the spectral measure associated with the Darboux transformation $\widehat{P} = \widetilde{P}_U \widetilde{P}_U$ is the Christoffel transformation of the Jacobi weight, i.e.

$$\widehat{w}(x) = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)} x^{\alpha+1}(1-x)^{\beta}, \quad x \in [0,1]$$

AN URN MODEL

Fix $\alpha = 2, \beta = 0$ (for any α, β the model is similar as long as we take α and β nonnegative integers). Consider the random walk $\{X_n : n = 0, 1, \ldots\}$ with transition probability matrix P with the coefficients of the Jacobi polynomials, i.e.

$$a_n = \frac{(n+1)(n+3)}{(2n+3)(2n+4)}, \quad n \geq 0$$

$$b_n = \frac{(n+1)^2}{(2n+3)(2n+4)} + \frac{(n+2)^2}{(2n+2)(2n+3)}, \quad n \geq 0$$

$$c_n = \frac{n(n+2)}{(2n+2)(2n+3)}, \quad n \geq 1$$

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Consider the UL factorization $P = P_U P_L$ for $y_0 = \frac{\alpha}{\alpha+\beta+1}$, i.e.

$$x_n = \frac{n+1}{2n+3}, \quad y_n = \frac{n+2}{2n+3}, \quad n \geq 0$$

$$s_n = \frac{n+2}{2n+2}, \quad r_n = \frac{n}{2n+2}, \quad n \geq 1, \quad s_0 = 1$$

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$$

Let us call $\{X_n^U : n = 0, 1, \ldots\}$ the Markov chain (pure birth random walk) generated by the coefficients x_n, y_n in P_U. This model will be the Experiment 1.

Similarly, let us call $\{X_n^L : n = 0, 1, \ldots\}$ the Markov chain (pure death random walk) generated by the coefficients s_n, r_n in P_L. This model will be the Experiment 2.

AN URN MODEL

Fix $\alpha = 2, \beta = 0$ (for any α, β the model is similar as long as we take α and β nonnegative integers). Consider the random walk $\{X_n : n = 0, 1, \ldots\}$ with transition probability matrix P with the coefficients of the Jacobi polynomials, i.e.

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Consider the UL factorization $P = P_U P_L$ for $y_0 = \frac{\alpha}{\alpha+\beta+1}$, i.e.

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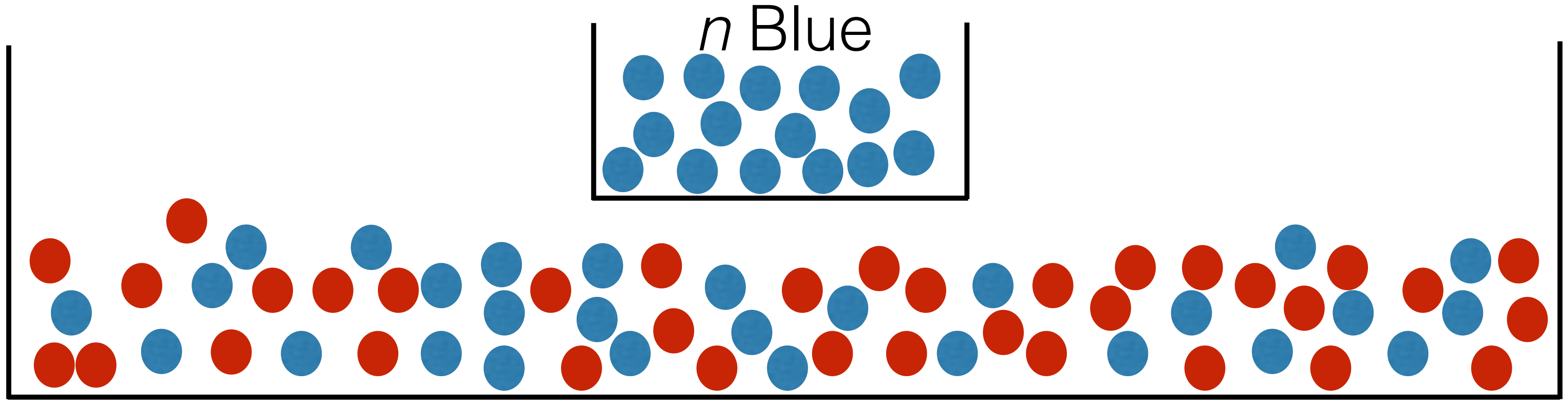
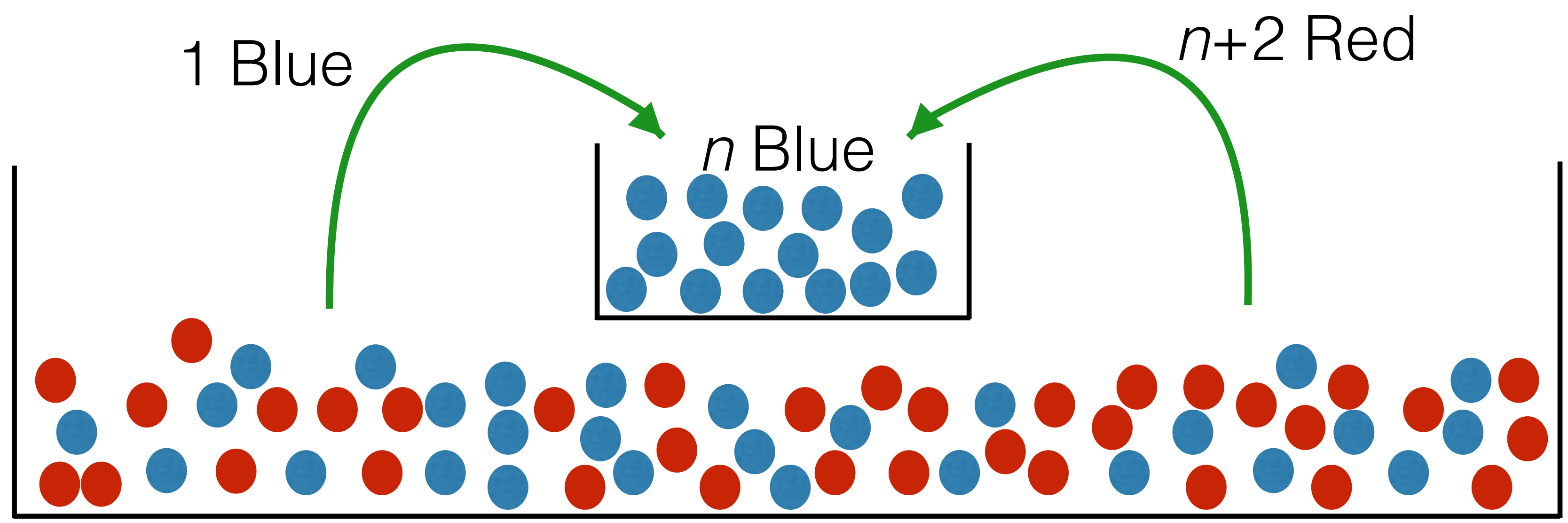
Similarly, let us call $\{X_n^L : n = 0, 1, \ldots\}$ the Markov chain (pure death random walk) generated by the coefficients s_n, r_n in P_L. This model will be the Experiment 2.

The urn model for $\{X_n : n = 0, 1, \ldots\}$ will be the composition of the urn model for $\{X_n^U : n = 0, 1, \ldots\}$ (Experiment 1) and then the urn model for $\{X_n^L : n = 0, 1, \ldots\}$ (Experiment 2).

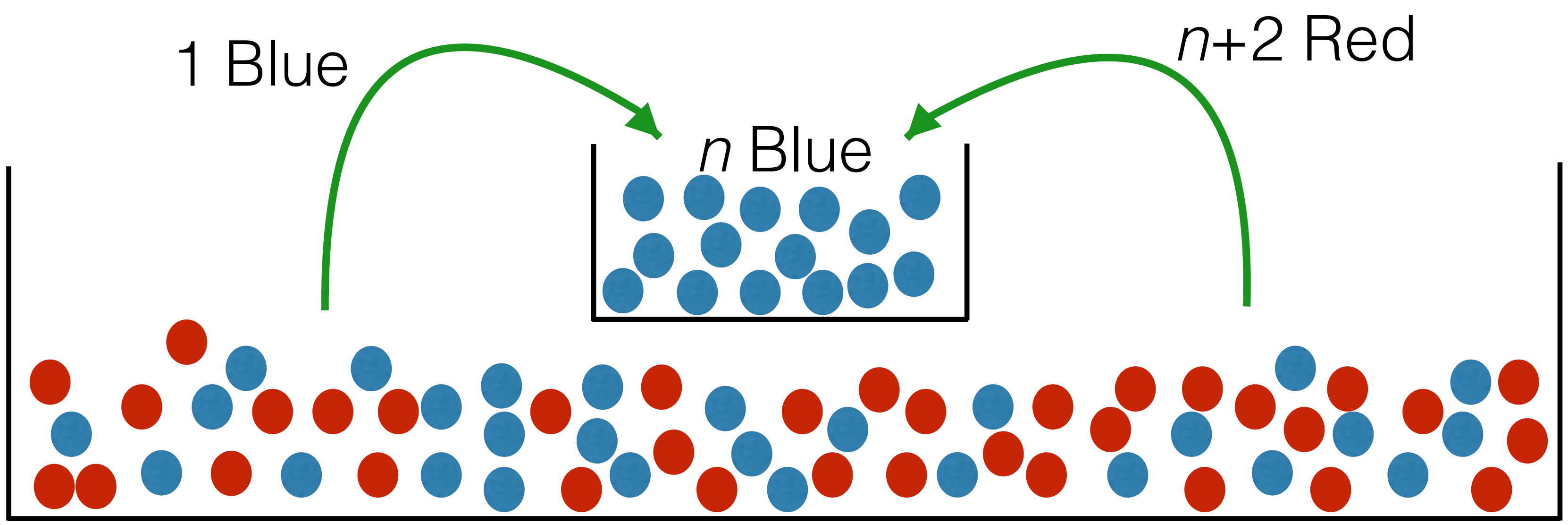
EXPERIMENT 1



EXPERIMENT 1



EXPERIMENT 1



In total, there will be $2n+3$ balls in the urn. Pick one ball at random (uniformly). The probability of having a blue or a red ball is given by

$$\mathbb{P}(B) = \frac{n+1}{2n+3} = x_n, \quad \mathbb{P}(R) = \frac{n+2}{2n+3} = y_n$$

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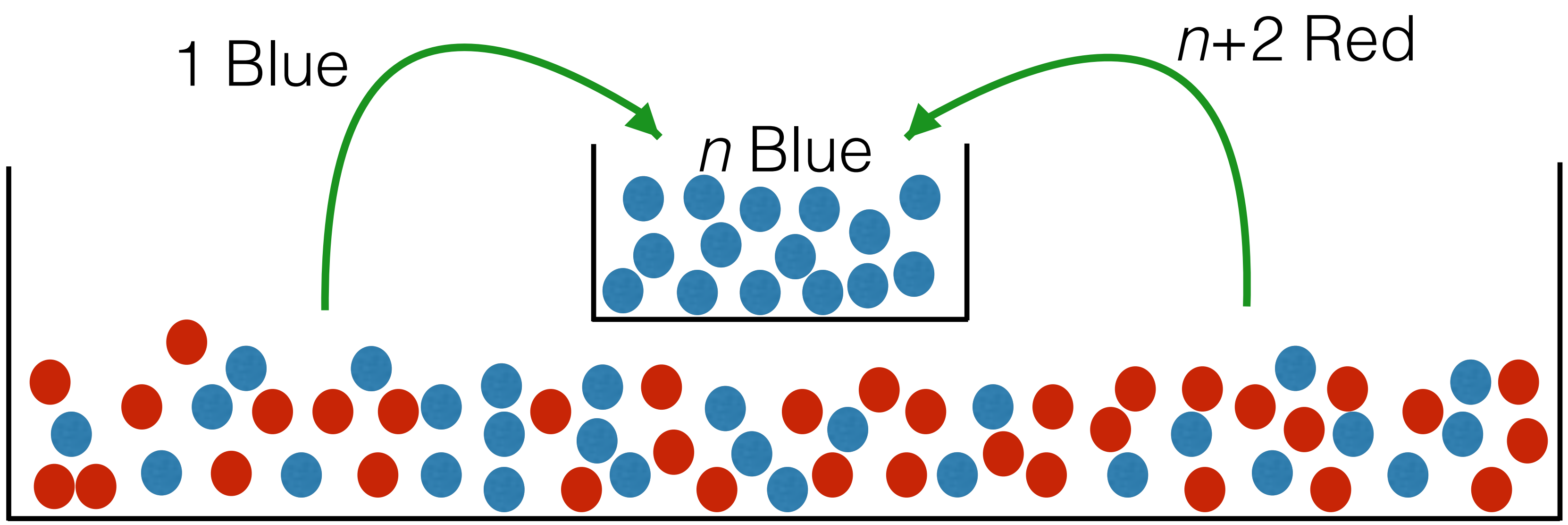

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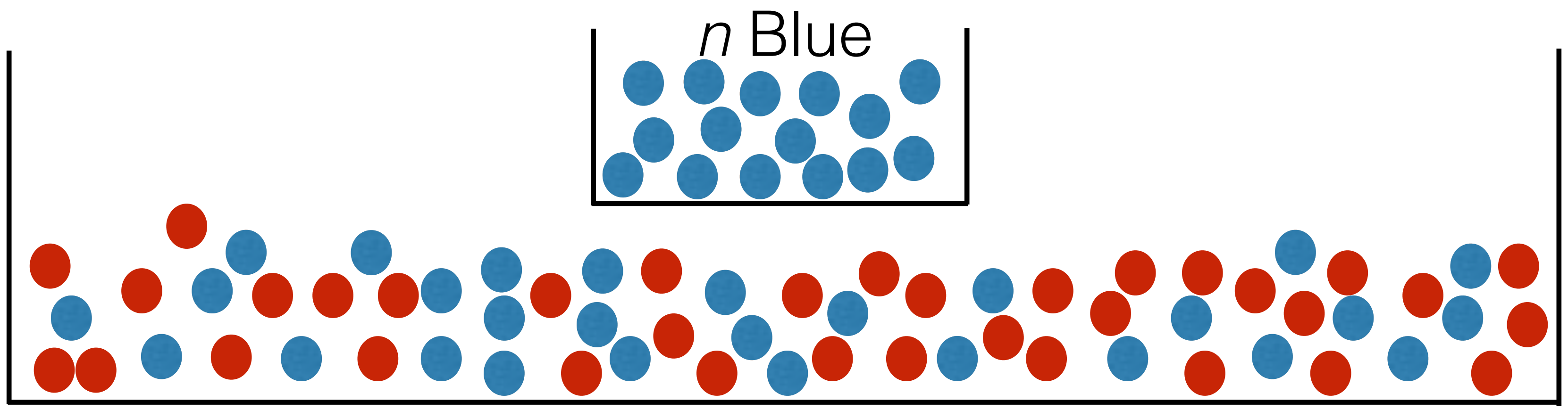
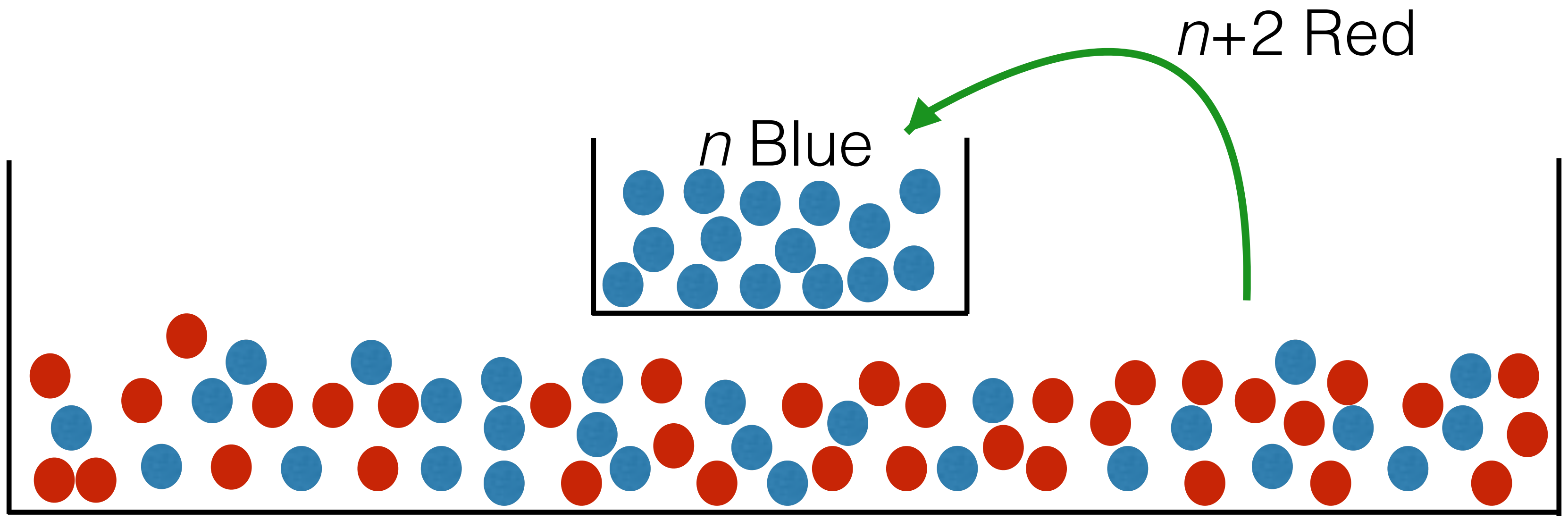
We follow a strategy depending if we pick a blue or a red ball:

- If we get a blue ball: then the blue ball goes back to the urn and we remove all red balls from the urn, and start over.
- If we get a red ball: then we remove 1 blue ball and all red balls from the urn and start over.

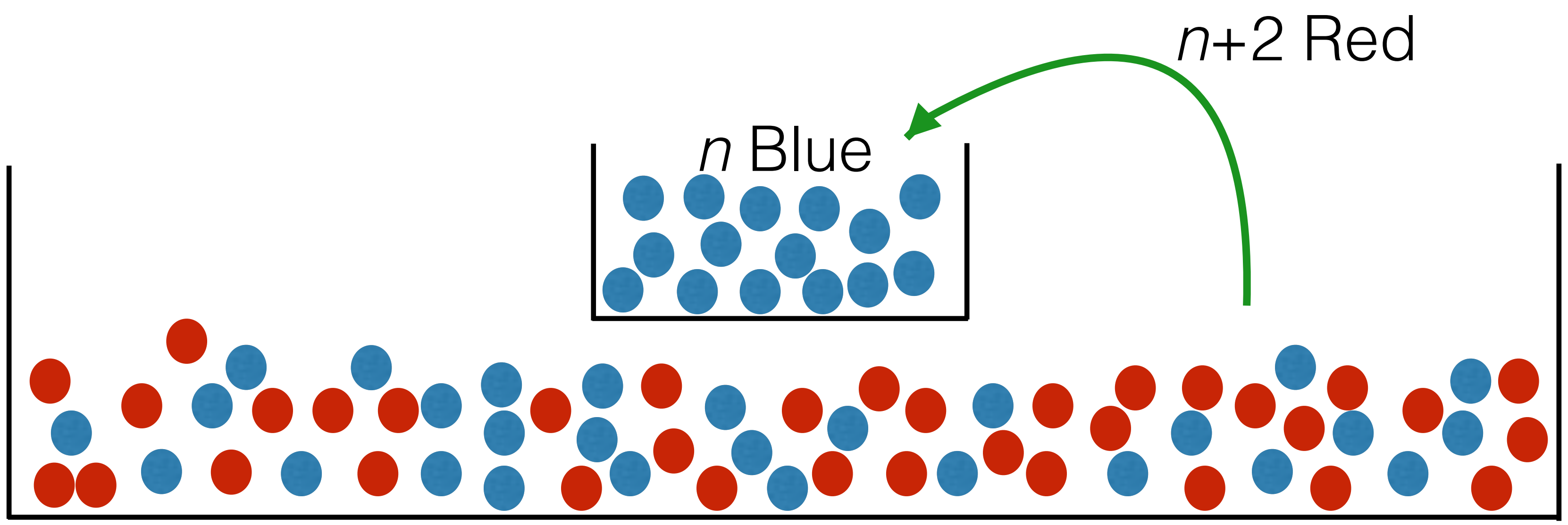
EXPERIMENT 2



EXPERIMENT 2



EXPERIMENT 2



In total, there will be $2n+2$ balls in the urn. Pick one ball at random (uniformly). The probability of having a blue or a red ball is given by

$$\mathbb{P}(B) = \frac{n}{2n+2} = r_n, \quad \mathbb{P}(R) = \frac{n+2}{2n+2} = s_n$$

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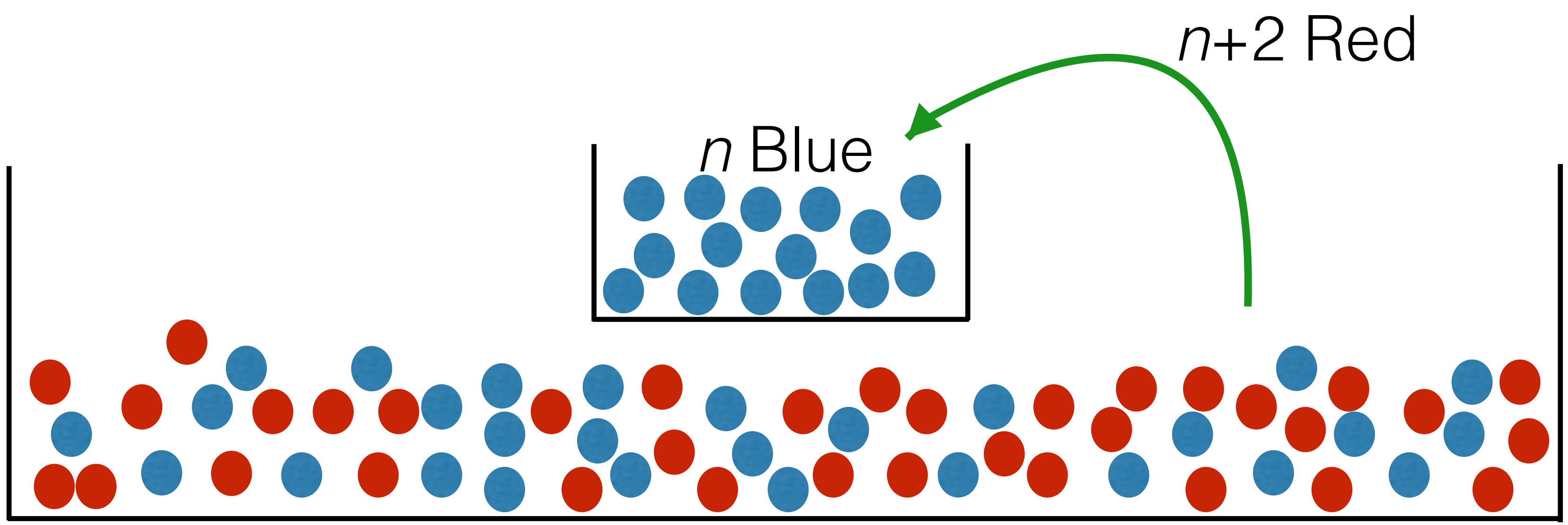

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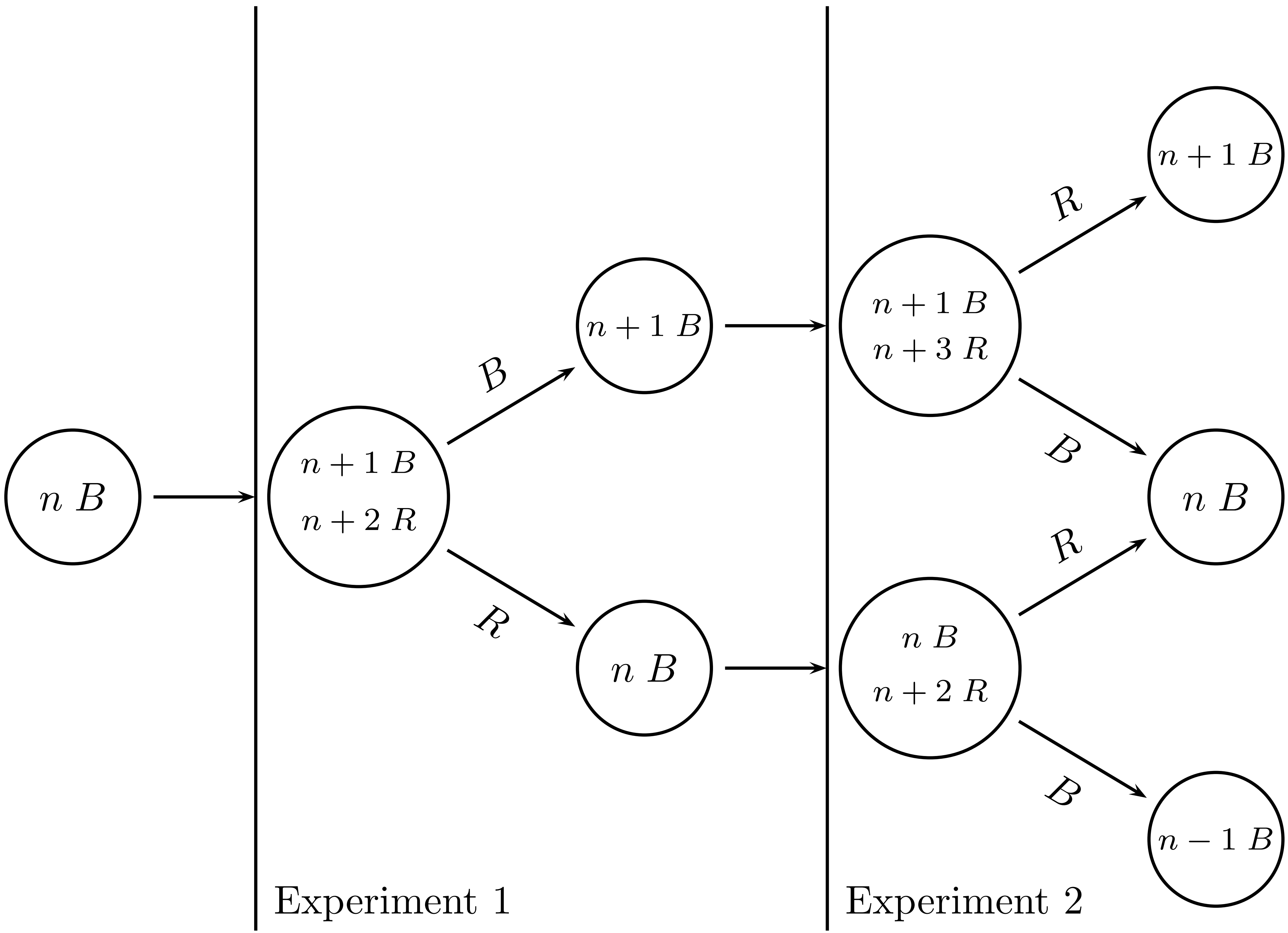
We follow a strategy depending if we pick a blue or a red ball:

- If we get a blue ball: then we remove that blue ball and all the red balls from the urn, and start over.
- If we get a red ball: then we remove all red balls from the urn and start over.

COMPOSITION OF BOTH EXPERIMENTS




THANK YOU