Bispectral Jacobi type polynomials*

Manuel Domínguez de la Iglesia

Instituto de Matemáticas, UNAM, México

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*Joint work with A.J. Durán

MAIN RESULTS

Let $G = \{g_1, \ldots, g_{m_1}\}$ and $H = \{h_1, \ldots, h_{m_2}\}$ be finite sets of positive integers ordered in increasing size and monic polynomials $\mathcal{R}_g, g \in G$, with $\deg \mathcal{R}_g = g$ and $\mathcal{S}_h, h \in H$, with $\deg \mathcal{S}_h = h$ and call $m = m_1 + m_2$.

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Denote $\mathcal{Z}_i(x), i=1,\dots,m$ the polynomials

$$\mathcal{Z}_i(x)=\begin{cases}\mathcal{R}_{g_i}(x), & \text{for } i=1,\dots,m_1\\ \mathcal{S}_{h_{i-m_1}}(x), & \text{for } i=m_1+1,\dots,m\end{cases}$$

and for $\alpha-m_2\neq-1,-2,\dots,$ and $\beta-m_1\neq-1,-2,\dots,$ we write

$$\rho^i_{x,j}=\begin{cases}(-1)^{m-j}\Gamma^{\alpha-j,\beta-1}_{\alpha-m,\beta-j}(x), & \text{for } i=1,\dots,m_1\\ 1, & \text{for } i=m_1+1,\dots,m\end{cases}$$

where $\Gamma^{a,b}_{c,d}(x)=\frac{\Gamma(x+a+1)\Gamma(x+b+1)}{\Gamma(x+c+1)\Gamma(x+d+1)}$.

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and for $\alpha - m_2 \neq -1, -2, \ldots,$ and $\beta - m_1 \neq -1, -2, \ldots,$ we write

$$\rho_{x,j}^{i} = \begin{cases} (-1)^{m-j} \Gamma_{\alpha-m,\beta-j}^{\alpha-j,\beta-1}(x), & \text{for } i = 1, \ldots, m_1 \\ 1, & \text{for } i = m_1 + 1, \ldots, m \end{cases}$$

where $\Gamma_{c,d}^{a,b}(x) = \frac{\Gamma(x+a+1)\Gamma(x+b+1)}{\Gamma(x+c+1)\Gamma(x+d+1)}$.

We will also denote $\mathfrak{p}, \mathfrak{q}$ the following polynomials

$$\mathfrak{p}(x) = \prod_{i=1}^{m_1-1} (-1)^{m_1-i} (x + \alpha - m + 1)_{m_1-i} (x + \beta - m_1 + i)_{m_1-i}$$

$$\mathfrak{q}(x) = (-1)^{\binom{m}{2}} \prod_{h=1}^{m-1} \left(\prod_{i=1}^{h} (2(x - m) + \alpha + \beta + i + h) \right)$$

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We associate to G and H the following $m \times m$ quasi-Casoratian determinant

$$\Lambda_{G,H}(n) = \frac{\begin{vmatrix} \rho^1_{n,1}\mathcal{Z}_1(\theta_{n-1}) & \rho^1_{n,2}\mathcal{Z}_1(\theta_{n-2}) & \cdots & \rho^1_{n,m}\mathcal{Z}_1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \rho^m_{n,1}\mathcal{Z}_m(\theta_{n-1}) & \rho^m_{n,2}\mathcal{Z}_m(\theta_{n-2}) & \cdots & \rho^m_{n,m}\mathcal{Z}_m(\theta_{n-m}) \end{vmatrix}}{\mathfrak{p}(n)\mathfrak{q}(n)}$$

where $\theta_x = x(x+\alpha+\beta+1)$ (Jacobi eigenvalue), and assume that $\Lambda_{G,H}(n) \neq 0, n \geq 0$.

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Finally define the polynomials $(q_n)_n$ by

$$q_n(x) = \frac{\begin{vmatrix} J^{\alpha,\beta}_n(x) & -J^{\alpha,\beta}_{n-1}(x) & \dots & (-1)^m J^{\alpha,\beta}_{n-m}(x) \\ \rho^1_{n,0}\mathcal{Z}_1(\theta_n) & \rho^1_{n,1}\mathcal{Z}_1(\theta_{n-1}) & \dots & \rho^1_{n,m}\mathcal{Z}_1(\theta_{n-m}) \\ \vdots & \vdots & \ddots & \vdots \\ \rho^m_{n,0}\mathcal{Z}_m(\theta_n) & \rho^m_{n,1}\mathcal{Z}_m(\theta_{n-1}) & \dots & \rho^m_{n,m}\mathcal{Z}_m(\theta_{n-m}) \end{vmatrix}}{\mathfrak{p}(n)\mathfrak{q}(n)}$$

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$$J^{\alpha,\beta}_n(x) = \frac{(-1)^n(\alpha+\beta+1)_n}{2^n(\beta+1)_n}\sum_{j=0}^{n}\binom{n+\alpha}{j}\binom{n+\beta}{n-j}(x-1)^{n-j}(x+1)^j$$

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1. Using the $\mathcal{D}$-operator method we already proved that there exists a higher-order differential operator D_x with polynomial coefficients such that $D_x(q_n) = \lambda_n q_n$. We will prove that $(q_n)_n$ are bispectral, i.e. they also satisfy a higher-order recurrence relation of the form

$$Q(x)q_n(x) = \sum_{i=s}^{r} \gamma_{n,i} q_{n+i}(x), \quad s \leq r$$

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$$\mathcal{R}_{g_k}(\theta_x) = u^{\alpha}_{\beta+k-1}(x) + \sum_{l=0}^{k-1} \frac{(\beta+k-l)_l \binom{k-1}{l} a_{k-l-1}}{(-1)^l(\beta-l)_l} u^{\alpha}_l(x), \quad k = 1, \dots, m_1$$

$$\mathcal{S}_{h_k}(\theta_x) = u^{\alpha}_{\alpha+k-1}(x) + \sum_{l=0}^{k-1} \frac{(\alpha+k-l)_l \binom{k-1}{l} b_{k-l-1}}{(-1)^l(\alpha-l)_l} u^{\alpha}_l(x), \quad k = 1, \dots, m_2$$

where α and β are nonnegative integers with $0 \leq \alpha \leq \max H$, $0 \leq \beta \leq \max G$, and

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   $(q_n)_n$ are the Krall-Jacobi polynomials, orthogonal with respect to the measure

$$(1-x)^{\alpha-m_2}(1+x)^{\beta-m_1} + \sum_{h=0}^{m_2-1} c_h \delta^{(h)}_1 + \sum_{h=0}^{m_1-1} d_h \delta^{(h)}_{-1}$$

   for certain parameters c_k, $k = 0, \dots, m_2-1$, d_k, $k = 0, \dots, m_1-1$.

OUTLINE

1. The case $\alpha - \max G \neq 0, -1, -2, \ldots$ and $\beta - \max H \neq 0, -1, -2, \ldots$

- Orthogonality properties and recurrence relations
- Algebra of difference operators

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- Orthogonality properties and recurrence relations
- Three-term recurrence relation

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ORTHOGONALITY PROPERTIES

We can always write the polynomials $\mathcal{R}_g$ and $\mathcal{S}_g$ in the following form

$$\mathcal{R}_g(\theta_x) = \sum_{s=0}^{g} \nu_s^g u_s^\alpha(x), \quad \mathcal{S}_h(\theta_x) = \sum_{s=0}^{h} \omega_s^h u_s^\alpha(x),$$

for certain numbers $\nu_s^g, s = 0, 1, \ldots, g$, and $\omega_s^h, s = 0, 1, \ldots, h$, $(\nu_g^g = 1, \omega_h^h = 1)$ which are uniquely determined from the polynomials $\mathcal{R}_g$ and $\mathcal{S}_h$.

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Let σ_i be a permutation of the set $\{1, 2, \ldots, m_i\}, i = 1, 2$, and write $\mathfrak{g}_l = g_{\sigma_1(l)}$ and $\mathfrak{h}_l = h_{\sigma_2(l)}$. Define the rational functions U_l and V_l by

$$U_l(x) = \phi_l(x) + \frac{\kappa_l}{\Gamma(\beta)} \sum_{s=0}^{\mathfrak{g}_{l+1}} \frac{(\beta - s)_s 2^s s! \nu_s^{\mathfrak{g}_{l+1}}}{(1+x)^{s+1}}, \quad V_l(x) = \psi_l(x) + \frac{\tau_l}{\Gamma(\alpha)} \sum_{s=0}^{\mathfrak{h}_{l+1}} \frac{(\alpha - s)_s 2^s s! \omega_s^{\mathfrak{h}_{l+1}}}{(1-x)^{s+1}},$$

where $\phi_l(x)$ and $\psi_l(x)$ are rational functions satisfying certain algebraic relations. Here κ_l, $l = 0, \ldots, m_1 - 1$ and τ_l, $l = 0, \ldots, m_2 - 1$ are in principle real numbers.

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For real numbers α and β such that $\alpha - \max H \neq 0, -1, -2, \ldots,$ and $\beta - \max G \neq 0, -1, -2, \ldots,$ we consider

$$\begin{aligned}\langle p, q \rangle &= \langle p, q \rangle_1 + \langle p, q \rangle_2 + \langle p, q \rangle_3 \\ &= \int_{-1}^{1} p(x) q(x) \mu_{\alpha - m_2, \beta - m_1}(x) dx \\ &\quad + \sum_{l=0}^{m_1 - 1} \frac{q^{(l)}(-1)}{2^{m_2} l!} \int_{-1}^{1} p(x) U_l(x) \mu_{\alpha,\beta}(x) dx + \sum_{l=0}^{m_2 - 1} \frac{q^{(l)}(1)}{(-1)^{m_1 + l} l!} \int_{-1}^{1} p(x) V_l(x) \mu_{\alpha,\beta}(x) dx\end{aligned}$$

where $\mu_{\alpha,\beta} = (1-x)^\alpha (1+x)^\beta$ denotes the Jacobi weight.

ORTHOGONALITY PROPERTIES

Theorem. Let α and β be real numbers with $\alpha - \max H \neq 0, -1, -2, \ldots,$ and $\beta - \max G \neq 0, -1, -2, \ldots$. Assume that $\Lambda_{G,H}(n) \neq 0, n \geq 0$. Then for $n \geq m$, the polynomials $(q_n)_n$ satisfy the following orthogonality properties with respect to the previous bilinear form:

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\begin{aligned}
\langle q_n, q_i \rangle &= 0, \quad i = 0, 1, \ldots, n-1, \\
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$$\begin{aligned} \langle q_n, q_i \rangle &= 0, \quad i = 0, 1, \ldots, n-1, \\ \langle q_n, q_n \rangle &\neq 0. \end{aligned}$$

The proof is based on the fact that for $k, n \geq 0$, $j = 0, 1, \ldots, n$, $n - j \geq k$ and $i = 1, \ldots, m_1$, we have

$$\langle (1+x)^k J_{n-j}^{\alpha,\beta}, b_i \rangle = \frac{c_{n,i} \rho_{n,j}^i}{(-1)^j} \left[\sum_{l=i-1}^{[m_1 \wedge (m_2+i)]-1} \frac{\kappa_l \binom{m_2}{l-i+1}}{(-2)^{l+1}} \sum_{s=k}^{\mathfrak{g}_{l+1}} 2^k \nu_s^{\mathfrak{g}_{l+1}} (\beta - s)_k (s-k+1)_k u_{s-k}^{\alpha}(n-j) \right]$$

where $c_{n,i} = (-1)^{m+i} 2^{\alpha+\beta+i} \Gamma(\beta+1)\Gamma(n+\alpha-m+1)/(\Gamma(\alpha+\beta+1)\Gamma(n+\beta))$ and for $i = m_1 + 1, \ldots, m$, we have

$$\langle (1-x)^k J_{n-j}^{\alpha,\beta}, b_i \rangle = \frac{d_{n,i}}{(-1)^j} \left[\sum_{l=i-m_1-1}^{[m_2 \wedge i]-1} \frac{\tau_l \binom{m_1}{i-l-1}}{(-2)^{l+1}} \sum_{s=k}^{\mathfrak{h}_{l+1}} 2^k \omega_s^{\mathfrak{h}_{l+1}} (\alpha - s)_k (s-k+1)_k u_{s-k}^{\alpha}(n-j) \right]$$

where $d_{n,i} = (-1)^{n+i} 2^{\alpha+\beta+i} \Gamma(\beta+1)/\Gamma(\alpha+\beta+1)$. Here b_i are defined by

$$b_s(x) = \begin{cases} (1+x)^{s-1}(1-x)^{m_2}, & \text{for } s = 1, \ldots, m_1, \\ (1+x)^{m_1}(1-x)^{s-m_1-1}, & \text{for } s = m_1+1, \ldots, m, \\ (1+x)^{m_1}(1-x)^{m_2} x^{s-m-1}, & \text{for } s = m+1, \ldots \end{cases}$$

RECURRENCE RELATIONS

Theorem. Let α and β be real numbers with $\alpha - \max H \neq 0, -1, -2, \ldots$, and $\beta - \max G \neq 0, -1, -2, \ldots$. Let $Q \in \mathbb{R}[x]$ be a polynomial of degree s satisfying that $(1+x)^{\max G}(1-x)^{\max H}$ divides Q'. Then the sequence $(q_n)_n$ satisfies the recurrence relation

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It is possible to prove that under the conditions of the previous two theorems, the polynomials $(q_n)_n$ never satisfy a three-term recurrence relation (i.e. $s = 1$) and hence they are not orthogonal with respect to any measure.

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Let $\mathcal{A}_n$ be the algebra formed by all higher-order difference operators of the form $D_n = \sum_{i=s}^{r} \gamma_{n,i} \mathfrak{S}_i$, where $\mathfrak{S}_l(f(n)) = f(n+l)$. Then we would like to study

$$\mathfrak{D}_n = \{D_n \in \mathcal{A}_n : D_n(q_n) = Q(x)q_n,\ Q \in \mathbb{R}[x]\}$$

$\mathfrak{D}_n$ is characterized by the algebra of polynomials from the corresponding eigenvalues

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In general it is difficult to study this algebra $\tilde{\mathfrak{D}}_n$, but in some situations we can characterize completely this algebra of difference operators.

ALGEBRA OF DIFFERENCE OPERATORS

Theorem. Let α, β be real numbers satisfying $\alpha - \max H \neq 0, -1, -2, \dots,$ and $\beta - \max G \neq 0, -1, -2, \dots$, and assume that $\Lambda_{G,H}(n) \neq 0$. If G and H are segments, i.e. $G = \{g, g+1, \dots, g+a\}$ and $H = \{h, h+1, \dots, h+b\}$ for some integers $f, h > 0$ and $a, b \geq 0$ then

$$\tilde{\mathfrak{D}}_n = \{Q \in \mathbb{R}[x] : (1+x)^{\max G}(1-x)^{\max H} \text{ divides } Q'\}.$$

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If G or H are not segments the algebra $\tilde{\mathfrak{D}}_n$ can have a more complicated structure, as the following example shows. Take $G = \{1, 3\}$, $H = \{1\}$, $\alpha = 1/2, \beta = 1/3$, and

$$R_1(x) = x + 1, \quad R_3(x) = x^3 + x^2/3 + 2x/3 + 1, \qquad S_1(x) = x + 1/2$$

Using Maple one can see that the polynomials $(q_n)_n$ satisfy recurrence relations for

$$Q_0(x) = \frac{x}{135}(135x^3 + 244x^2 - 270x - 732)$$

and Q_0' is not divisible by $(1+x)^3(1-x)$. Computational evidence suggests that

$$\tilde{\mathfrak{D}}_n = \{Q(x) + c_0 Q_0(x) : (1+x)^3(1-x) \text{ divides } Q' \text{ and } c_0 \in \mathbb{R}\}.$$

2. The case $\alpha = 0, \ldots, \max G$ and/or $\beta = 0, \ldots, \max H$

- Orthogonality properties and recurrence relations
- Three-term recurrence relation

ORTHOGONALITY PROPERTIES

We have to change $\langle p,q\rangle_2$ and $\langle p,q\rangle_3$ in the previous bilinear form by transforming a portion of the integral in a discrete Sobolev inner product. More precisely, we define the bilinear form

$$\langle p,q\rangle_S = \langle p,q\rangle_1 + \langle p,q\rangle_2 + \langle p,q\rangle_{2S} + \langle p,q\rangle_3 + \langle p,q\rangle_{3S}$$

where

$$\langle p,q\rangle_{2S} = 2^{\alpha+\beta}\Gamma(\alpha+1)\sum_{l=0}^{m_1-1}\kappa_l\frac{q^{(l)}(-1)}{2^{m_2}l!}\sum_{j=0}^{\mathfrak{g}_{l+1}-\beta}\frac{2^j p^{(j)}(-1)}{j!}\sum_{s=\beta+j}^{\mathfrak{g}_{l+1}}\frac{(\beta-s)_j s!}{\Gamma(\alpha+\beta-s+j+1)}\nu_s^{\mathfrak{g}_{l+1}}$$

and

$$\langle p,q\rangle_{3S} = 2^{\alpha+\beta}\Gamma(\beta+1)\sum_{l=0}^{m_2-1}\tau_l\frac{q^{(l)}(1)}{(-1)^{m_1+l}l!}\sum_{j=0}^{\mathfrak{h}_{l+1}-\alpha}\frac{2^j p^{(j)}(1)}{(-1)^j j!}\sum_{s=\alpha+j}^{\mathfrak{h}_{l+1}}\frac{(\alpha-s)_j s!}{\Gamma(\alpha+\beta-s+j+1)}\omega_s^{\mathfrak{h}_{l+1}}$$

The discrete part of this bilinear form can not be represented in general as the discrete Jacobi-Sobolev bilinear form.

ORTH. PROPERTIES AND REC. RELATIONS

Theorem. Let α and β be positive integers satisfying $\alpha = m_2, \ldots, \max H$ and $\beta = m_1, \ldots, \max G$ and assume that $\Lambda_{G,H}(n) \neq 0$. Then, for $n \geq m$, the polynomials $(q_n)_n$ defined satisfy the following orthogonality properties with respect to the previous bilinear from:

$$\langle q_n, q_i \rangle_S = 0, \quad i = 0, 1, \ldots, n-1,$$
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Theorem. Let α, β be positive integers satisfying $\alpha = m_2, \ldots, \max H$, and $\beta = m_1, \ldots, \max G$, and assume that $\Lambda_{G,H}(n) \neq 0$. Write

$$\rho_1 = \max\{\max G - \beta + 1, \beta\}, \quad \rho_2 = \max\{\max H - \alpha + 1, \alpha\}$$

Let $Q \in \mathbb{R}[x]$ be a polynomial of degree s satisfying that $(1+x)^{\rho_1}(1-x)^{\rho_2}$ divides Q'. Then the sequence of polynomials $(q_n)_n$ satisfies the recurrence relation of the form

$$Q(x)q_n(x) = \sum_{j=-s}^{s} \gamma_{n,j} q_{n+j}(x), \quad \gamma_{n,s}, \gamma_{n,-s} \neq 0$$

where $s = \deg Q$.

THREE-TERM RECURRENCE RELATION

Theorem. Let α, β be two positive integers satisfying $\alpha = m_2, \dots, \max H$, and $\beta = m_1, \dots, \max G$, and assume that $\Lambda_{G,H}(n) \neq 0$. Then the sequence $(q_n)_n$ only satisfies a three-term recurrence relation when G and H are segments of the form $G = \{\beta, \beta + 1, \dots, \beta + m_1 - 1\}$, and $H = \{\alpha, \alpha + 1, \dots, \alpha + m_2 - 1\}$, respectively, and the polynomials $\mathcal{R}_g(x)$ and $\mathcal{S}_h(x)$ have the form

$$\mathcal{R}_{g_k}(\theta_x) = u^{\alpha}_{\beta+k-1}(x) + \sum_{l=0}^{k-1} \frac{(\beta + k - l)_l \binom{k-1}{l} a_{k-l-1}}{(-1)^l (\beta - l)_l} u^{\alpha}_l(x), \quad k = 1, \dots, m_1$$

$$\mathcal{S}_{h_k}(\theta_x) = u^{\alpha}_{\alpha+k-1}(x) + \sum_{l=0}^{k-1} \frac{(\alpha + k - l)_l \binom{k-1}{l} b_{k-l-1}}{(-1)^l (\alpha - l)_l} u^{\alpha}_l(x), \quad k = 1, \dots, m_2$$

where

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$(q_n)_n$ are the Krall-Jacobi polynomials, orthogonal with respect to the measure

$$(1-x)^{\alpha - m_2}(1+x)^{\beta - m_1} + \sum_{h=0}^{m_2-1} c_h \delta^{(h)}_1 + \sum_{h=0}^{m_1-1} d_h \delta^{(h)}_{-1}$$

for certain parameters c_k, $k = 0, \dots, m_2 - 1$, d_k, $k = 0, \dots, m_1 - 1$.


THANK YOU