SOME EXAMPLES OF MATRIX-VALUED ORTHOGONAL FUNCTIONS HAVING A DIFFERENTIAL AND AN INTEGRAL OPERATOR AS EIGENFUNCTIONS

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Hermite or wave functions

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$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x)$$

where $H_n(x) = (-1)^n e^{x^2} (e^{-x^2})^{(n)}$, are the Hermite polynomials

$(\psi_n)_n$ is a complete orthonormal set in $L^2(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} \psi_n(x)\psi_m^*(x)dx = \delta_{nm}, \quad n, m \geq 0$$

$(\psi_n)_n$ are eigenfunctions of the Schrödinger operator

$$\psi_n''(x) - x^2\psi_n(x) = -(2n+1)\psi_n(x), \quad x \in \mathbb{R}$$

and the Fourier transform

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_n(t) e^{ixt} dt = (i)^n \psi_n(x), \quad x \in \mathbb{R}$$

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Hermite polynomials

Some consequences for the Hermite polynomials:

Differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

Real integral equations

$$e^{-x^2/2} H_{2n}(x) = (-1)^n \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2} H_{2n}(t) \cos(xt) dt$$

$$e^{-x^2/2} H_{2n+1}(x) = (-1)^n \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2} H_{2n+1}(t) \sin(xt) dt$$

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Matrix-valued function spaces

We will say that $F \in L^2(\mathbb{R}, \mathbb{C}^{N\times N})$ if

$$\langle F, F\rangle \doteq \int_{-\infty}^{\infty} F(x)F^*(x)dx < \infty,$$

$(L^2(\mathbb{R}, \mathbb{C}^{N\times N}), \|\cdot\|)$ with $\|F\| = \mathrm{Tr}\,(\langle F, F\rangle)^{1/2}$ is a Hilbert space

Similarly for the weighted spaces $L^2_W(\mathbb{R}, \mathbb{C}^{N\times N})$ with the inner product

$$\langle F, G\rangle_W \doteq \int_{-\infty}^{\infty} F(x)W(x)G^*(x)dx$$

Typically

$$W(x) = \rho(x)T(x)T^*(x),$$

where T satisfies

$$T'(x) = H(x)T(x), \quad T(c) = I, \quad c \in \mathbb{R}$$

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Some notation

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ J = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

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For any infinitely differentiable f in a neighborhood of $x = 0$ we have

$$f(x) = \sum_{j=0}^{\infty} f^{(j)}(0) \frac{x^j}{j!}$$

Whenever we write $f(X)$, for any $N \times N$ matrix X, we mean

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Some examples $f(x) = \frac{1}{1+x}$

$$f(A) = (I + A)^{-1} = \sum_{j=0}^{N-1} (-1)^j A^j$$

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Some examples $f(x) = e^{i\frac{\pi}{2}x}$

$$f(J) = e^{i\frac{\pi}{2}J} = \begin{pmatrix} (i)^{N-1} & & & \\ & \ddots & & \\ & & i & \\ & & & 1 \end{pmatrix}$$

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Some examples $\quad f(x) = \sin\left(\frac{\pi}{2}x\right) = \frac{1}{2i}(e^{i\frac{\pi}{2}x} - e^{-i\frac{\pi}{2}x})$

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First example

Let W be the following weight matrix

$$W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R}$$

We know from Durán-Grünbaum (2004) that

$$\widehat{P}_n''(x) - 2\widehat{P}_n'(x)(xI - A) + \widehat{P}_n(x)(A^2 - 2J) = (-2nI + A^2 - 2J)\widehat{P}_n(x)$$

Let

$$P_n(x) = e^{-A^2/4}\widehat{P}_n(x)$$

$(P_n)_n$ satisfies the same differential equation
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Differential and integral equation

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CONSEQUENCES FOR $(P_n)_n$

SYMMETRY CONDITIONS

$$P_n(x) = (-1)^n e^{i\pi J} P_n(-x) e^{i\pi J}, \quad W(x) = e^{i\pi J} W(-x) e^{i\pi J}$$

REAL INTEGRAL EQUATIONS

Denote $C_- = \sin\left(\frac{\pi}{2}J\right)$, $C_+ = \cos\left(\frac{\pi}{2}J\right)$ and
$k_n(x,t) = \cos(xt)$ if n is even and $k_n(x,t) = \sin(xt)$ if n is odd

$$e^{-x^2/2}(e^{i\pi J} \pm I)P_n(x)e^{Ax}C_\pm = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{\sqrt{2\pi}} C_\pm \int_{-\infty}^{\infty} e^{-t^2/2} k_n(x,t) P_n(t) e^{At} dt (e^{i\pi J} \pm I),$$

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Case $N = 2$

The *normalized* family is

$$\Phi_n(x) = \begin{pmatrix} \psi_n(x)/\sqrt{\gamma_{n+1}} & \nu_1\sqrt{\frac{n+1}{2\gamma_{n+1}}}\psi_{n+1}(x) \\ -\nu_1\sqrt{\frac{n}{2\gamma_n}}\psi_{n-1}(x) & \psi_n(x)/\sqrt{\gamma_n} \end{pmatrix}, \quad \gamma_n = 1 + \frac{n}{2}\nu_1^2$$

That means $\int_{-\infty}^{\infty}\Phi_n(x)\Phi_m^*(x)dx = \delta_{nm}I$, so the diagonal entries are probability distributions on $\mathbb{R}$. The plots for the first values of n and $\nu_1 = 1$ are given by

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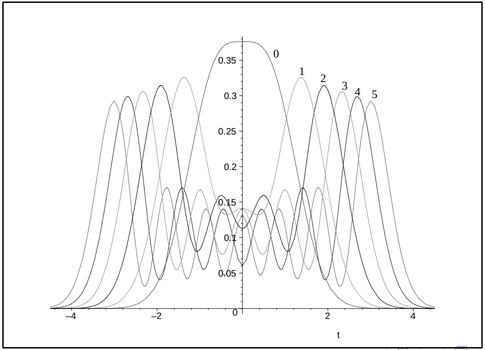

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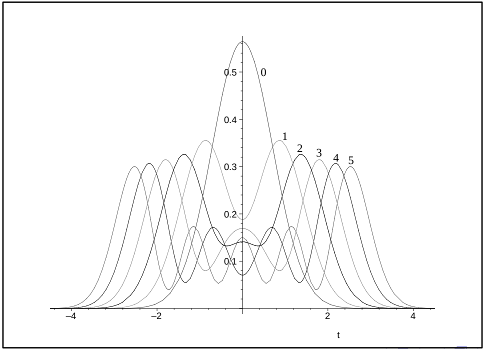

We also have

$$(xI)_{nm} \doteq \int_{-\infty}^{\infty} x\Phi_n(x)\Phi_m^*(x)dx = \begin{pmatrix} \sqrt{\frac{n\gamma_{n+1}}{2\gamma_n}} & 0 \\ 0 & \sqrt{\frac{n\gamma_{n-1}}{2\gamma_n}} \end{pmatrix} \delta_{m,n-1}$$

$$+ \begin{pmatrix} 0 & \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} \\ \frac{\nu_1}{2\sqrt{\gamma_n\gamma_{n+1}}} & 0 \end{pmatrix} \delta_{m,n} + \begin{pmatrix} \sqrt{\frac{(n+1)\gamma_{n+2}}{2\gamma_{n+1}}} & 0 \\ 0 & \sqrt{\frac{(n+1)\gamma_n}{2\gamma_{n+1}}} \end{pmatrix} \delta_{m,n+1}$$

Therefore (xI) is the matrix of the homomorphism $F \mapsto xF$ in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$ with respect to the basis $(\Phi_n)_n$

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Second example

Let W be the following weight matrix

$$W(x) = e^{-x^2} e^{Bx^2} e^{B^* x^2}, \quad B = A(I+A)^{-1}$$

We know from Durán-Grünbaum (2004) that

$$\widehat{P}_n''(x) - 2x\widehat{P}_n'(x)(I-2B) + 2\widehat{P}_n(x)(B-2J) = -2((I-2B)n - B + 2J)\widehat{P}_n(x)$$

Let

$$P_n(x) = [(I+A)^{-1/2}]^{2n+1}\widehat{P}_n(x)$$

$(P_n)_n$ satisfies the same differential equation
but now with diagonal eigenvalue $-2nI - 4J$

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Differential and integral equation

Consider the family of matrix-valued functions

$$\Phi_n(x) = e^{-x^2/2} P_n(x) e^{Bx^2}$$

$(\Phi_n)_n$ is an orthogonal family in $L^2(\mathbb{R}, \mathbb{C}^{N\times N})$

$(\Phi_n)_n$ are simultaneously eigenfunctions
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$$\Phi_n''(x) - \Phi_n(x)(x^2 I + 4J) + ((2n+1)I + 4J)\Phi_n(x) = 0, \quad x \in \mathbb{R}$$

and an integral operator of Fourier type

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_n(t) e^{ixt} e^{i\pi J} dt = (i)^n e^{i\pi J} \Phi_n(x), \quad x \in \mathbb{R}$$

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Consequences for $(P_n)_n$

Symmetry conditions

$$P_n(x) = (-1)^n P_n(-x)$$

Real integral equations

$$e^{-x^2/2} e^{i\pi J} P_{2n}(x) e^{Bx^2} = (-1)^n \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2} P_{2n}(t) e^{Bt^2} \cos xt e^{i\pi J} dt,$$

and

$$e^{-x^2/2} e^{i\pi J} P_{2n+1}(x) e^{Bx^2} = (-1)^n \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2} P_{2n+1}(t) e^{Bt^2} \sin xt e^{i\pi J} dt$$

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