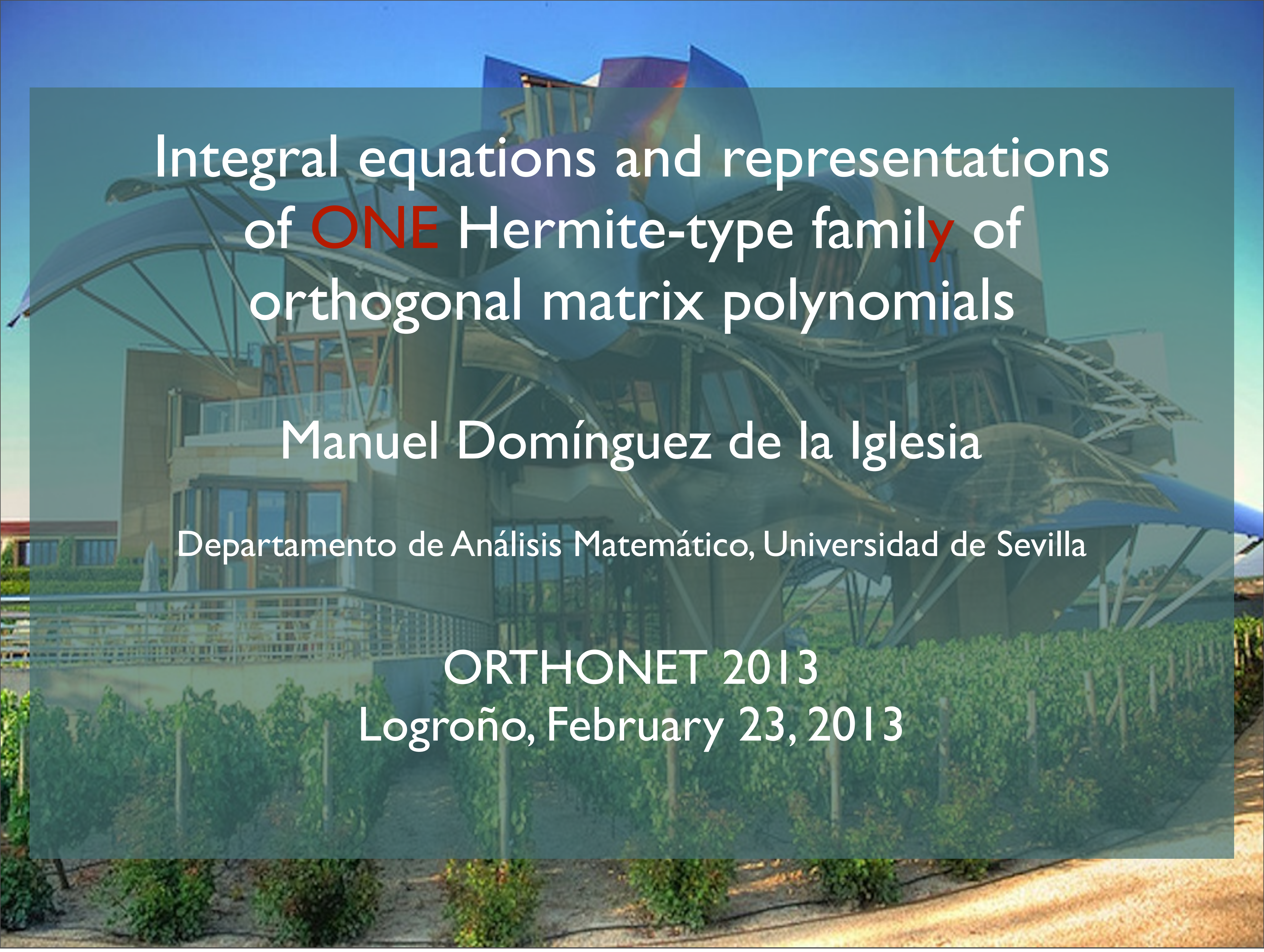
Integral equations and representations of some Hermite-type families of orthogonal matrix polynomials

Manuel Domínguez de la Iglesia

Departamento de Análisis Matemático, Universidad de Sevilla

ORTHONET 2013
Logroño, February 23, 2013



Integral equations and representations of ONE Hermite-type family of orthogonal matrix polynomials

Manuel Domínguez de la Iglesia

Departamento de Análisis Matemático, Universidad de Sevilla

ORTHONET 2013
Logroño, February 23, 2013

HERMITE POLYNOMIALS

Consider the classical Hermite polynomials

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \left(\mathrm{e}^{-x^2}\right)^{(n)}$$

satisfying

$$\int_{\mathbb{R}} H_n(x) H_m(x) \mathrm{e}^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

HERMITE POLYNOMIALS

Consider the classical Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}$$

satisfying

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

1810

1859

HERMITE POLYNOMIALS

Consider the classical Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}$$

satisfying

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

SUR UN NOUVEAU DÉVELOPPEMENT

EN SÉRIE DES FONCTIONS.

Comptes rendus de l'Académie des Sciences, t. LVIII, 1864 (I), p. 93 et 266.

Les fonctions uniformes de plusieurs variables à périodes simultanées par lesquelles MM. Weierstrass et Riemann ont résolu le problème de l'inversion des intégrales de différentielles algébriques quelconques sont représentées, comme l'on sait, par le quotient de deux séries telles que

$$\sum e^{-\varphi(x+m,\, y+n,\, z+p,\, \dots)},$$

HERMITE OR WAVE FUNCTIONS

HERMITE OR WAVE FUNCTIONS

The Hermite or wave functions are defined by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \mathrm{e}^{-x^2/2} H_n(x)$$

HERMITE OR WAVE FUNCTIONS

The Hermite or wave functions are defined by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \mathrm{e}^{-x^2/2} H_n(x)$$

$(\psi_n)_n$ is a *complete* orthonormal set in $L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} \psi_n(x) \psi_m^*(x) dx = \delta_{nm}$$

HERMITE OR WAVE FUNCTIONS

The Hermite or wave functions are defined by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x)$$

Let $\mathcal{D} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the Schrödinger operator with quadratic potential

$$[\mathcal{D}f](x) = \frac{d^2}{dx^2} f(x) - x^2 f(x)$$

and $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the continuous Fourier transform

$$[\mathcal{F}f](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt} dt$$

HERMITE OR WAVE FUNCTIONS

The Hermite or wave functions are defined by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \mathrm{e}^{-x^2/2} H_n(x)$$

Let $\mathcal{D} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the Schrödinger operator with quadratic potential

$$[\mathcal{D}f](x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) - x^2 f(x)$$

and $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the continuous Fourier transform

$$[\mathcal{F}f](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \mathrm{e}^{ixt} \mathrm{d}t$$

$$\Rightarrow \mathcal{D} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{D}$$

HERMITE OR WAVE FUNCTIONS

The Hermite or wave functions are defined by

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} \mathrm{e}^{-x^2/2} H_n(x)$$

$(\psi_n)_n$ are eigenfunctions of the Schrödinger operator

$$[\mathcal{D}\psi_n](x) = \psi_n''(x) - x^2\psi_n(x) = -(2n+1)\psi_n(x)$$

and the Fourier transform

$$[\mathcal{F}\psi_n](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_n(t) e^{ixt} dt = (i)^n \psi_n(x)$$

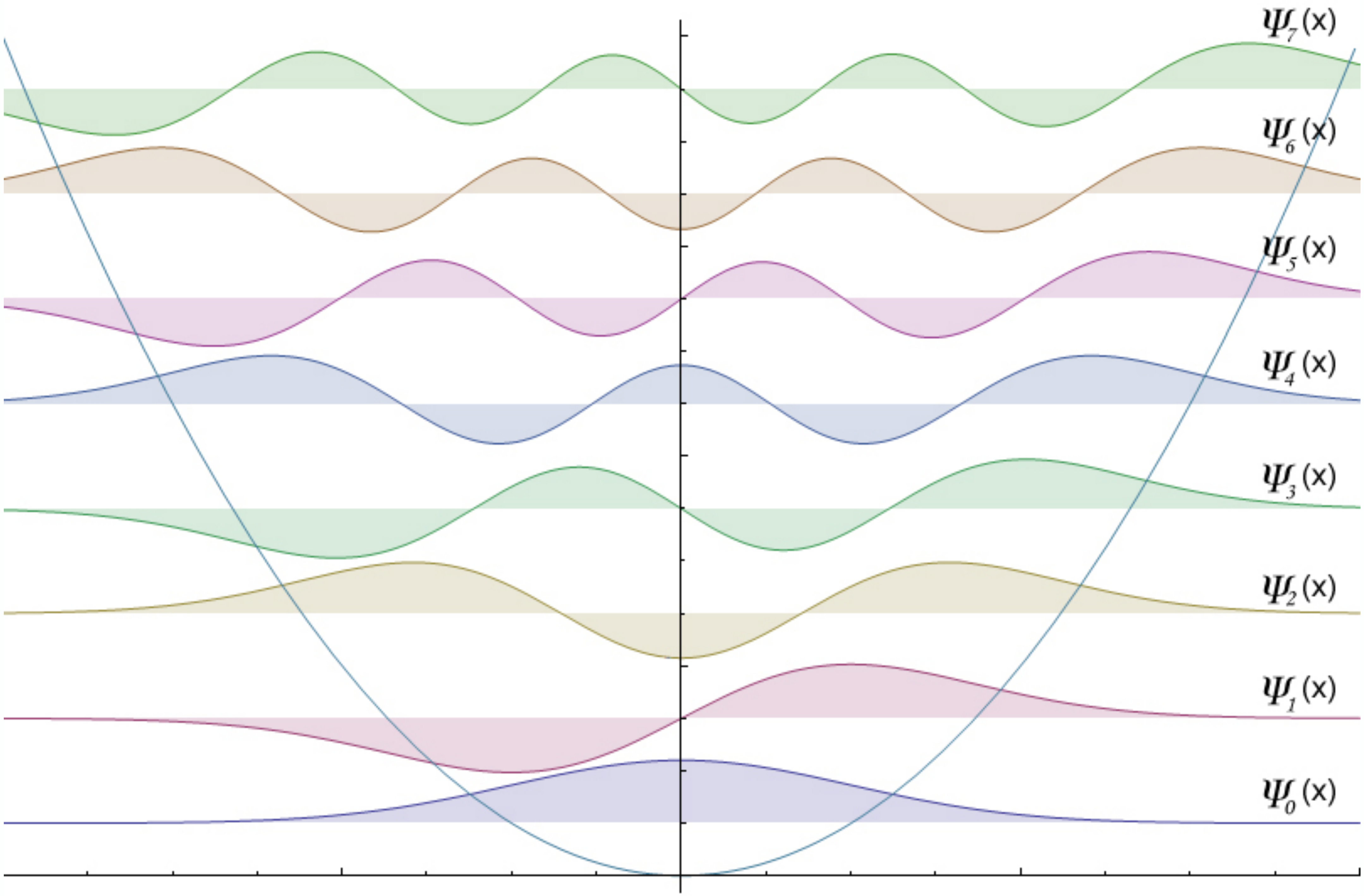
$$\Rightarrow \mathcal{D} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{D}$$

HERMITE OR WAVE FUNCTIONS

$(\psi_n)_n$ are the eigenfunctions of the one-dimensional time-independent quantum harmonic oscillator

HERMITE OR WAVE FUNCTIONS

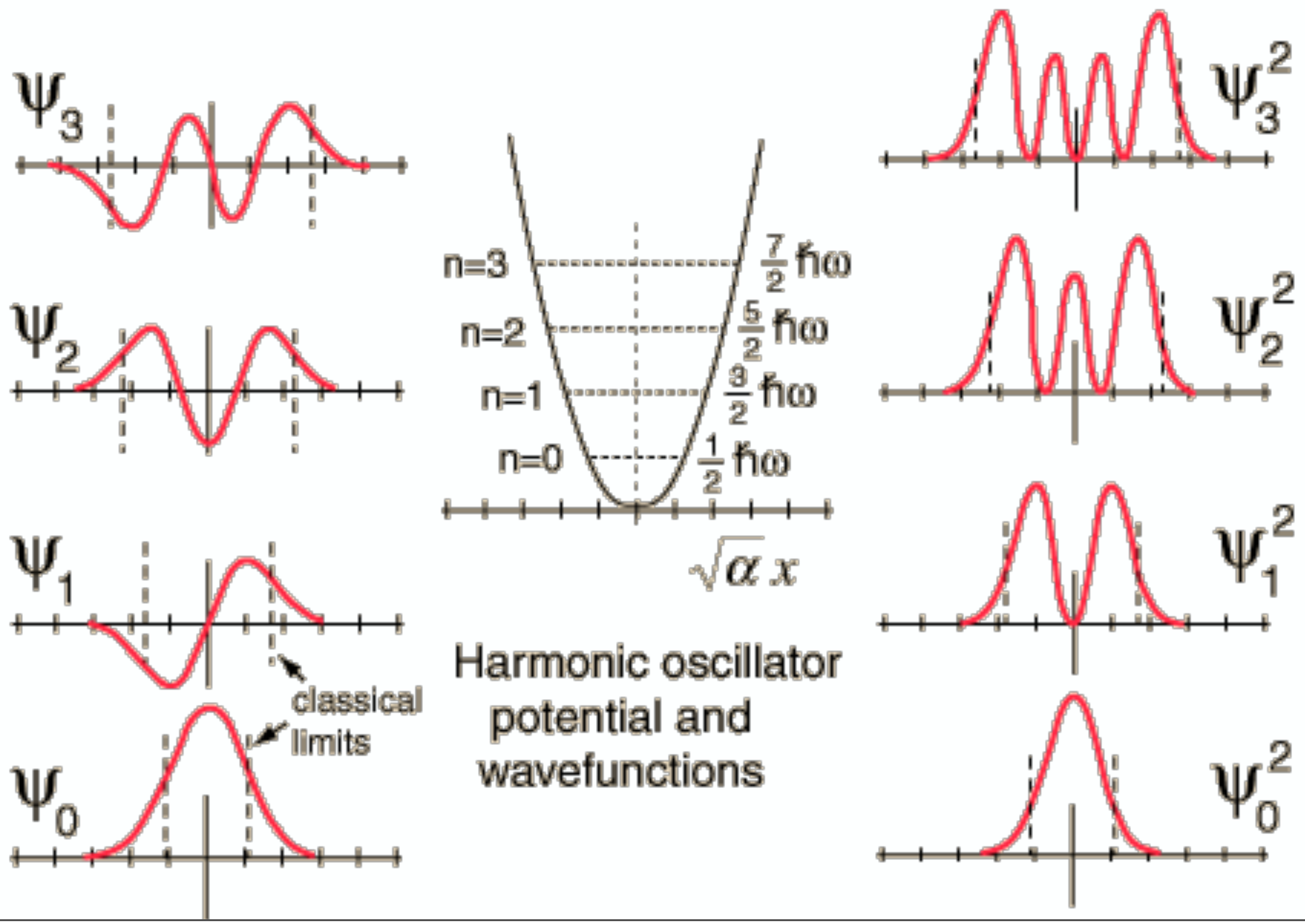
$(\psi_n)_n$ are the eigenfunctions of the one-dimensional time-independent quantum harmonic oscillator



HERMITE OR WAVE FUNCTIONS

$(\psi_n)_n$ are the eigenfunctions of the one-dimensional time-independent quantum harmonic oscillator

$|\psi_n|^2$ is a probability distribution for every n and gives the probability of finding the oscillator at a particular value of x



HERMITE OR WAVE FUNCTIONS

$(\psi_n)_n$ are the eigenfunctions of the one-dimensional time-independent quantum harmonic oscillator

$|\psi_n|^2$ is a probability distribution for every n and gives the probability of finding the oscillator at a particular value of x

We also have that

$$(x)_{nm} \doteq \int_{\mathbb{R}} x\psi_n(x)\psi_m^*(x)dx = \sqrt{\frac{n}{2}}\delta_{m,n-1} + \sqrt{\frac{n+1}{2}}\delta_{m,n+1}$$

(x) is the matrix of the homomorphism $f \mapsto xf$ in $L^2(\mathbb{R})$ with respect to the basis $(\psi_n)_n$ (three-term recurrence relation).

This also means that transitions occur only between adjacent energy level of the quantum harmonic oscillator.

HERMITE POLYNOMIALS

Some consequences for the Hermite polynomials:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

HERMITE POLYNOMIALS

Some consequences for the Hermite polynomials:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

HERMITE POLYNOMIALS

Some consequences for the Hermite polynomials:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

From the integral equation we get

$$\mathrm{e}^{-x^2/2}H_{2n}(x) = (-1)^n\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{e}^{-t^2/2}H_{2n}(t)\cos(xt)dt$$

$$\mathrm{e}^{-x^2/2}H_{2n+1}(x) = (-1)^n\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{e}^{-t^2/2}H_{2n+1}(t)\sin(xt)dt$$

HERMITE POLYNOMIALS

Some consequences for the Hermite polynomials:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

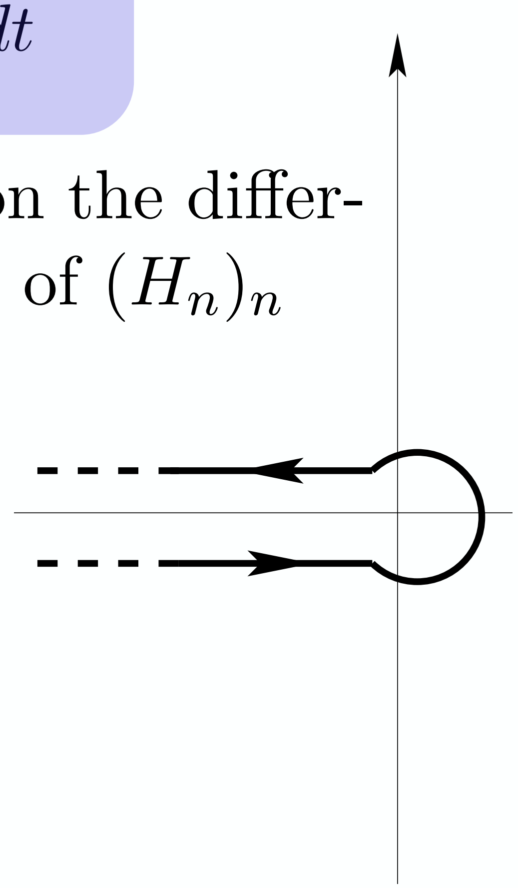
From the integral equation we get

$$\mathrm{e}^{-x^2/2}H_{2n}(x) = (-1)^n\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{e}^{-t^2/2}H_{2n}(t)\cos(xt)dt$$

$$\mathrm{e}^{-x^2/2}H_{2n+1}(x) = (-1)^n\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{e}^{-t^2/2}H_{2n+1}(t)\sin(xt)dt$$

Additionally, using the Laplace and the Fourier transform on the differential equation we obtain the following integral representations of $(H_n)_n$

$$H_n(x) = \frac{n!}{2\pi i}\oint_\gamma \mathrm{e}^{-z^2+2zx}\frac{dz}{z^{n+1}}$$



$$H_n(x) = \frac{(-2i)^n\mathrm{e}^{x^2}}{\sqrt{\pi}}\int_{\mathbb{R}} \mathrm{e}^{-u^2+2ixu}u^n du$$

CHRISTOFFEL-DARBOUX KERNEL

The Christoffel-Darboux (CD) kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y)$$

describes the statistical properties of the eigenvalues of a random matrix $\boldsymbol{M}$ in the space of $(n \times n)$ Hermitian matrices with the measure $\mu(\boldsymbol{M}) = \mathrm{e}^{-\mathrm{Tr}(\boldsymbol{M}^2)} d\boldsymbol{M}$ (GUE, Mehta).

CHRISTOFFEL-DARBOUX KERNEL

The Christoffel-Darboux (CD) kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y)$$

describes the statistical properties of the eigenvalues of a random matrix $\boldsymbol{M}$ in the space of $(n \times n)$ Hermitian matrices with the measure $\mu(\boldsymbol{M}) = \mathrm{e}^{-\mathrm{Tr}(\boldsymbol{M}^2)} d\boldsymbol{M}$ (GUE, Mehta).

The last particle distribution is given by the Fredholm determinant

$$F(s) = \mathbb{P}\left[\lambda_{\max} \leq s\right] = \det(I - \chi_s \mathbb{K}_n)$$

where χ_s is the indicator function of the interval $[s,\infty)$ and $\mathbb{K}_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the integral operator

$$[\mathbb{K}_n f](x) = \int_{\mathbb{R}} K_n(x,y) f(y) dy \quad \forall\, f \in L^2(\mathbb{R})$$

CHRISTOFFEL-DARBOUX KERNEL

The Christoffel-Darboux (CD) kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y)$$

describes the statistical properties of the eigenvalues of a random matrix $\boldsymbol{M}$ in the space of $(n \times n)$ Hermitian matrices with the measure $\mu(\boldsymbol{M}) = \mathrm{e}^{-\mathrm{Tr}(\boldsymbol{M}^2)} d\boldsymbol{M}$ (GUE, Mehta).

The last particle distribution is given by the Fredholm determinant

$$F(s) = \mathbb{P}\left[\lambda_{\max} \le s\right] = \det(I - \chi_s \mathbb{K}_n)$$

where χ_s is the indicator function of the interval $[s, \infty)$ and $\mathbb{K}_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the integral operator

$$[\mathbb{K}_n f](x) = \int_{\mathbb{R}} K_n(x,y) f(y) dy \quad \forall\, f \in L^2(\mathbb{R})$$

THEOREM (Tracy-Widom, 1994)

The log derivative of the Fredholm determinant $R(s) = \partial_s \log(\det(I - \chi_s \mathbb{K}_n))$ solves the sigma-form of the PIV equation

$$(R'')^2 + 4(R')^2(R' + 2n) - 4(sR' - R)^2 = 0$$

MATRIX VALUED FUNCTIONS

Let $\boldsymbol{W}$ be a weight matrix (positive definite and finite moments). Consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ the weighted space with the inner product

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{F}(x)\boldsymbol{W}(x)\boldsymbol{G}^*(x)dx$$

MATRIX VALUED FUNCTIONS

Let $\boldsymbol{W}$ be a weight matrix (positive definite and finite moments). Consider $L^2_{\boldsymbol{W}}(\mathbb{R},\mathbb{C}^{N\times N})$ the weighted space with the inner product

$$\langle \boldsymbol{F},\boldsymbol{G}\rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{F}(x)\boldsymbol{W}(x)\boldsymbol{G}^*(x)dx$$

A sequence $(\boldsymbol{P}_n)_n$ of matrix orthonormal polynomials (MOP) with respect to $\boldsymbol{W}$ is a sequence satisfying

$$\deg \boldsymbol{P}_n = n, \quad \langle \boldsymbol{P}_n,\boldsymbol{P}_m\rangle_{\boldsymbol{W}} = \boldsymbol{I}_N\delta_{nm}$$

MATRIX VALUED FUNCTIONS

Let $\boldsymbol{W}$ be a weight matrix (positive definite and finite moments). Consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ the weighted space with the inner product

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{F}(x)\boldsymbol{W}(x)\boldsymbol{G}^*(x)dx$$

A sequence $(\boldsymbol{P}_n)_n$ of matrix orthonormal polynomials (MOP) with respect to $\boldsymbol{W}$ is a sequence satisfying

$$\deg \boldsymbol{P}_n = n, \quad \langle \boldsymbol{P}_n, \boldsymbol{P}_m \rangle_{\boldsymbol{W}} = \boldsymbol{I}_N \delta_{nm}$$

They were studied for the first time by M. G. Krein (1949)

FUNDAMENTAL ASPECTS OF THE REPRESENTATION THEORY OF HERMITIAN OPERATORS WITH DEFICIENCY INDEX (m, m) *

M. G. KREĬN

To the memory of my honored teacher, Nikolaĭ Grigor'evič Čebotarev.

In studying the spectral properties of selfadjoint extensions of Hermitian operators we can always limit ourselves to the case of a simple operator (see §1.3).

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

For any infinitely differentiable f in a neighborhood of $x = 0$ we have

$$f(x) = \sum_{j=0}^{\infty} f^{(j)}(0) \frac{x^j}{j!}$$

Whenever we write $f(\boldsymbol{X})$, for any $N \times N$ matrix $\boldsymbol{X}$, we mean

$$f(\boldsymbol{X}) = \sum_{j=0}^{\infty} f^{(j)}(0) \frac{\boldsymbol{X}^j}{j!}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

For example $f(x) = e^{i\frac{\pi}{2}x}$

$$f(\boldsymbol{J}) = e^{i\frac{\pi}{2}\boldsymbol{J}} = \begin{pmatrix} (i)^{N-1} & & & \\ & \ddots & & \\ & & i & \\ & & & 1 \end{pmatrix}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

- $f(x) = e^{i\pi x}$

$$f(\boldsymbol{J}) = e^{i\pi J} = \begin{pmatrix} (-1)^{N-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; \nu_i \in \mathbb{R}, \; \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

- $f(x) = \sin\left(\frac{\pi}{2}x\right) = \frac{1}{2i}\left(e^{i\frac{\pi}{2}x} - e^{-i\frac{\pi}{2}x}\right)$

$$f(\boldsymbol{J}) = \sin\left(\frac{\pi}{2}J\right) = \begin{pmatrix} \sin\left(\frac{\pi}{2}(N-1)\right) & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 0 \end{pmatrix}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

- $f(x) = \cos\left(\frac{\pi}{2}x\right) = \frac{1}{2}(e^{i\frac{\pi}{2}x} + e^{-i\frac{\pi}{2}x})$

$$f(\boldsymbol{J}) = \cos\left(\frac{\pi}{2}\boldsymbol{J}\right) = \begin{pmatrix} \cos\left(\frac{\pi}{2}(N-1)\right) & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & 1 \end{pmatrix}$$

SOME NOTATION

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \boldsymbol{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$z^{\boldsymbol{J}} = \begin{pmatrix} z^{N-1} & & & \\ & \ddots & & \\ & & z & \\ & & & 1 \end{pmatrix}, \quad z^{-\boldsymbol{J}} = \begin{pmatrix} \frac{1}{z^{N-1}} & & & \\ & \ddots & & \\ & & \frac{1}{z} & \\ & & & 1 \end{pmatrix}$$

THE EXAMPLE

THE EXAMPLE

IMRN International Mathematics Research Notices
2004, No. 10

Orthogonal Matrix Polynomials Satisfying Second-Order Differential Equations

Antonio J. Durán and F. Alberto Grünbaum

THE EXAMPLE

IMRN International Mathematics Research Notices
2004, No. 10

Orthogonal Matrix Polynomials Satisfying Second-Order Differential Equations

Antonio J. Durán and F. Alberto Grünbaum

1 Introduction

The aim of this paper is to develop a general method that leads us to introduce and study classes of examples of orthonormal matrix polynomials $(P_n)_n$ satisfying a right-hand side second-order differential equations of the form

$$P_n''(t)A_2(t) + P_n'(t)A_1(t) + P_n(t)A_0(t) = \Gamma_n P_n(t), \quad n \geq 0, \tag{1.1}$$

where A_2, A_1, and A_0 are matrix polynomials (which do not depend on n) of degrees not bigger than 2, 1, and 0, respectively, and Γ_n are Hermitian matrices (i.e., each orthonormal matrix polynomial P_n is an eigenvector of the right-hand side second-order differential operator $\ell_{2,R} = D^2A_2(t) + D^1A_1(t) + D^0A_0(t)$).

THE EXAMPLE

Let us consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ with the weight matrix

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\boldsymbol{A}x}\mathrm{e}^{\boldsymbol{A}^*x}, \quad x \in \mathbb{R}$$

and the family of MOP $(\boldsymbol{P}_n)_n$ satisfying

$$\boldsymbol{P}_n''(x) + \boldsymbol{P}_n'(x)(-2x\boldsymbol{I} + 2\boldsymbol{A}) + \boldsymbol{P}_n(x)(\boldsymbol{A}^2 - 2\boldsymbol{J}) = (-2n\boldsymbol{I} - 2\boldsymbol{J})\boldsymbol{P}_n(x)$$

THE EXAMPLE

Let us consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ with the weight matrix

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\boldsymbol{A}x}\mathrm{e}^{\boldsymbol{A}^*x}, \quad x \in \mathbb{R}$$

and the family of MOP $(\boldsymbol{P}_n)_n$ satisfying

$$\boldsymbol{P}_n''(x) + \boldsymbol{P}_n'(x)(-2x\boldsymbol{I} + 2\boldsymbol{A}) + \boldsymbol{P}_n(x)(\boldsymbol{A}^2 - 2\boldsymbol{J}) = (-2n\boldsymbol{I} - 2\boldsymbol{J})\boldsymbol{P}_n(x)$$

The orthogonal matrix-valued functions (complete in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$)

$$\boldsymbol{\Phi}_n(x) = \mathrm{e}^{-x^2/2}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$$

are simultaneously eigenfunctions

THE EXAMPLE

Let us consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ with the weight matrix

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\boldsymbol{A}x}\mathrm{e}^{\boldsymbol{A}^*x}, \quad x \in \mathbb{R}$$

and the family of MOP $(\boldsymbol{P}_n)_n$ satisfying

$$\boldsymbol{P}_n''(x) + \boldsymbol{P}_n'(x)(-2x\boldsymbol{I} + 2\boldsymbol{A}) + \boldsymbol{P}_n(x)(\boldsymbol{A}^2 - 2\boldsymbol{J}) = (-2n\boldsymbol{I} - 2\boldsymbol{J})\boldsymbol{P}_n(x)$$

The orthogonal matrix-valued functions (complete in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$)

$$\boldsymbol{\Phi}_n(x) = \mathrm{e}^{-x^2/2}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$$

are simultaneously eigenfunctions

1) of a differential equation of Schrödinger type

$$\boldsymbol{\Phi}_n''(x) - \boldsymbol{\Phi}_n(x)(x^2\boldsymbol{I} + 2\boldsymbol{J}) + ((2n+1)\boldsymbol{I} + 2\boldsymbol{J})\boldsymbol{\Phi}_n(x) = \boldsymbol{0}$$

THE EXAMPLE

Let us consider $L^2_{\boldsymbol{W}}(\mathbb{R},\mathbb{C}^{N\times N})$ with the weight matrix

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\boldsymbol{A}x}\mathrm{e}^{\boldsymbol{A}^*x}, \quad x\in\mathbb{R}$$

and the family of MOP $(\boldsymbol{P}_n)_n$ satisfying

$$\boldsymbol{P}_n''(x) + \boldsymbol{P}_n'(x)(-2x\boldsymbol{I} + 2\boldsymbol{A}) + \boldsymbol{P}_n(x)(\boldsymbol{A}^2 - 2\boldsymbol{J}) = (-2n\boldsymbol{I} - 2\boldsymbol{J})\boldsymbol{P}_n(x)$$

The orthogonal matrix-valued functions (complete in $L^2(\mathbb{R},\mathbb{C}^{N\times N})$)

$$\boldsymbol{\Phi}_n(x) = \mathrm{e}^{-x^2/2}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$$

are simultaneously eigenfunctions

1) of a differential equation of Schrödinger type

$$\boldsymbol{\Phi}_n''(x) - \boldsymbol{\Phi}_n(x)(x^2\boldsymbol{I} + 2\boldsymbol{J}) + ((2n+1)\boldsymbol{I} + 2\boldsymbol{J})\boldsymbol{\Phi}_n(x) = \boldsymbol{0}$$

2) and an integral operator of Fourier type

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\boldsymbol{\Phi}_n(t)\mathrm{e}^{ixt}\mathrm{e}^{i\frac{\pi}{2}\boldsymbol{J}}dt = (i)^n\mathrm{e}^{i\frac{\pi}{2}\boldsymbol{J}}\boldsymbol{\Phi}_n(x)$$

THE EXAMPLE

Let us consider $L^2_{\boldsymbol{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ with the weight matrix

$$\boldsymbol{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\boldsymbol{A}x}\mathrm{e}^{\boldsymbol{A}^*x}, \quad x \in \mathbb{R}$$

and the family of MOP $(\boldsymbol{P}_n)_n$ satisfying

$$\boldsymbol{P}_n''(x) + \boldsymbol{P}_n'(x)(-2x\boldsymbol{I} + 2\boldsymbol{A}) + \boldsymbol{P}_n(x)(\boldsymbol{A}^2 - 2\boldsymbol{J}) = (-2n\boldsymbol{I} - 2\boldsymbol{J})\boldsymbol{P}_n(x)$$

That means that the second-order differential operator $\mathcal{D} : L^2(\mathbb{R}, \mathbb{C}^{N\times N}) \to L^2(\mathbb{R}, \mathbb{C}^{N\times N})$

$$[\boldsymbol{F}\mathcal{D}](x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{F}(x) - \boldsymbol{F}(x)(x^2\boldsymbol{I} + 2\boldsymbol{J})$$

and the integral operator $\mathcal{F} : L^2(\mathbb{R}, \mathbb{C}^{N\times N}) \to L^2(\mathbb{R}, \mathbb{C}^{N\times N})$

$$[\boldsymbol{F}\mathcal{F}](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \boldsymbol{F}(t)\mathrm{e}^{ixt}\mathrm{e}^{i\frac{\pi}{2}\boldsymbol{J}}dt$$

commute in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$, i.e. $\mathcal{D} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{D}$

CONSEQUENCES FOR $(\boldsymbol{P}_n)_n$

CONSEQUENCES FOR $(\boldsymbol{P}_n)_n$

We have the following symmetry conditions

$$\boldsymbol{P}_n(x) = (-1)^n e^{i\pi \boldsymbol{J}} \boldsymbol{P}_n(-x) \mathrm{e}^{i\pi \boldsymbol{J}}, \quad \boldsymbol{W}(x) = \mathrm{e}^{i\pi \boldsymbol{J}} \boldsymbol{W}(-x) \mathrm{e}^{i\pi \boldsymbol{J}}$$

CONSEQUENCES FOR $(\boldsymbol{P}_n)_n$

We have the following symmetry conditions

$$\boldsymbol{P}_n(x) = (-1)^n e^{i\pi \boldsymbol{J}} \boldsymbol{P}_n(-x) \mathrm{e}^{i\pi \boldsymbol{J}}, \quad \boldsymbol{W}(x) = \mathrm{e}^{i\pi \boldsymbol{J}} \boldsymbol{W}(-x) \mathrm{e}^{i\pi \boldsymbol{J}}$$

Denote $\boldsymbol{C}_- = \sin\left(\frac{\pi}{2}\boldsymbol{J}\right)$, $\boldsymbol{C}_+ = \cos\left(\frac{\pi}{2}\boldsymbol{J}\right)$ and
$k_n(x,t) = \cos(xt)$ if n is even and $k_n(x,t) = \sin(xt)$ if n is odd
We have the following real integral equations

$$\mathrm{e}^{-x^2/2}(\mathrm{e}^{i\pi \boldsymbol{J}} \pm \boldsymbol{I})\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}\boldsymbol{C}_\pm = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{\sqrt{2\pi}} \boldsymbol{C}_\pm \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} k_n(x,t) \boldsymbol{P}_n(t) \mathrm{e}^{\boldsymbol{A}t} dt (\mathrm{e}^{i\pi \boldsymbol{J}} \pm \boldsymbol{I})$$

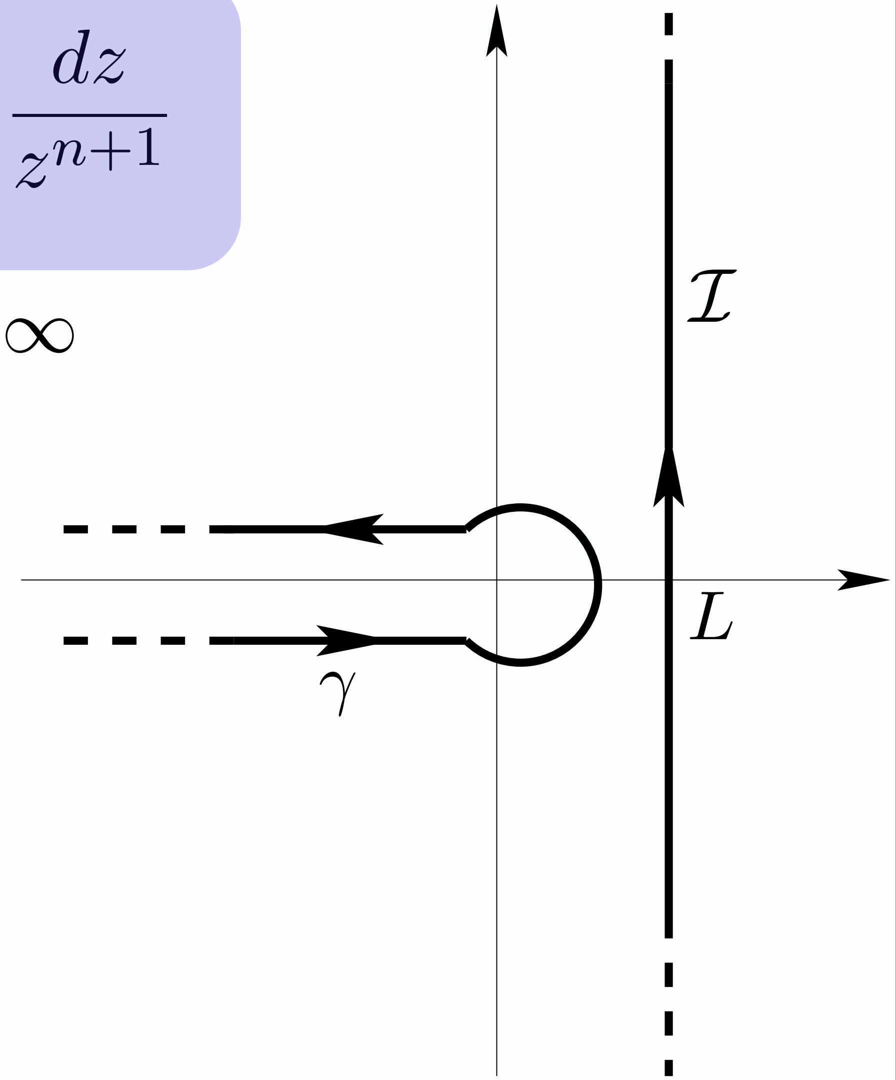
$$\mathrm{e}^{-x^2/2}(\mathrm{e}^{i\pi \boldsymbol{J}} \pm \boldsymbol{I})\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}\boldsymbol{C}_\mp = \pm \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{\sqrt{2\pi}} \boldsymbol{C}_\pm \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} k_{n+1}(x,t) \boldsymbol{P}_n(t) \mathrm{e}^{\boldsymbol{A}t} dt (\mathrm{e}^{i\pi \boldsymbol{J}} \mp \boldsymbol{I})$$

INTEGRAL REPRESENTATIONS

INTEGRAL REPRESENTATIONS

Applying the Laplace transform to the differential equation satisfied by the family $\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ there exists a suitable constant matrices $\boldsymbol{C}_n$ such that

$$\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x} = \oint_\gamma z^{-\boldsymbol{J}}\boldsymbol{C}_n z^{\boldsymbol{J}} \mathrm{e}^{-z^2+2zx}\frac{dz}{z^{n+1}}$$



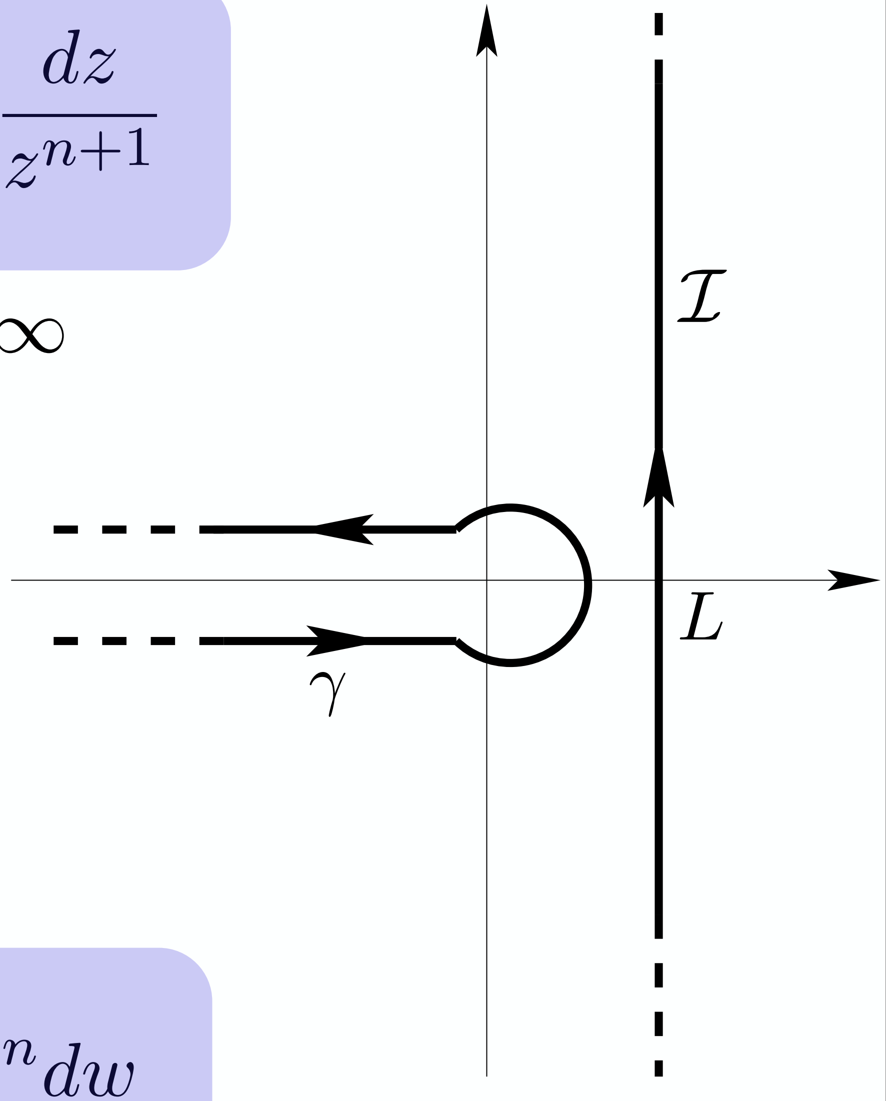
where the contour γ encloses the origin, closes at $-\infty$ and it is traversed in a counterclockwise direction

INTEGRAL REPRESENTATIONS

Applying the Laplace transform to the differential equation satisfied by the family $\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ there exists a suitable constant matrices $\boldsymbol{C}_n$ such that

$$\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x} = \oint_\gamma z^{-\boldsymbol{J}}\boldsymbol{C}_n z^{\boldsymbol{J}}\mathrm{e}^{-z^2+2zx}\frac{dz}{z^{n+1}}$$

where the contour γ encloses the origin, closes at $-\infty$ and it is traversed in a counterclockwise direction



Similarly, applying the Fourier transform to the differential equation satisfied by the family $\mathrm{e}^{-x^2}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ there exists a suitable constant matrix $\boldsymbol{D}_n$ such that

$$\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x} = \mathrm{e}^{x^2}\int_{\mathcal{I}} w^{\boldsymbol{J}}\boldsymbol{D}_n w^{-\boldsymbol{J}}\mathrm{e}^{w^2-2xw}w^n dw$$

where $\mathcal{I} = L + i\mathbb{R}$, and $L > 0$ is chosen so to have no intersection with γ

CASE *N*=2

The *normalized* family is

$$\mathbf{\Phi}_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

CASE N=2

The *normalized* family is

$$\mathbf{\Phi}_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

That means $\int_{\mathbb{R}} \mathbf{\Phi}_n(x)\mathbf{\Phi}_m^*(x)dx = \delta_{nm}\boldsymbol{I}_2$, so the diagonal entries are probability distributions on $\mathbb{R}$.

The plots for the first values of n and $\nu_1 = 1$ are given by

CASE N=2

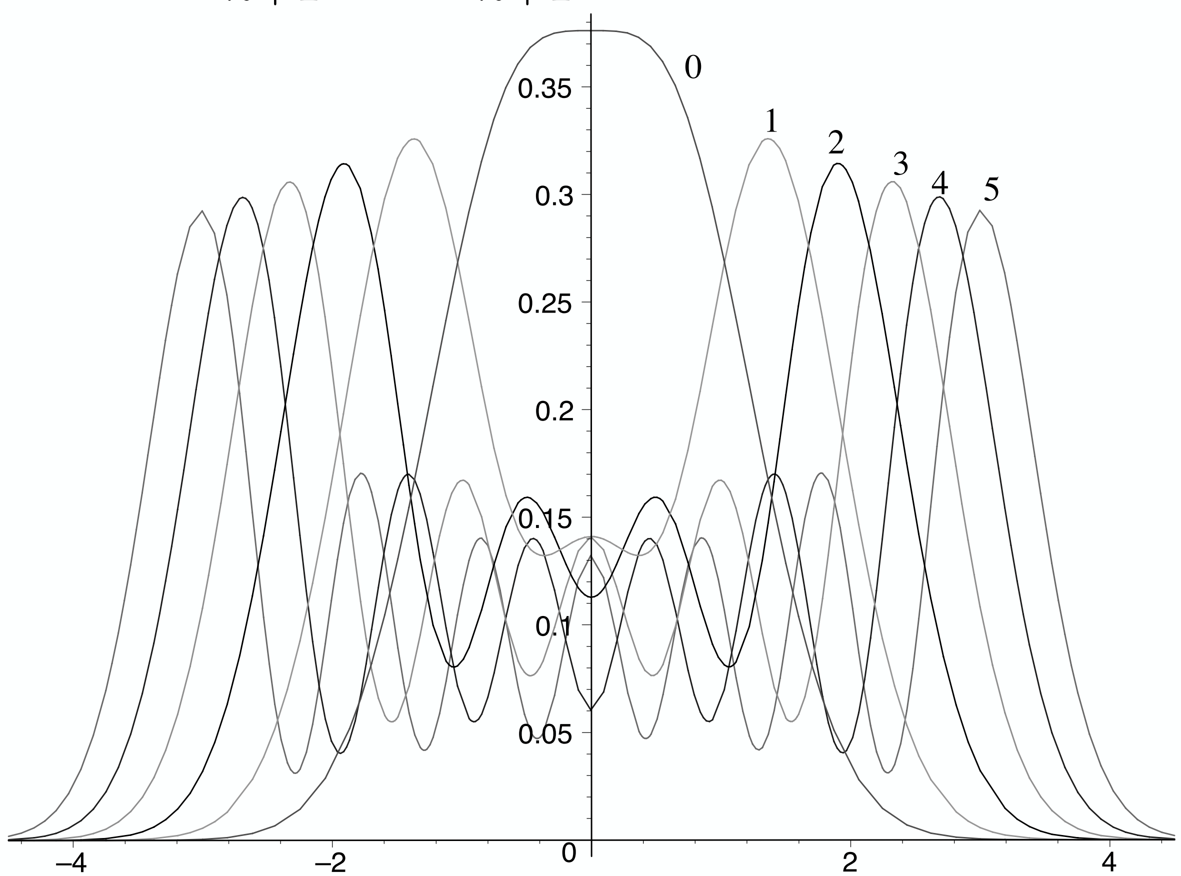
The *normalized* family is

$$\boldsymbol{\Phi}_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

That means $\int_{\mathbb{R}} \boldsymbol{\Phi}_n(x)\boldsymbol{\Phi}_m^*(x)dx = \delta_{nm}\boldsymbol{I}_2$, so the diagonal entries are probability distributions on $\mathbb{R}$.

The plots for the first values of n and $\nu_1 = 1$ are given by

$\frac{1}{\gamma_{n+1}}(\psi_n^2 - \psi_{n+1}^2) + \psi_{n+1}^2$



CASE N=2

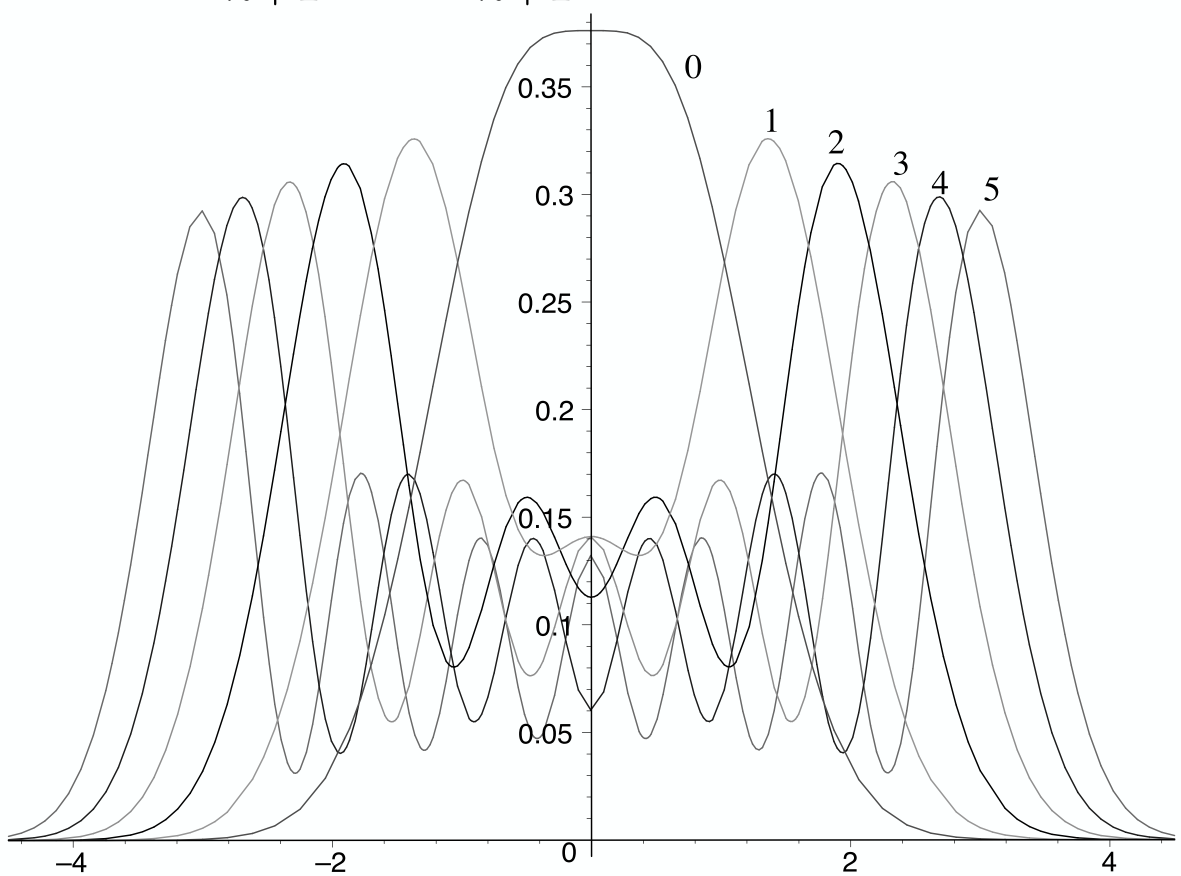
The *normalized* family is

$$\mathbf{\Phi}_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

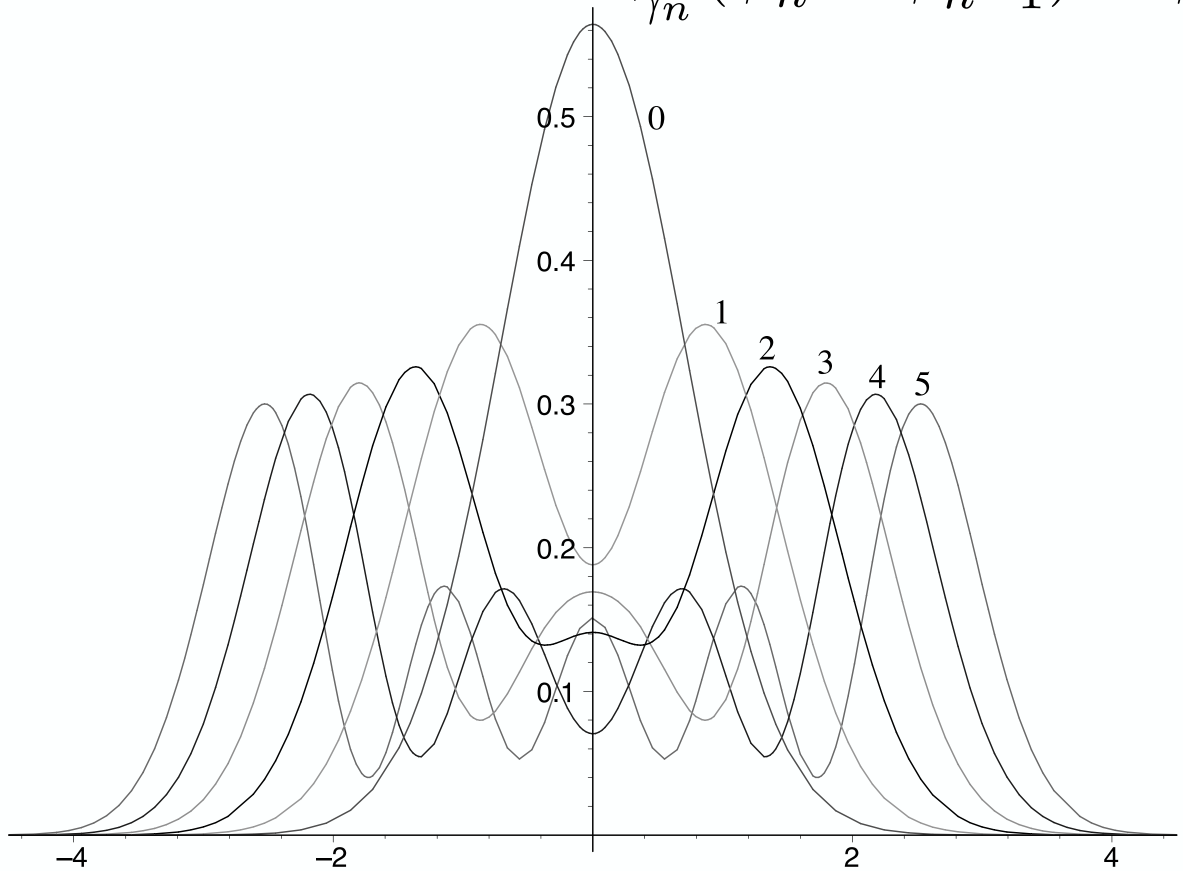
That means $\int_{\mathbb{R}} \mathbf{\Phi}_n(x)\mathbf{\Phi}_m^*(x)dx = \delta_{nm}\boldsymbol{I}_2$, so the diagonal entries are probability distributions on $\mathbb{R}$.

The plots for the first values of n and $\nu_1 = 1$ are given by

$\frac{1}{\gamma_{n+1}}(\psi_n^2 - \psi_{n+1}^2) + \psi_{n+1}^2$



$\frac{1}{\gamma_n}(\psi_n^2 - \psi_{n-1}^2) + \psi_{n-1}^2$



CASE N=2

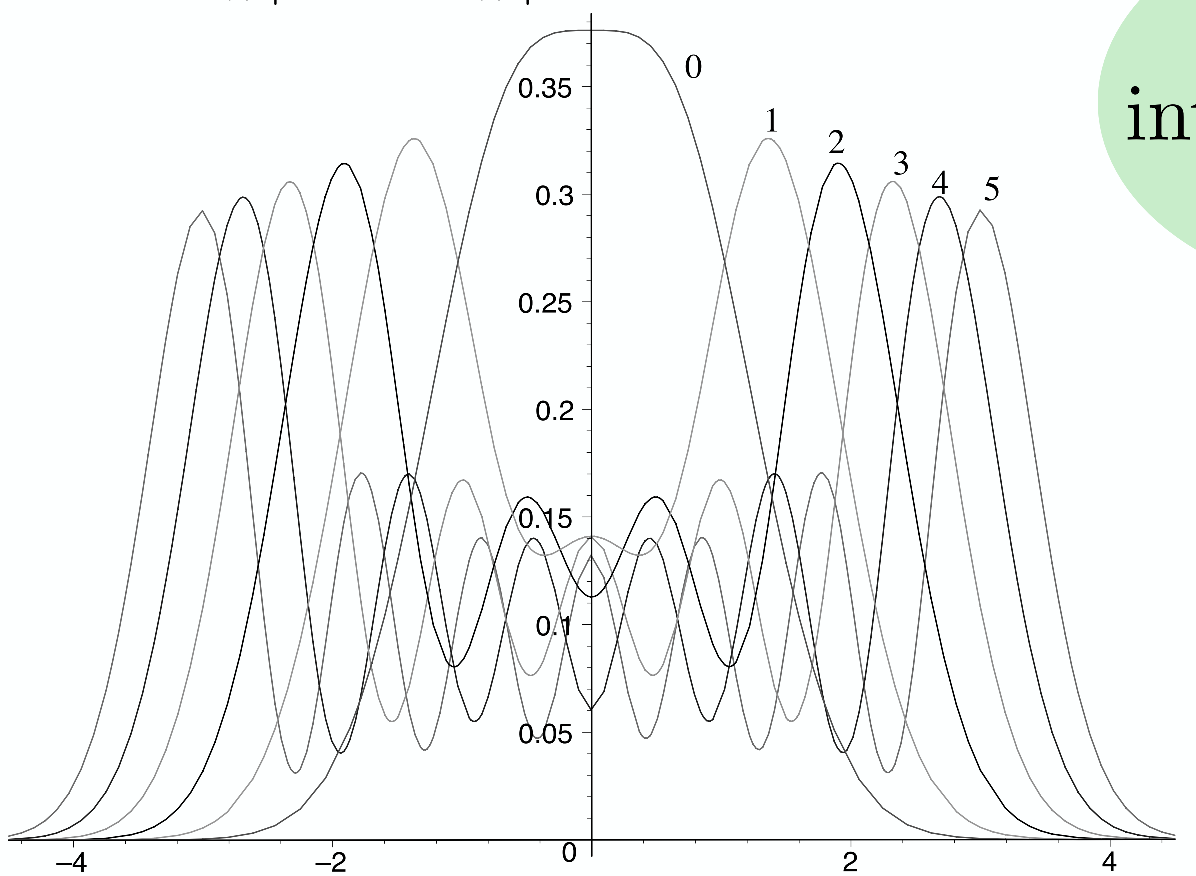
The *normalized* family is

$$\mathbf{\Phi}_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

That means $\int_{\mathbb{R}} \mathbf{\Phi}_n(x)\mathbf{\Phi}_m^*(x)dx = \delta_{nm}\boldsymbol{I}_2$, so the diagonal entries are probability distributions on $\mathbb{R}$.

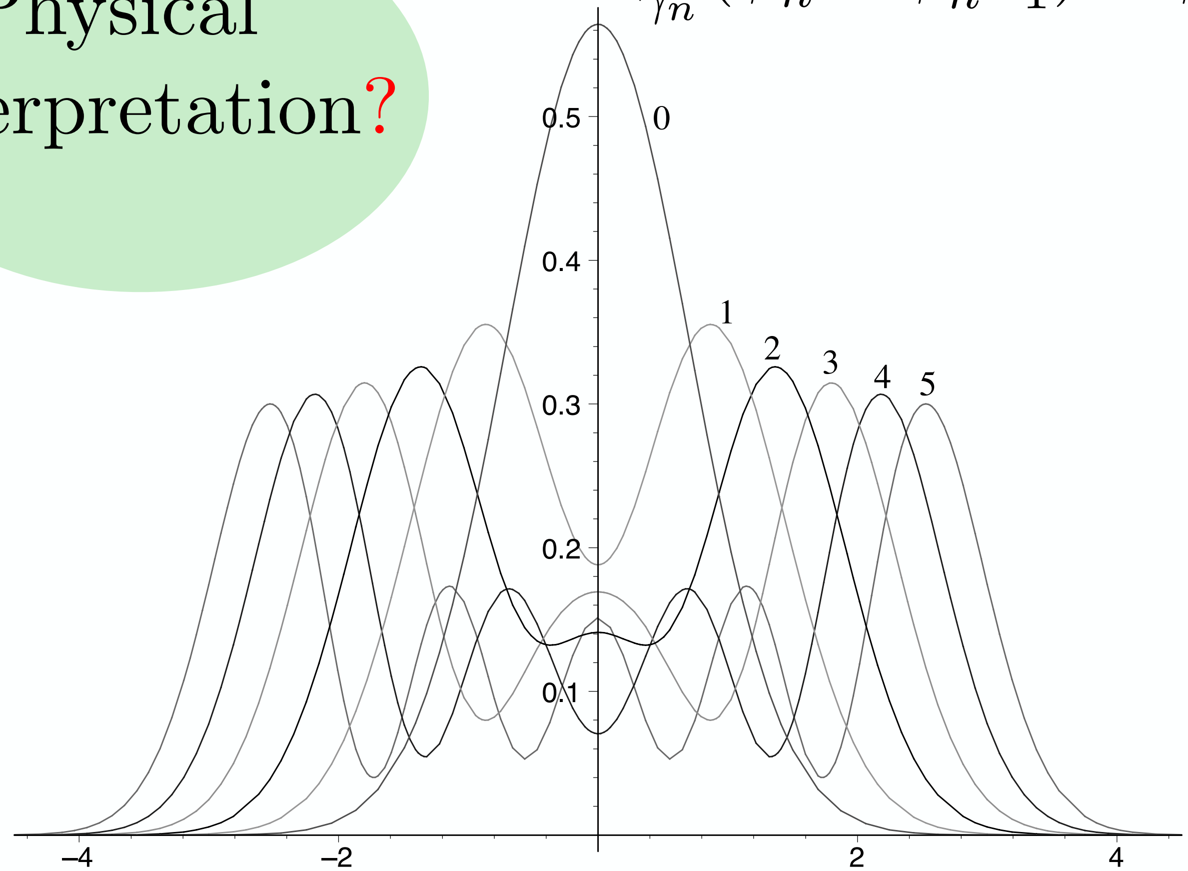
The plots for the first values of n and $\nu_1 = 1$ are given by

$\frac{1}{\gamma_{n+1}}(\psi_n^2 - \psi_{n+1}^2) + \psi_{n+1}^2$



Physical interpretation?

$\frac{1}{\gamma_n}(\psi_n^2 - \psi_{n-1}^2) + \psi_{n-1}^2$



THREE-TERM RECURRENCE RELATION

We also have

$$(x\boldsymbol{I})_{nm} \doteq \int_{\mathbb{R}} x\boldsymbol{\Phi}_n(x)\boldsymbol{\Phi}_m^*(x)dx = \begin{pmatrix} \sqrt{\frac{n\gamma_{n+1}}{2\gamma_n}} & 0 \\ 0 & \sqrt{\frac{n\gamma_{n-1}}{2\gamma_n}} \end{pmatrix} \delta_{m,n-1}$$

$$+ \begin{pmatrix} 0 & \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} \\ \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} & 0 \end{pmatrix} \delta_{m,n} + \begin{pmatrix} \sqrt{\frac{(n+1)\gamma_{n+2}}{2\gamma_{n+1}}} & 0 \\ 0 & \sqrt{\frac{(n+1)\gamma_n}{2\gamma_{n+1}}} \end{pmatrix} \delta_{m,n+1}$$

THREE-TERM RECURRENCE RELATION

We also have

$$(x\boldsymbol{I})_{nm} \doteq \int_{\mathbb{R}} x\boldsymbol{\Phi}_n(x)\boldsymbol{\Phi}_m^*(x)dx = \begin{pmatrix} \sqrt{\frac{n\gamma_{n+1}}{2\gamma_n}} & 0 \\ 0 & \sqrt{\frac{n\gamma_{n-1}}{2\gamma_n}} \end{pmatrix} \delta_{m,n-1}$$

$$+ \begin{pmatrix} 0 & \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} \\ \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} & 0 \end{pmatrix} \delta_{m,n} + \begin{pmatrix} \sqrt{\frac{(n+1)\gamma_{n+2}}{2\gamma_{n+1}}} & 0 \\ 0 & \sqrt{\frac{(n+1)\gamma_n}{2\gamma_{n+1}}} \end{pmatrix} \delta_{m,n+1}$$

Therefore $(x\boldsymbol{I})$ is the matrix of the homomorphism $F \to xF$ in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$ with respect to the basis $(\boldsymbol{\Phi}_n)_n$

$$(x\boldsymbol{I}) = \left(\begin{array}{cc|cc|cc|cc} 0 & \star & \star & & & & & \\ \star & 0 & 0 & \star & & & & \\ \hline \star & 0 & 0 & \star & \star & & & \\ & \star & \star & 0 & 0 & \star & & \\ \hline & & \star & 0 & 0 & \star & \star & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right)$$

THREE-TERM RECURRENCE RELATION

We also have

$$(x\boldsymbol{I})_{nm} \doteq \int_{\mathbb{R}} x\boldsymbol{\Phi}_n(x)\boldsymbol{\Phi}_m^*(x)dx = \begin{pmatrix} \sqrt{\frac{n\gamma_{n+1}}{2\gamma_n}} & 0 \\ 0 & \sqrt{\frac{n\gamma_{n-1}}{2\gamma_n}} \end{pmatrix} \delta_{m,n-1}$$

$$+ \begin{pmatrix} 0 & \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} \\ \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} & 0 \end{pmatrix} \delta_{m,n} + \begin{pmatrix} \sqrt{\frac{(n+1)\gamma_{n+2}}{2\gamma_{n+1}}} & 0 \\ 0 & \sqrt{\frac{(n+1)\gamma_n}{2\gamma_{n+1}}} \end{pmatrix} \delta_{m,n+1}$$

Therefore $(x\boldsymbol{I})$ is the matrix of the homomorphism $F \to xF$ in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$ with respect to the basis $(\boldsymbol{\Phi}_n)_n$

$$(x\boldsymbol{I}) = \left(\begin{array}{cc|cc|cc|cc} 0 & \star & \star & & & & & \\ \star & 0 & 0 & \star & & & & \\ \hline \star & 0 & 0 & \star & \star & & & \\ & \star & \star & 0 & 0 & \star & & \\ \hline & & \star & 0 & 0 & \star & \star & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right)$$

Physical interpretation?

INTEGRAL RELATIONS

There will be 8 real integral equations

INTEGRAL RELATIONS

There will be 8 real integral equations

If we call $\boldsymbol{Q}_n(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{P}_n(x) \begin{pmatrix} 0 & \nu_1 x \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \mathrm{e}^{-x^2/2}\boldsymbol{Q}_{2n}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \cos(xt) \boldsymbol{Q}_{2n}(t) dt$$

$$\mathrm{e}^{-x^2/2}\boldsymbol{Q}_{2n+1}(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \sin(xt) \boldsymbol{Q}_{2n+1}(t) dt$$

INTEGRAL RELATIONS

There will be 8 real integral equations

If we call $\boldsymbol{Q}_n(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{P}_n(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \mathrm{e}^{-x^2/2} \boldsymbol{Q}_{2n}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \cos(xt) \boldsymbol{Q}_{2n}(t) dt$$

$$\mathrm{e}^{-x^2/2} \boldsymbol{Q}_{2n+1}(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \sin(xt) \boldsymbol{Q}_{2n+1}(t) dt$$

INTEGRAL RELATIONS

There will be 8 real integral equations

If we call $\boldsymbol{Q}_n(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{P}_n(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \mathrm{e}^{-x^2/2}\boldsymbol{Q}_{2n}(x) = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-t^2/2}\sin(xt)\boldsymbol{Q}_{2n}(t)dt$$

$$\mathrm{e}^{-x^2/2}\boldsymbol{Q}_{2n+1}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-t^2/2}\cos(xt)\boldsymbol{Q}_{2n+1}(t)dt$$

INTEGRAL RELATIONS

There will be 8 real integral equations

If we call $\boldsymbol{Q}_n(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{P}_n(x) \begin{pmatrix} 0 & \nu_1 x \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \mathrm{e}^{-x^2/2} \boldsymbol{Q}_{2n}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \sin(xt) \boldsymbol{Q}_{2n}(t) dt$$

$$\mathrm{e}^{-x^2/2} \boldsymbol{Q}_{2n+1}(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \cos(xt) \boldsymbol{Q}_{2n+1}(t) dt$$

INTEGRAL RELATIONS

There will be 8 real integral equations

If we call $\boldsymbol{Q}_n(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{P}_n(x) \begin{pmatrix} 0 & \nu_1 x \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \mathrm{e}^{-x^2/2} \boldsymbol{Q}_{2n}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \sin(xt) \boldsymbol{Q}_{2n}(t) dt$$

$$\mathrm{e}^{-x^2/2} \boldsymbol{Q}_{2n+1}(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \cos(xt) \boldsymbol{Q}_{2n+1}(t) dt$$

Additionally we have

$$\boldsymbol{P}_n(x) \begin{pmatrix} 1 & \nu_1 x \\ 0 & 1 \end{pmatrix} = \frac{n!}{2^{n+1}\pi i} \oint_{\gamma} \begin{pmatrix} 1 & \dfrac{(n+1)\nu_1}{2z} \\ -\dfrac{z\nu_1}{\gamma_n^2} & \dfrac{1}{\gamma_n^2} \end{pmatrix} \mathrm{e}^{-z^2+2zx} \frac{dz}{z^{n+1}}$$

$$\boldsymbol{P}_n(x) \begin{pmatrix} 1 & \nu_1 x \\ 0 & 1 \end{pmatrix} = \frac{\mathrm{e}^{x^2}}{i\sqrt{\pi}} \int_{\mathcal{I}} \begin{pmatrix} 1 & w\nu_1 \\ -\dfrac{n\nu_1}{2w\gamma_n^2} & \dfrac{1}{\gamma_n^2} \end{pmatrix} \mathrm{e}^{w^2-2xw} w^n dw$$

CHRISTOFFEL-DARBOUX KERNEL

The CD kernel

$$\boldsymbol{K}_n(x,y) = \sum_{k=0}^{n-1} \boldsymbol{\Phi}_k^*(y)\boldsymbol{\Phi}_k(x)$$

where $\boldsymbol{\Phi}_n(x) = \mathrm{e}^{-x^2/2}\|\boldsymbol{P}_n\|_{\boldsymbol{W}}^{-1}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ can be written as

CHRISTOFFEL-DARBOUX KERNEL

The CD kernel

$$\boldsymbol{K}_n(x,y)=\sum_{k=0}^{n-1}\boldsymbol{\Phi}_k^*(y)\boldsymbol{\Phi}_k(x)$$

where $\boldsymbol{\Phi}_n(x)=\mathrm{e}^{-x^2/2}\|\boldsymbol{P}_n\|_{\boldsymbol{W}}^{-1}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ can be written as

$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}}dw\oint_\gamma dz\; z^{\boldsymbol{J}_2}\boldsymbol{B}_n z^{-\boldsymbol{J}_2}w^{\boldsymbol{J}_2}\boldsymbol{B}_n^{-1}w^{-\boldsymbol{J}_2}\frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

where

$$\boldsymbol{B}_n=\begin{pmatrix}1 & -\nu_1\\ \dfrac{n\nu_1}{2} & 1\end{pmatrix}$$

CHRISTOFFEL-DARBOUX KERNEL

The CD kernel

$$\boldsymbol{K}_n(x,y)=\sum_{k=0}^{n-1}\boldsymbol{\Phi}_k^*(y)\boldsymbol{\Phi}_k(x)$$

where $\boldsymbol{\Phi}_n(x)=\mathrm{e}^{-x^2/2}\|\boldsymbol{P}_n\|_{\boldsymbol{W}}^{-1}\boldsymbol{P}_n(x)\mathrm{e}^{\boldsymbol{A}x}$ can be written as

$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}}dw\oint_{\gamma}dz\; z^{\boldsymbol{J}_2}\boldsymbol{B}_n z^{-\boldsymbol{J}_2}w^{\boldsymbol{J}_2}\boldsymbol{B}_n^{-1}w^{-\boldsymbol{J}_2}\frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

where

$$\boldsymbol{B}_n=\begin{pmatrix}1 & -\nu_1\\ \dfrac{n\nu_1}{2} & 1\end{pmatrix}$$

In other words

$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}}dw\oint_{\gamma}dz\begin{pmatrix}\frac{z(\gamma_n^2-1)+w}{w\gamma_n^2} & \frac{\nu_1(w-z)}{\gamma_n^2}\\[2ex] \frac{n\nu_1(z-w)}{2\gamma_n^2} & \frac{w(\gamma_n^2-1)+z}{z\gamma_n^2}\end{pmatrix}\frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

NON-COMMUTATIVE PAINLEVÉ IV

NON-COMMUTATIVE PAINLEVÉ IV

$$\mathbb{K}_n \sim \boldsymbol{K}_n(x,y)$$

NON-COMMUTATIVE PAINLEVÉ IV

$$\mathbb{K}_n \sim \boldsymbol{K}_n(x,y)$$

$$[\mathbb{K}_n \boldsymbol{F}](x) = \int_{\mathbb{R}} \boldsymbol{F}(y)\boldsymbol{K}_n(x,y)dy$$

NON-COMMUTATIVE PAINLEVÉ IV

IIKS theory

$$\mathbb{K}_n \sim \boldsymbol{K}_n(x,y)$$

$$[\mathbb{K}_n \boldsymbol{F}](x) = \int_{\mathbb{R}} \boldsymbol{F}(y) \boldsymbol{K}_n(x,y) dy$$

NON-COMMUTATIVE PAINLEVÉ IV

IIKS theory

$\mathbb{K}_n \sim \boldsymbol{K}_n(x,y)$ → 4×4 RHP $\boldsymbol{\Gamma}(\lambda)$ on $\mathbb{C}\backslash\{\gamma \cup \mathcal{I}\}$

$$[\mathbb{K}_n \boldsymbol{F}](x) = \int_{\mathbb{R}} \boldsymbol{F}(y)\boldsymbol{K}_n(x,y)dy$$

NON-COMMUTATIVE PAINLEVÉ IV

IIKS theory

$\mathbb{K}_n \sim \boldsymbol{K}_n(x,y)$ → 4×4 RHP $\boldsymbol{\Gamma}(\lambda)$ on $\mathbb{C}\backslash\{\gamma \cup \mathcal{I}\}$

$$\begin{cases} \boldsymbol{\Gamma}_+(\lambda) = \boldsymbol{\Gamma}_-(\lambda)(\boldsymbol{I} - \boldsymbol{G}(\lambda)), \quad \lambda \in \eta \cup \gamma \\ \boldsymbol{\Gamma}(\lambda) = \boldsymbol{I} + \dfrac{\boldsymbol{\Gamma}_1}{\lambda} + \dfrac{\boldsymbol{\Gamma}_2}{\lambda^2} + \cdots, \quad \lambda \to \infty \end{cases}$$

$\boldsymbol{G}(\lambda)$ depends on $\lambda^{\boldsymbol{J}_2}\boldsymbol{B}_n\lambda^{-\boldsymbol{J}_2}$

NON-COMMUTATIVE PAINLEVÉ IV

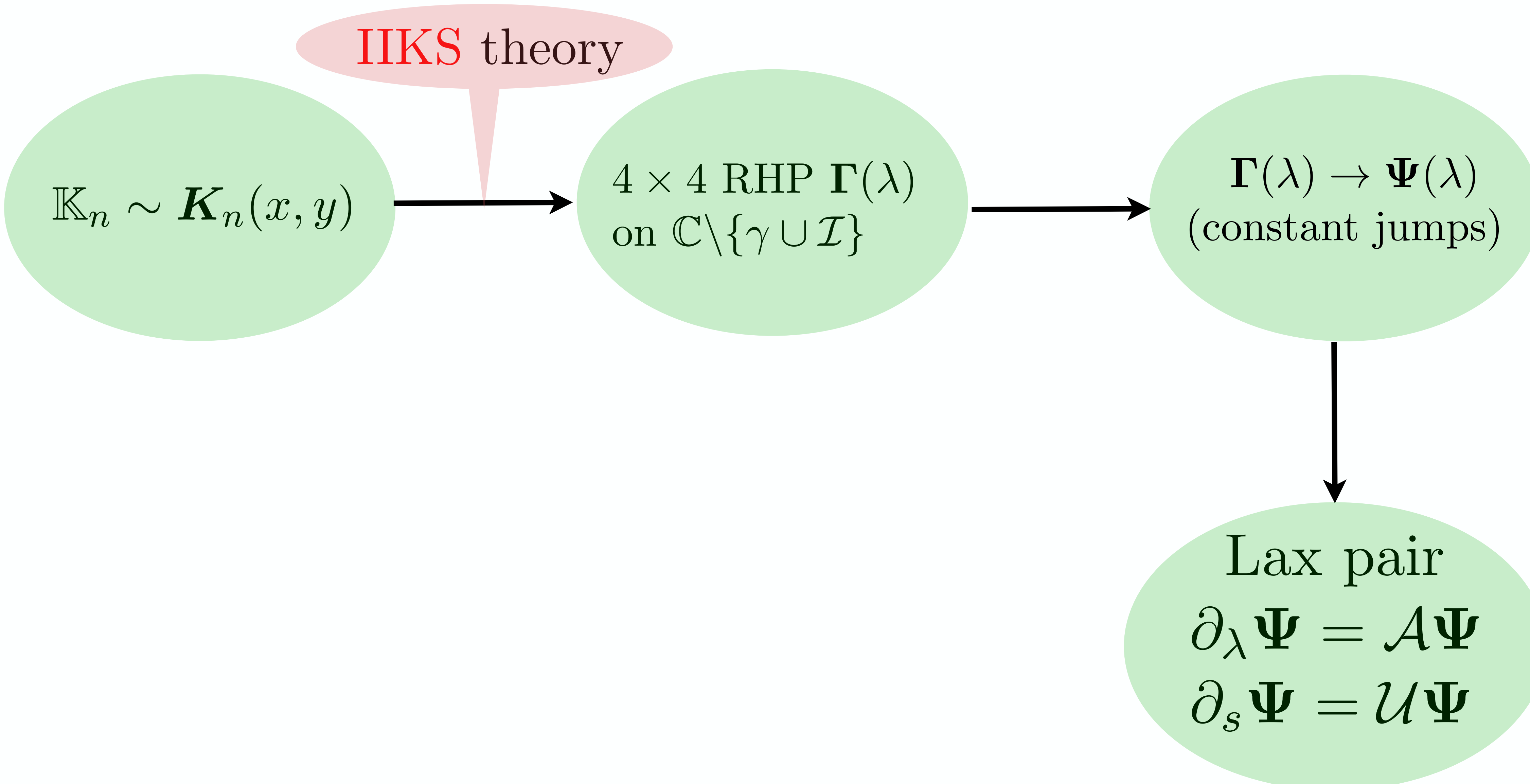
IIKS theory

$\mathbb{K}_n \sim \boldsymbol{K}_n(x,y)$ → 4×4 RHP $\boldsymbol{\Gamma}(\lambda)$ on $\mathbb{C}\backslash\{\gamma\cup\mathcal{I}\}$ → $\boldsymbol{\Gamma}(\lambda)\to\boldsymbol{\Psi}(\lambda)$ (constant jumps)

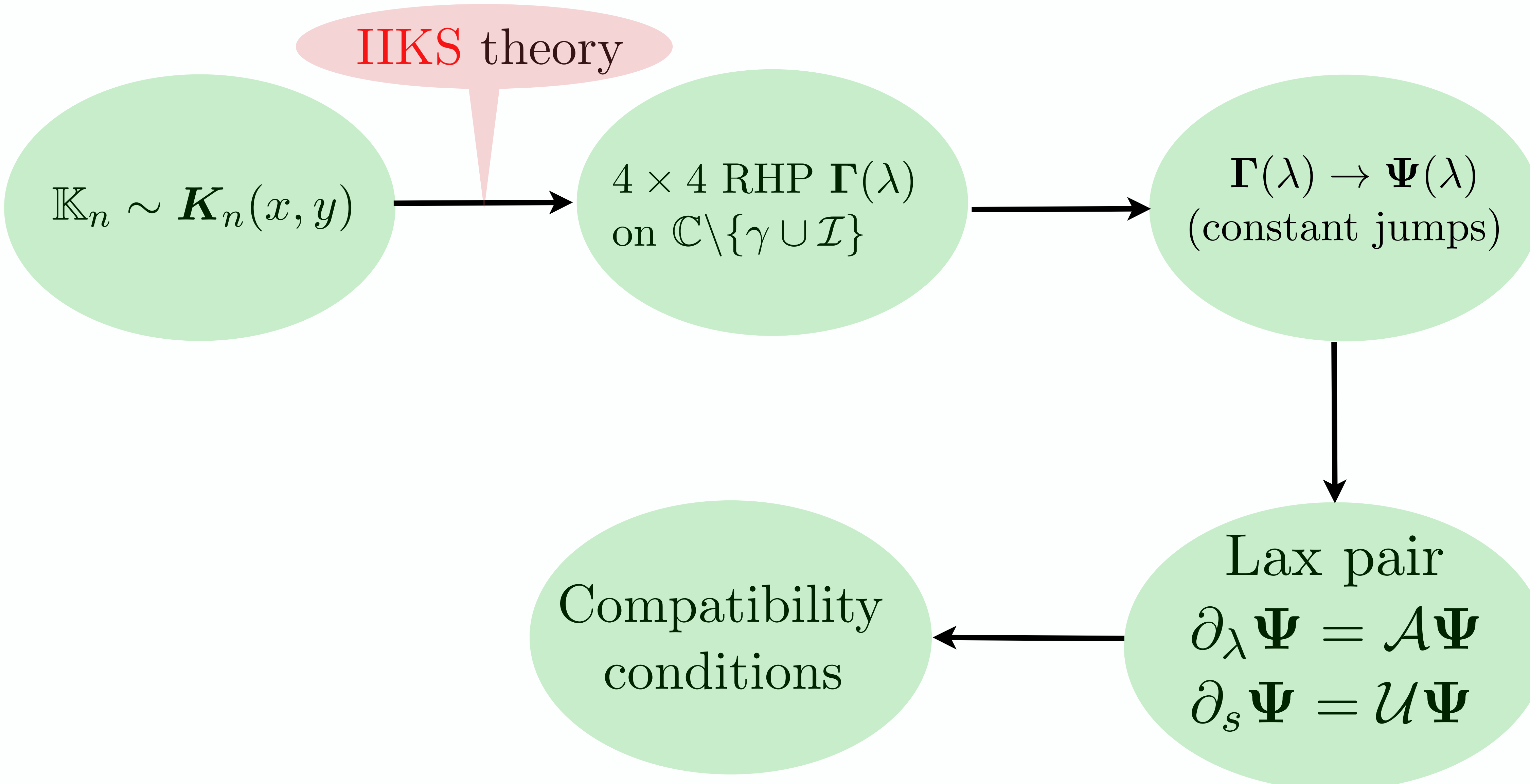
$$\begin{cases} \boldsymbol{\Gamma}_+(\lambda)=\boldsymbol{\Gamma}_-(\lambda)(\boldsymbol{I}-\boldsymbol{G}(\lambda)), & \lambda\in\eta\cup\gamma \\ \boldsymbol{\Gamma}(\lambda)=\boldsymbol{I}+\dfrac{\boldsymbol{\Gamma}_1}{\lambda}+\dfrac{\boldsymbol{\Gamma}_2}{\lambda^2}+\cdots, & \lambda\to\infty \end{cases}$$

$\boldsymbol{G}(\lambda)$ depends on $\lambda^{\boldsymbol{J}_2}\boldsymbol{B}_n\lambda^{-\boldsymbol{J}_2}$

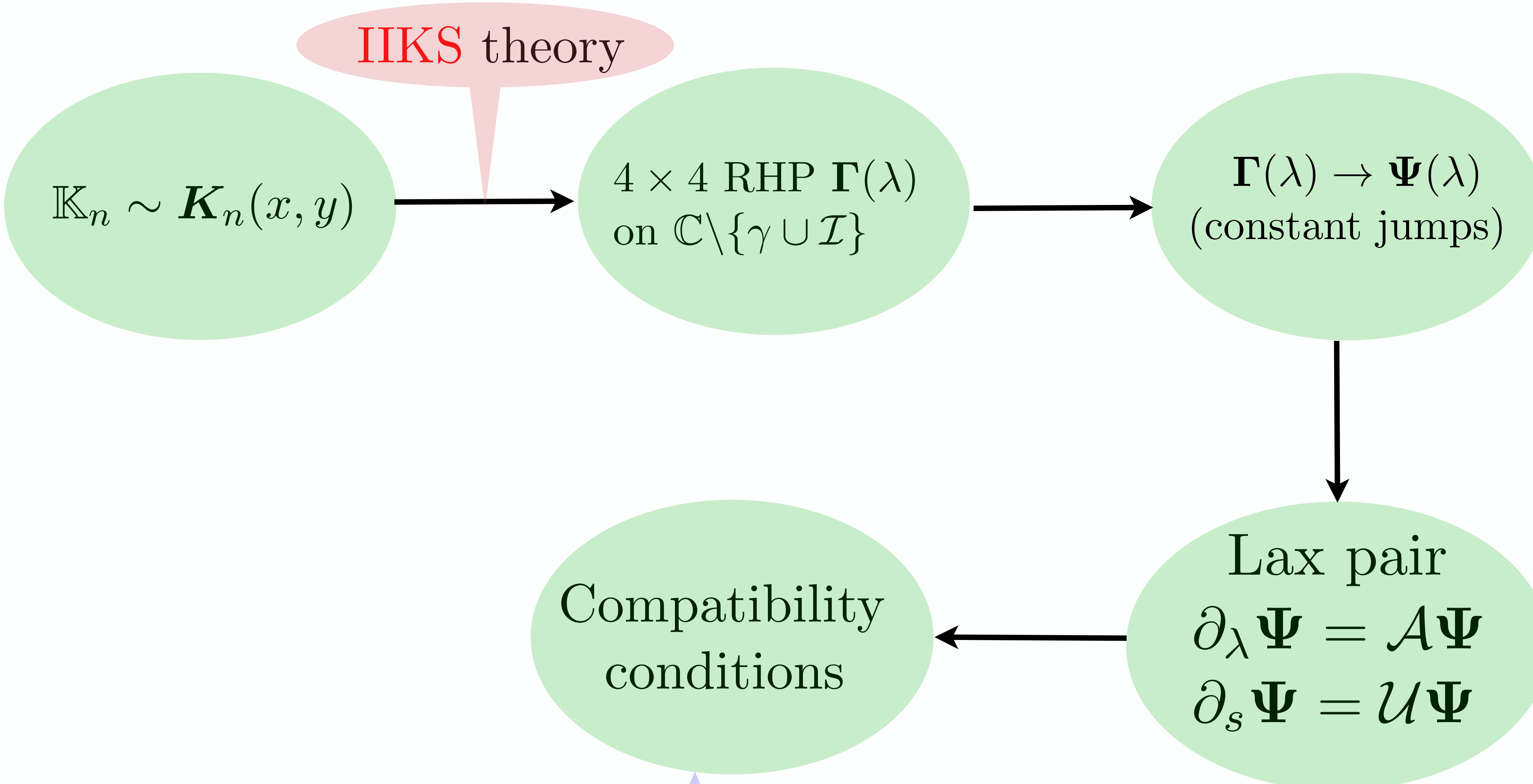
NON-COMMUTATIVE PAINLEVÉ IV



NON-COMMUTATIVE PAINLEVÉ IV



NON-COMMUTATIVE PAINLEVÉ IV

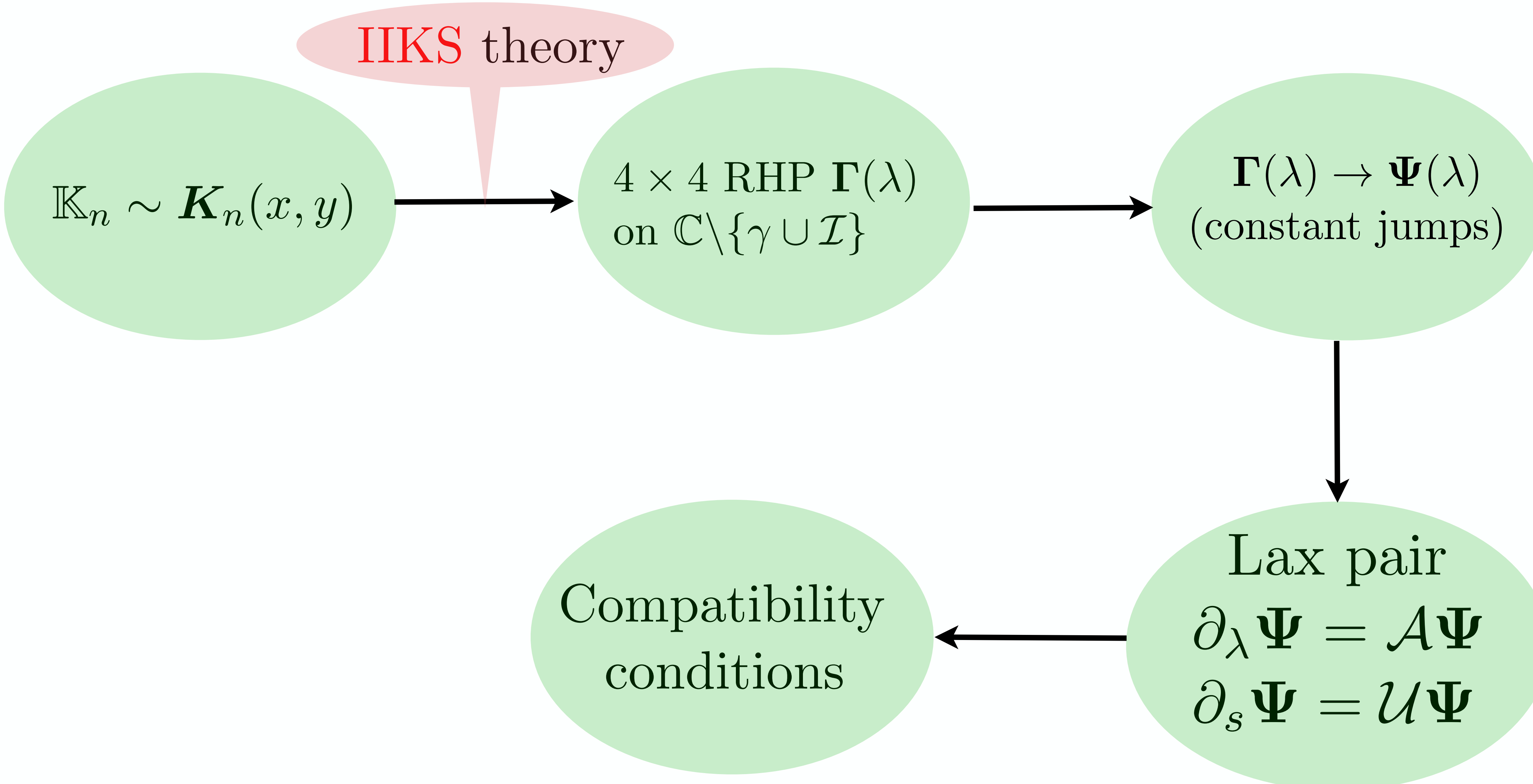


$$\begin{cases} \boldsymbol{u}' = -\boldsymbol{u}^2 + 2s\boldsymbol{u} + 4\boldsymbol{z} - 2n\boldsymbol{I}_2 + \boldsymbol{V_A} \\ \boldsymbol{z}'' = 2\boldsymbol{u}'\boldsymbol{z} + 2\boldsymbol{u}\boldsymbol{z}' - 2s\boldsymbol{z}' \end{cases}$$

where $\boldsymbol{V_A} = 2[\boldsymbol{J}_2,\boldsymbol{y}]\boldsymbol{y}^{-1}$ $([\boldsymbol{x},\boldsymbol{y}]=\boldsymbol{x}\boldsymbol{y}-\boldsymbol{y}\boldsymbol{x})$ and

$$\boldsymbol{z} = -(\boldsymbol{\Gamma}_1)'_{11}, \quad \boldsymbol{y} = -2(\boldsymbol{\Gamma}_1)_{12}, \quad \boldsymbol{u} = (\boldsymbol{\Gamma}_1)'_{12}(\boldsymbol{\Gamma}_1)^{-1}_{12} + 2s\boldsymbol{I}_2$$

NON-COMMUTATIVE PAINLEVÉ IV



Non-commutative version of the derived Painlevé IV equation

$$\begin{aligned}&\boldsymbol{u}'''+[\boldsymbol{u}'',\boldsymbol{u}]-4(n+1+s^2)\boldsymbol{u}'-2\left(\{\boldsymbol{u}',\boldsymbol{u}^2\}+\boldsymbol{u}\boldsymbol{u}'\boldsymbol{u}\right)\\&\quad+6s\{\boldsymbol{u}',\boldsymbol{u}\}+4\boldsymbol{u}(\boldsymbol{u}-s\boldsymbol{I}_2)+(\boldsymbol{V}_{\boldsymbol{A}}'-2(\boldsymbol{u}\boldsymbol{V}_{\boldsymbol{A}}))'+2s\boldsymbol{V}_{\boldsymbol{A}}'=\boldsymbol{0}\end{aligned}$$

NON-COMMUTATIVE PAINLEVÉ IV

Non-commutative version of the derived Painlevé IV equation

$$\boldsymbol{u}''' + [\boldsymbol{u}'', \boldsymbol{u}] - 4(n+1+s^2)\boldsymbol{u}' - 2\left(\{\boldsymbol{u}', \boldsymbol{u}^2\} + \boldsymbol{u}\boldsymbol{u}'\boldsymbol{u}\right) \\ +6s\{\boldsymbol{u}', \boldsymbol{u}\} + 4\boldsymbol{u}(\boldsymbol{u} - s\boldsymbol{I}_2) + (\boldsymbol{V}_{\boldsymbol{A}}' - 2(\boldsymbol{u}\boldsymbol{V}_{\boldsymbol{A}}))' + 2s\boldsymbol{V}_{\boldsymbol{A}}' = \boldsymbol{0}$$

If we assume that all the variables commute, we get the equation

$$u''' - 4u' - 6u^2u' + 12u'u - 4nu' + 4u^2 - 4su - 4s^2u' = 0$$

and this equation is the derivative of the Painlevé IV equation

$$u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 - 4su^2 + 2(s^2+1+n)u - \frac{2n^2}{u}$$

Joint work with Mattia Cafasso


THANK YOU