Algebraic aspects of the Riemann-Hilbert problem for matrix orthogonal polynomials[1]

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[1]joint work with F. A. Grünbaum and A. Martínez-Finkelshtein

Outline

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Orthogonal polynomials

Let $d\mu$ be a positive Borel measure supported on $\mathbb{R}$.
We will assume $d\mu(x) = \omega(x)dx$, $\omega \geq 0$ and $x^i\omega, x^j\omega' \in L^1(\mathbb{R})$.
We can then construct a family of orthonormal polynomials $(p_n)_n$ s.t.

$$(p_n, p_m)_\omega = \int_{\mathbb{R}} p_n(x)p_m(x)\omega(x)dx = \delta_{n,m}, \quad n, m \geq 0$$

$$p_n(x) = \kappa_n(x^n + a_{n,n-1}x^{n-1} + \cdots) = \kappa_n\widehat{p}_n(x)$$

The monic polynomials $\widehat{p}_n(x)$ satisfy a three-term recurrence relation

$$x\widehat{p}_n(x) = \widehat{p}_{n+1}(x) + \alpha_n\widehat{p}_n(x) + \beta_n\widehat{p}_{n-1}(x)$$

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Solution of the RHP for orthogonal polynomials

We try to find a 2×2 matrix-valued function $\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2\times 2}$ such that

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_2 + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RHP above is given by

Fokas-Its-Kitaev, 1990

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{p}_n(z) & C(\widehat{p}_n\omega)(z) \\ -2\pi i \gamma_{n-1}\widehat{p}_{n-1}(z) & -2\pi i \gamma_{n-1} C(\widehat{p}_{n-1}\omega)(z) \end{pmatrix}$$

where $C(f)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z} dt$ is the Cauchy transform and $\gamma_n = \kappa_n^2$.
The existence and unicity is a consequence of the Morera's theorem, Liouville's theorem and $\det \mathbf{Y}^n(z) = 1$.

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The Lax pair I

We look for a pair of first-order difference/differential equations of the form

$$\mathbf{Y}^{n+1}(z) = \mathbf{E}_n(z)\mathbf{Y}^n(z), \quad \frac{d}{dz}\mathbf{Y}^n(z) = \mathbf{F}_n(z)\mathbf{Y}^n(z)$$

Problem. Typically, the coefficient $\mathbf{F}_n(z)$ is difficult to obtain. We can avoid that by transforming the RHP in another RHP with constant jump.
Consider the transformation

$$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}$$

We observe that $\mathbf{X}^n$ is invertible and that

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$\Rightarrow$ $\mathbf{E}_n(z)$ and $\mathbf{F}_n(z)$ are completely determined by their behavior at $z \to \infty$

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The Lax pair II

If we additionally assume that $\frac{(\omega^{1/2})'}{\omega^{1/2}}$ is a polynomial of degree m, then

$$\mathbf{X}^{n+1}(z) = \underbrace{\begin{pmatrix} z-\alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i\gamma_n & 0 \end{pmatrix}}_{\mathbf{E}_n(z)} \mathbf{X}^n(z)$$

$$\frac{d}{dz}\mathbf{X}^n(z) = \underbrace{\begin{pmatrix} -\mathcal{B}_n(z) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathcal{A}_n(z) \\ 2\pi i\,\mathcal{A}_{n-1}(z)\gamma_{n-1} & \mathcal{B}_n(z) \end{pmatrix}}_{\mathbf{F}_n(z)} \mathbf{X}^n(z)$$

where $\mathcal{A}_n(z)$ and $\mathcal{B}_n(z)$ are polynomials of degree $m-1$ and m respectively.

Cross-differentiating the Lax pair yield

Compatibility conditions

$$\mathbf{E}_n'(z) + \mathbf{E}_n(z)\mathbf{F}_n(z) = \mathbf{F}_{n+1}(z)\mathbf{E}_n(z)$$

also known as string equations.

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Example I: Hermite polynomials

Consider $\omega(x) = e^{-x^2} \Rightarrow$ Hermite polynomials $(H_n)_n$.

The transformation $\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} e^{-z^2/2} & 0 \\ 0 & e^{z^2/2} \end{pmatrix}$ gives the following Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z), \quad \frac{d}{dz}\mathbf{X}^n(z) = \begin{pmatrix} -z & -\frac{1}{\pi i}\gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & z \end{pmatrix} \mathbf{X}^n(z)$$

The difference equation gives (using $\beta_n = \gamma_n/\gamma_{n+1}$) the TTRR

$$x\widehat{H}_n(x) = \widehat{H}_{n+1}(x) + \beta_n \widehat{H}_{n-1}(x),$$

while the differential equation gives the ladder operators

$$\widehat{H}_n'(x) = 2\beta_n \widehat{H}_{n-1}(x), \quad \widehat{H}_n'(x) - 2x\widehat{H}_n(x) = -2\widehat{H}_{n+1}(x).$$

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$$\beta_{n+1} - \beta_n = \frac{1}{2} \Rightarrow \beta_n = \frac{n}{2}$$

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Example II: Freud orthogonal polynomials

Consider $\omega(x) = e^{-x^4} \Rightarrow$ Freud polynomials $(P_n)_n$.

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$\mathbf{X}^n(z) = \mathbf{Y}^n(z) \begin{pmatrix} e^{-z^4/2} & 0 \\ 0 & e^{z^4/2} \end{pmatrix}$ satisfies the following Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z)$$

$$\frac{d}{dz}\mathbf{X}^n(z) = \begin{pmatrix} -2z^3 - 4\beta_n z & -\frac{2}{\pi i}\gamma_n^{-1}(z^2 + \beta_n + \beta_{n+1}) \\ 8\pi i \gamma_{n-1}(z^2 + \beta_n + \beta_{n+1}) & 2z^3 + 4\beta_n z \end{pmatrix} \mathbf{X}^n(z)$$

The ladder operators are

$$\widehat{P}_n'(x) + 4\beta_n x \widehat{P}_n(x) = 4(x^2 + \beta_n + \beta_{n+1})\beta_n \widehat{P}_{n-1}(x)$$

$$\widehat{P}_n'(x) + 4x^3 \widehat{P}_n(x) = -4(x^2 + \beta_n + \beta_{n+1})\widehat{P}_{n+1}(x)$$

The compatibility conditions are

$$n = 4\beta_n(\beta_{n+1} + \beta_n + \beta_{n-1})$$

Outline

Matrix orthogonal polynomials

The theory of matrix orthogonal polynomials on the real line (MOP) was introduced by Krein in 1949.

A $N \times N$ matrix polynomial on the real line is

$$\mathbf{P}(x) = \mathbf{A}_n x^n + \mathbf{A}_{n-1} x^{n-1} + \cdots + \mathbf{A}_0, \quad x \in \mathbb{R} \quad \mathbf{A}_i \in \mathbb{C}^{N\times N}$$

Let $\mathbf{W}$ be a $N \times N$ a matrix of measures or weight matrix.
We will assume $d\mathbf{W}(x) = \mathbf{W}(x)dx$ and $\mathbf{W}$ smooth and positive definite on $\mathbb{R}$.
We can construct a family of MOP with respect to the inner product

$$(\mathbf{P},\mathbf{Q})_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{P}(x)\mathbf{W}(x)\mathbf{Q}^*(x)dx \in \mathbb{C}^{N\times N}$$

such that

$$(\mathbf{P}_n,\mathbf{P}_m)_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{P}_n(x)\mathbf{W}(x)\mathbf{P}_m^*(x)dx = \delta_{n,m}\mathbf{I}_N, \quad n,m \geq 0$$

$$\mathbf{P}_n(x) = \boldsymbol{\kappa}_n(x^n + \mathbf{a}_{n,n-1}x^{n-1} + \cdots) = \boldsymbol{\kappa}_n \widehat{\mathbf{P}}_n(x)$$

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Solution of the RHP for MOP

$\mathbf{Y}^n : \mathbb{C} \to \mathbb{C}^{2N\times 2N}$ such that

1. $\mathbf{Y}^n$ is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\mathbf{Y}^n_+(x) = \mathbf{Y}^n_-(x) \begin{pmatrix} \mathbf{I}_N & \mathbf{W}(x) \\ \mathbf{0} & \mathbf{I}_N \end{pmatrix}$ when $x \in \mathbb{R}$
3. $\mathbf{Y}^n(z) = (\mathbf{I}_{2N} + \mathcal{O}(1/z)) \begin{pmatrix} z^n\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & z^{-n}\mathbf{I}_N \end{pmatrix}$ as $z \to \infty$

For $n \geq 1$ the unique solution of the RH problem above is given by

$$\mathbf{Y}^n(z) = \begin{pmatrix} \widehat{\mathbf{P}}_n(z) & C(\widehat{\mathbf{P}}_n\mathbf{W})(z) \\ -2\pi i\gamma_{n-1}\widehat{\mathbf{P}}_{n-1}(z) & -2\pi i\gamma_{n-1}C(\widehat{\mathbf{P}}_{n-1}\mathbf{W})(z) \end{pmatrix}$$

where $C(\mathbf{F})(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\mathbf{F}(t)}{t-z}dt$ and $\gamma_n = \boldsymbol{\kappa}_n^*\boldsymbol{\kappa}_n$.

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The Lax pair I

We look for a pair of first-order difference/differential equations of the form

$$\mathbf{Y}^{n+1}(z) = \mathbf{E}_n(z)\mathbf{Y}^n(z), \quad \frac{d}{dz}\mathbf{Y}^n(z) = \mathbf{F}_n(z)\mathbf{Y}^n(z)$$

Goal: obtain an invertible transformation $\mathbf{Y}^n \to \mathbf{X}^n$ such that $\mathbf{X}^n$ has a constant jump across $\mathbb{R}$. Consider $\mathbf{X}^n(z) = \mathbf{Y}^n(z)\mathbf{V}(z)$ where

$$\mathbf{V}(z) = \begin{pmatrix} \mathbf{T}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-*}(z) \end{pmatrix}$$

where $\mathbf{T}$ is an invertible $N \times N$ smooth matrix function.
This motivates to consider a factorization of the weight in the form

$$\mathbf{W}(x) = \mathbf{T}(x)\mathbf{T}^*(x), \quad x \in \mathbb{R}.$$

This factorization is not unique since

$$\mathbf{T}(x) = \widehat{\mathbf{T}}(x)\mathbf{S}(x), \quad x \in \mathbb{R}$$

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The Lax pair II

We additionally assume

$$\mathbf{T}'(z) = \mathbf{G}(z)\mathbf{T}(z),$$

where $\mathbf{G}$ is a matrix polynomial of degree m (most of our examples)

$$\mathbf{Y}^{n+1}(z) = \underbrace{\begin{pmatrix} z - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix}}_{\mathbf{E}_n(z;\mathbf{G})} \mathbf{Y}^n(z)$$

$$\frac{d}{dz}\mathbf{Y}^n(z) = \underbrace{\begin{pmatrix} -\mathcal{B}_n(z;\mathbf{G}) & -\frac{1}{2\pi i}\gamma_n^{-1}\mathcal{A}_n(z;\mathbf{G}) \\ 2\pi i\,\mathcal{A}_{n-1}(z;\mathbf{G})\gamma_{n-1} & \mathcal{B}_n^*(z;\mathbf{G}) \end{pmatrix}}_{\mathbf{F}_n(z;\mathbf{G})} \mathbf{Y}^n(z)$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are matrix polynomials of degree $m-1$ and m respectively. Cross-differentiating the Lax pair yield the compatibility conditions

$$\mathbf{E}_n'(z;\mathbf{G}) + \mathbf{E}_n(z;\mathbf{G})\mathbf{F}_n(z;\mathbf{G}) = \mathbf{F}_{n+1}(z;\mathbf{G})\mathbf{E}_n(z;\mathbf{G})$$

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The Lax pair III

If there exists a non-trivial matrix-valued function $\mathbf{S}$, non-singular on $\mathbb{C}$, smooth and unitary on $\mathbb{R}$, s.t.

$$\mathbf{H}(z) = \mathbf{T}(z)\mathbf{S}'(z)\mathbf{S}^*(z)\mathbf{T}^{-1}(z)$$

is also a polynomial, then $\widetilde{\mathbf{T}} = \mathbf{T}\mathbf{S}$ satisfies

$$\mathbf{W}(x) = \widetilde{\mathbf{T}}(x)\widetilde{\mathbf{T}}^*(x), \quad x \in \mathbb{R}, \qquad \widetilde{\mathbf{T}}'(z) = \widetilde{\mathbf{G}}(z)\widetilde{\mathbf{T}}(z), \quad z \in \mathbb{C},$$

with $\widetilde{\mathbf{G}}(z) = \mathbf{G}(z) + \mathbf{H}(z)$ and the matrix $\mathbf{X}^n$ satisfies

$$\frac{d}{dz}\mathbf{X}^n(z) = \mathbf{F}_n(z;\mathbf{G})\mathbf{X}^n(z) + \mathbf{F}_n(z;\mathbf{H})\mathbf{X}^n(z) - \mathbf{X}^n(z)\begin{pmatrix} \chi(z) & \mathbf{0} \\ \mathbf{0} & -\chi^*(z) \end{pmatrix}$$

with $\chi(z) = \mathbf{S}'(z)\mathbf{S}^*(z)$.

Consequences: We have a class of ladder operators.

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Example I: Hermite type MOP

Let us consider $\mathbf{T}(x) = e^{-x^2/2} e^{\mathbf{A}x}$ and

$$\mathbf{W}(x) = e^{-x^2} e^{\mathbf{A}x} e^{\mathbf{A}^* x}, \quad \mathbf{A} \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}.$$

Lax pair

$$\mathbf{X}^{n+1}(z) = \begin{pmatrix} z\mathbf{I}_N - \alpha_n & \frac{1}{2\pi i}\gamma_n^{-1} \\ -2\pi i \gamma_n & 0 \end{pmatrix} \mathbf{X}^n(z)$$

$$\frac{d}{dz}\mathbf{X}^n(z) = \begin{pmatrix} -z\mathbf{I}_N + \mathbf{A} & -\frac{1}{\pi i}\gamma_n^{-1} \\ 4\pi i \gamma_{n-1} & z\mathbf{I}_N - \mathbf{A}^* \end{pmatrix} X^n(z)$$

Compatibility conditions

$$\alpha_n = (\mathbf{A} + \gamma_n^{-1}\mathbf{A}^*\gamma_n)/2, \quad 2(\beta_{n+1} - \beta_n) = \mathbf{A}\alpha_n - \alpha_n\mathbf{A} + \mathbf{I}_N$$

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Ladder operators

$$\widehat{\mathbf{P}}_n'(x) + \widehat{\mathbf{P}}_n(x)\mathbf{A} - \mathbf{A}\widehat{\mathbf{P}}_n(x) = 2\beta_n \widehat{\mathbf{P}}_{n-1}(x),$$

$$-\widehat{\mathbf{P}}_n'(x) + 2x\widehat{\mathbf{P}}_n(x) + \mathbf{A}\widehat{\mathbf{P}}_n(x) - \widehat{\mathbf{P}}_n(x)\mathbf{A} - 2\alpha_n \widehat{\mathbf{P}}_n(x) = 2\widehat{\mathbf{P}}_{n+1}(x).$$

Combining them we get a second order differential equation

Second order differential equation

$$\widehat{\mathbf{P}}_n''(x) + 2\widehat{\mathbf{P}}_n'(x)(\mathbf{A} - x\mathbf{I}_N) + \widehat{\mathbf{P}}_n(x)(\mathbf{A}^2 - 2x\mathbf{A})$$
$$= (-2x\mathbf{A} + \mathbf{A}^2 - 4\beta_n)\widehat{\mathbf{P}}_n(x) + 2(\mathbf{A} - \alpha_n)(\widehat{\mathbf{P}}_n'(x) + \widehat{\mathbf{P}}_n(x)\mathbf{A} - \mathbf{A}\widehat{\mathbf{P}}_n(x)).$$

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In order to use the freedom in the matrix case by a unitary matrix function $\mathbf{S}$ we have to impose additional constraints on the weight $\mathbf{W}$.

The matrix $\mathbf{H}$ can be written as

$$\mathbf{H}(x) = e^{\mathbf{A}x}\chi e^{-\mathbf{A}x} = \chi + \mathrm{ad}_{\mathbf{A}}(\chi)x + \mathrm{ad}_{\mathbf{A}}^2(\chi)\frac{x^2}{2} + \cdots ,$$

where $\chi(x) = \mathbf{S}'(x)\mathbf{S}^*(x)$ is skew-Hermitian on $\mathbb{R}$.

This matrix equation was considered already by Durán-Grünbaum (2004), when χ is a constant matrix.

- If $\deg \mathbf{H} = 0$ then $\chi = ia\mathbf{I}_N, a \in \mathbb{R} \Rightarrow$ No new ladder operators.
- If $\deg \mathbf{H} = 1$ then
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First case $\mathbf{A} = \mathbf{L}$

New compatibility conditions

$$\mathbf{J}\boldsymbol{\alpha}_n - \boldsymbol{\alpha}_n\mathbf{J} + \boldsymbol{\alpha}_n = \mathbf{L} + \frac{1}{2}(\mathbf{L}^2\boldsymbol{\alpha}_n - \boldsymbol{\alpha}_n\mathbf{L}^2), \quad \mathbf{J} - \gamma_n^{-1}\mathbf{J}\gamma_n = \mathbf{L}\boldsymbol{\alpha}_n + \boldsymbol{\alpha}_n\mathbf{L} - 2\boldsymbol{\alpha}_n^2$$

New ladder operators (0-th order)

$$\widehat{\mathbf{P}}_n(x)\mathbf{J} - \mathbf{J}\widehat{\mathbf{P}}_n(x) - x(\widehat{\mathbf{P}}_n(x)\mathbf{L} - \mathbf{L}\widehat{\mathbf{P}}_n(x)) + 2\beta_n\widehat{\mathbf{P}}_n(x) - n\widehat{\mathbf{P}}_n(x) = 2(\mathbf{L} - \boldsymbol{\alpha}_n)\beta_n\widehat{\mathbf{P}}_{n-1}(x)$$

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First-order differential equation

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Example II: Freud type MOP

Let us consider $\mathbf{W}(x) = e^{-x^4} e^{\mathbf{B}x^2} e^{\mathbf{B}^* x^2}, \quad \mathbf{B} \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}.$

Ladder operators

$$\widehat{\mathbf{P}}'_n(x) + 2x(\widehat{\mathbf{P}}_n(x)\mathbf{B} - \mathbf{B}\widehat{\mathbf{P}}_n(x)) + 4x\beta_n \widehat{\mathbf{P}}_n(x) = (4(x^2\mathbf{I} + \beta_{n+1} + \beta_n) - 2(\mathbf{B} + \gamma_n^{-1}\mathbf{B}^*\gamma_n))\beta_n \widehat{\mathbf{P}}_{n-1}(x)$$

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2. This method works for every weight matrix $\mathbf{W}$. The corresponding MOP satisfy differential equations, but not necessarily of Sturm-Liouville type

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