Integral representations of some Hermite type matrix-valued kernels and non-commutative Painlevé equations[1]

Manuel Domínguez de la Iglesia

Departamento de Análisis Matemático, Universidad de Sevilla

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[1] joint work with Mattia Cafasso

Outline

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Motivation

Let $(H_n)_n$ be the classical Hermite polynomials such that

$$\int_{\mathbb{R}} H_n(x)H_m(x)\mathrm{e}^{-x^2}dx = \delta_{nm}$$

The Christoffel-Darboux (CD) kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} H_k(x)H_k(y)\mathrm{e}^{-\frac{x^2+y^2}{2}}$$

describes the statistical properties of the eigenvalues of a random matrix $\mathbf{M}$ in the space of $(n \times n)$ Hermitian matrices with the measure $\mu(\mathbf{M}) = \mathrm{e}^{-\mathrm{Tr}(\mathbf{M}^2)}d\mathbf{M}$ (GUE, Mehta).
The last particle distribution is given by the Fredholm determinant

$$F(s) = \mathbb{P}\left[\lambda_{\max} \leq s\right] = \det(\mathrm{Id} - \chi_s \mathbb{K}_n)$$

where χ_s is the indicator function of the interval $[s,\infty)$ and $\mathbb{K}_n : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the integral operator

$$[\mathbb{K}_n f](x) = \int_{\mathbb{R}} K_n(x,y)f(y)dy \quad \forall\, f \in L^2(\mathbb{R})$$

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Theorem (Tracy-Widom, 1994)

The log derivative of the Fredholm determinant

$$R(s) = \partial_s \log(\det(\mathrm{Id} - \chi_s \mathbb{K}_n))$$

solves the sigma-form of the Painlevé IV equation

$$(R'')^2 + 4(R')^2(R' + 2n) - 4(sR' - R)^2 = 0$$

Goal of this talk

Extend these results to CD kernels associated to Hermite-type matrix-valued orthogonal polynomials (MOP).

- Double integral representations of some Hermite-type MOP

  $\Rightarrow$ Matrix-valued CD kernels.

- Relate the Fredholm determinant of this kernel to a Riemann-Hilbert problem (RHP) whose compatibility conditions lead to a derived version of a non-commutative Painlevé IV equation.

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Preliminaries

Let $\mathbf{W}$ be a weight matrix (positive definite and finite moments). Consider $L^2_{\mathbf{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$ the weighted space with the inner product

$$\langle \mathbf{F}, \mathbf{G} \rangle_{\mathbf{W}} = \int_{\mathbb{R}} \mathbf{F}(x)\mathbf{W}(x)\mathbf{G}^*(x)dx$$

A sequence $(\mathbf{P}_n)_n$ of matrix orthonormal polynomials (MOP) with respect to $\mathbf{W}$ is a sequence satisfying

$$\deg \mathbf{P}_n = n, \quad \langle \mathbf{P}_n, \mathbf{P}_m \rangle_{\mathbf{W}} = \mathbf{I}_N \delta_{nm}$$

If $(\mathbf{P}_n)_n$ is complete, the Christoffel-Darboux (CD) kernel is

$$\mathbf{K}_n(x,y) = \sum_{k=0}^{n-1} \mathbf{P}_k^*(y)\mathbf{P}_k(x), \quad x, y \in \mathbb{R}$$

with the properties ($\mathbf{F} \in L^2_{\mathbf{W}}(\mathbb{R}, \mathbb{C}^{N\times N})$)

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Some notation

$$\mathbf{A}_N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \nu_i \in \mathbb{R}, \ \mathbf{J}_N = \begin{pmatrix} N-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

We will remove the dependence of N whenever there is no confusion about the dimension of the matrices.

Let us denote $z^{\mathbf{M}} = e^{\mathbf{M} \log z}$. For example

$$z^{\mathbf{J}_N} = \begin{pmatrix} z^{N-1} & & & \\ & \ddots & & \\ & & z & \\ & & & 1 \end{pmatrix}, \quad z^{-\mathbf{J}_N} = \begin{pmatrix} \frac{1}{z^{N-1}} & & & \\ & \ddots & & \\ & & \frac{1}{z} & \\ & & & 1 \end{pmatrix}$$

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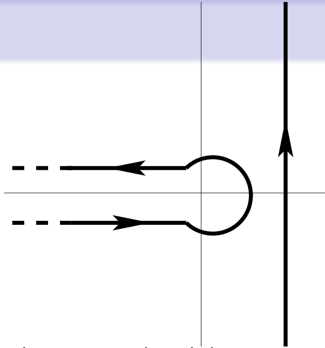
The first example

Let us consider
the weight matrix (Durán-Grünbaum, 2004)

$$\mathbf{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\mathbf{A}x}\mathrm{e}^{\mathbf{A}^* x}, \quad x \in \mathbb{R}$$

and the family of MOP $(\mathbf{P}_n)_n$
satisfying the second order differential equation



$$\mathbf{P}_n''(x) + \mathbf{P}_n'(x)(-2x\mathbf{I} + 2\mathbf{A}) + \mathbf{P}_n(x)(\mathbf{A}^2 - 2\mathbf{J}) = (-2n\mathbf{I} - 2\mathbf{J})\mathbf{P}_n(x)$$

Theorem (Cafasso-MdI, 2013)

There exist suitable constant matrices $\mathbf{C}_n$ and $\mathbf{D}_n$ such that

$$\mathbf{P}_n(x)\mathrm{e}^{\mathbf{A}x} = \oint_\gamma z^{-\mathbf{J}}\mathbf{C}_n z^{\mathbf{J}}\mathrm{e}^{-z^2+2zx}\frac{dz}{z^{n+1}}$$

$$\mathbf{P}_n(x)\mathrm{e}^{\mathbf{A}x} = \mathrm{e}^{x^2}\int_{\mathcal{I}} w^{\mathbf{J}}\mathbf{D}_n w^{-\mathbf{J}}\mathrm{e}^{w^2-2xw}w^n dw$$

Case $N = 2$

For the family of MOP $(\mathbf{P}_n)_n$ with respect to $(N = 2)$

$$\mathbf{W}(x) = \mathrm{e}^{-x^2} \begin{pmatrix} 1 + x^2\nu^2 & \nu x \\ \nu x & 1 \end{pmatrix}, \quad x \in \mathbb{R}$$

we have that, if $\gamma_n^2 = 1 + \frac{n}{2}\nu^2$

$$\mathbf{P}_n(x) \begin{pmatrix} 1 & \nu x \\ 0 & 1 \end{pmatrix} = \frac{n!}{2^{n+1}\pi i} \oint_\gamma \begin{pmatrix} 1 & \dfrac{(n+1)\nu}{2z} \\ -\dfrac{z\nu}{\gamma_n^2} & \dfrac{1}{\gamma_n^2} \end{pmatrix} \mathrm{e}^{-z^2+2zx} \frac{dz}{z^{n+1}}$$

$$\mathbf{P}_n(x) \begin{pmatrix} 1 & \nu x \\ 0 & 1 \end{pmatrix} = \frac{\mathrm{e}^{x^2}}{i\sqrt{\pi}} \int_{\mathcal{I}} \begin{pmatrix} 1 & w\nu \\ -\dfrac{n\nu}{2w\gamma_n^2} & \dfrac{1}{\gamma_n^2} \end{pmatrix} \mathrm{e}^{w^2-2xw} w^n dw$$

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Christoffel-Darboux kernel ($N = 2$)

The CD kernel

$$\mathbf{K}_n(x,y) = \sum_{k=0}^{n-1} \boldsymbol{\Phi}_k^*(y)\boldsymbol{\Phi}_k(x)$$

where $\boldsymbol{\Phi}_n(x) = \mathrm{e}^{-x^2/2}\|\mathbf{P}_n\|_{\mathbf{W}}^{-1}\mathbf{P}_n(x)\mathrm{e}^{\mathbf{A}x}$ can be written as

$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}} dw \oint_{\gamma} dz\, z^{\mathbf{J}_2}\mathbf{B}_n z^{-\mathbf{J}_2} w^{\mathbf{J}_2}\mathbf{B}_n^{-1} w^{-\mathbf{J}_2}\frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

where

$$\mathbf{B}_n = \begin{pmatrix} 1 & -\nu \\ \frac{n\nu}{2} & 1 \end{pmatrix}$$

In other words

$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}} dw \oint_{\gamma} dz \begin{pmatrix} \frac{z(\gamma_n^2-1)+w}{w\gamma_n^2} & \frac{\nu(w-z)}{\gamma_n^2} \\ \frac{n\nu(z-w)}{2\gamma_n^2} & \frac{w(\gamma_n^2-1)+z}{z\gamma_n^2} \end{pmatrix}\frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

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The second example

Let us consider the weight matrix (Durán-Grünbaum, 2004)

$$\mathbf{W}(x) = \mathrm{e}^{-x^2}\mathrm{e}^{\mathbf{B}x^2}\mathrm{e}^{\mathbf{B}^*x^2}, \quad x \in \mathbb{R}$$

where $\mathbf{B} = \mathbf{A}(\mathbf{I} + \mathbf{A})^{-1}$ and the family of MOP $(\mathbf{P}_n)_n$ satisfying the second-order differential equation

$$\mathbf{P}_n''(x) + \mathbf{2}xP_n'(x)(2\mathbf{B} - \mathbf{I}) + \mathbf{2}P_n(x)(\mathbf{B} - 2\mathbf{J}) = (-2n\mathbf{I} - 4\mathbf{J})\mathbf{P}_n(x)$$

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For the family of MOP $(\mathbf{P}_n)_n$ with respect to $(N = 2)$

$$\mathbf{W}(x) = \mathrm{e}^{-x^4} \begin{pmatrix} 1 + x^2\nu^2 & \nu x^2 \\ \nu x^2 & 1 \end{pmatrix}, \quad x \in \mathbb{R}$$

we have that, if $\delta_n^2 = 1 + \frac{n(n-1)}{4}\nu^2$

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$$\frac{2}{(2\pi i)^2}\mathrm{e}^{\frac{x^2-y^2}{2}}\int_{\mathcal{I}} dw \oint_{\gamma} dz\, z^{2\mathbf{J}_2}\mathbf{B}_n z^{-\mathbf{J}_3} w^{\mathbf{J}_3}\hat{\mathbf{B}}_n w^{-2\mathbf{J}_2}\frac{\mathrm{e}^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

where

$$\mathbf{B}_n = \begin{pmatrix} \dfrac{1}{\delta_{n+1}^2} & \dfrac{n\nu^2}{2\delta_{n+1}^2\delta_n^2} & -\nu \\ \dfrac{\nu n(n+1)}{4\delta_{n+1}^2} & -\dfrac{n\nu}{2\delta_{n+1}^2\delta_n^2} & 1 \end{pmatrix}, \quad \hat{\mathbf{B}}_n = \begin{pmatrix} 1 & \nu \\ 1 & \nu \\ -\dfrac{\nu n(n-1)}{4\delta_n^2} & \dfrac{1}{\delta_n^2} \end{pmatrix}$$

We have that $\hat{\mathbf{B}}_n$ is a *right inverse* of $\mathbf{B}_n$, i.e. $\mathbf{B}_n\hat{\mathbf{B}}_n = \mathbf{I}$.

Christoffel-Darboux kernel ($N=2$)

The CD kernel

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Outline

CD kernels and Riemann-Hilbert problems

The Christoffel-Darboux kernel in both cases can be written as

$$\mathbf{K}_n(x,y) = \frac{2}{(2\pi i)^2} e^{\frac{x^2-y^2}{2}} \int_{\mathcal{I}} dw \oint_{\gamma} dz\, \boldsymbol{\mathcal{B}}_n(z)\hat{\boldsymbol{\mathcal{B}}}_n(w) \frac{e^{w^2-2xw-z^2+2zy+n\log(w/z)}}{w-z}$$

where $\boldsymbol{\mathcal{B}}_n$ is $(N \times p)$ and $\hat{\boldsymbol{\mathcal{B}}}_n$ is $(p \times N)$ such that $\boldsymbol{\mathcal{B}}_n(z)\hat{\boldsymbol{\mathcal{B}}}_n(z) = \mathbf{I}_N$.

Consider $\mathbb{K}_n$ the integral operator with kernel $\mathbf{K}_n(x,y)$

$$[\mathbb{K}_n\mathbf{F}](x) = \int_{\mathbb{R}} \mathbf{F}(y)\mathbf{K}_n(x,y)dy \quad \forall\, \mathbf{F} \in L^2(\mathbb{R}, \mathbb{C}^{N\times N})$$

Its-Izergin-Korepin-Slavnov (IIKS) theory

The Fredholm determinant $\det(\mathrm{Id} - \chi_s\mathbb{K}_n)$ is equal to the Jimbo-Miwa-Ueno tau function τ_{JMU}. In particular

$$\partial_s \log\det(\mathrm{Id} - \chi_s\mathbb{K}_n) = \frac{1}{2\pi i}\int_{\gamma\cup\mathcal{I}} \mathrm{Tr}\left(\boldsymbol{\Gamma}_-^{-1}(\lambda)(\partial_\lambda\boldsymbol{\Gamma}_-)(\lambda)\boldsymbol{\Xi}(\lambda)\right) d\lambda$$

where we denoted

$$\boldsymbol{\Xi}(\lambda) = \partial_s(\mathbf{I} - \mathbf{G}(\lambda))(\mathbf{I} - \mathbf{G}(\lambda))^{-1} = -\partial_s\mathbf{G}(\lambda)(\mathbf{I} + \mathbf{G}(\lambda))$$

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CD kernels and Riemann-Hilbert problems

$\mathbf{\Gamma}(\lambda)$ solves the following

Riemann-Hilbert problem

Find $\mathbf{\Gamma}(\lambda) \in GL(N+p, \mathbb{C})$ analytic on $\mathbb{C} \backslash \{\gamma \cup \mathcal{I}\}$ such that

$$\begin{cases} \mathbf{\Gamma}_+(\lambda) = \mathbf{\Gamma}_-(\lambda)(\mathbf{I} - \mathbf{G}(\lambda)), & \lambda \in \eta \cup \gamma \\ \mathbf{\Gamma}(\lambda) = \mathbf{I} + \dfrac{\mathbf{\Gamma}_1}{\lambda} + \dfrac{\mathbf{\Gamma}_2}{\lambda^2} + \cdots, & \lambda \to \infty \end{cases}$$

with

$$\mathbf{G}(\lambda) = \begin{bmatrix} \mathbf{0} & \mathrm{e}^{\theta_n(\lambda,s)} \hat{\boldsymbol{\mathcal{B}}}_n^*(\lambda) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \chi_{\mathcal{I}}(\lambda) + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathrm{e}^{-\theta_n(\lambda,s)} \boldsymbol{\mathcal{B}}_n^*(\lambda) & \mathbf{0} \end{bmatrix} \chi_\gamma(\lambda)$$

and $\theta_n(\lambda, s) = \lambda^2 - 2\lambda s + n \log(\lambda)$.

The first example ($N = 2$)

Theorem (Cafasso-MdI, 2013)

Let $\mathbf{\Gamma}(\lambda)$ be the solution of the previous Riemann-Hilbert problem with $\boldsymbol{\mathcal{B}}_n(z) = z^{\mathbf{J}_2}\mathbf{B}_n z^{-\mathbf{J}_2}$ and $\hat{\boldsymbol{\mathcal{B}}}_n(w) = (\boldsymbol{\mathcal{B}}_n(w))^{-1}$ where

$$\mathbf{B}_n = \begin{pmatrix} 1 & -\nu \\ \dfrac{n\nu}{2} & 1 \end{pmatrix}$$

$$\Rightarrow \partial_s \log\det(\mathrm{Id} - \chi_s \mathbb{K}_n) = \mathrm{Tr}\Big((\mathbf{\Gamma}_1)_{22} - (\mathbf{\Gamma}_1)_{11}\Big)$$

Consider the transformation $\mathbf{\Psi}(\lambda) = \mathbf{\Gamma}(\lambda)\mathrm{e}^{\mathbf{T_A}(\lambda)}$ where

$$\mathbf{T_A}(\lambda) = \begin{pmatrix} \frac{\theta_n(\lambda,s)}{2}\mathbf{I}_2 - \mathbf{J}_2\log(\lambda) & \mathbf{0} \\ \mathbf{0} & -\frac{\theta_n(\lambda,s)}{2}\mathbf{I}_2 - \mathbf{J}_2\log(\lambda) \end{pmatrix}$$

This transformation allows $\mathbf{\Psi}(\lambda)$ to satisfy a Riemann-Hilbert problem like the previous one but with constant jump.

The first example ($N = 2$)

Theorem (Continuation)

$\boldsymbol{\Psi}(\lambda)$ satisfies the Lax equations

$$\partial_\lambda \boldsymbol{\Psi} = \left(\lambda \boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{A}}_0 + \lambda^{-1} \boldsymbol{\mathcal{A}}_{-1}\right) \boldsymbol{\Psi}, \quad \partial_s \boldsymbol{\Psi} = \left(\lambda \boldsymbol{\mathcal{U}}_1 + \boldsymbol{\mathcal{U}}_0\right) \boldsymbol{\Psi}$$

The compatibility conditions give the following coupled system of ODEs:

$$\begin{cases} \mathbf{u}' &= -\mathbf{u}^2 + 2s\mathbf{u} + 4\mathbf{z} - 2n\mathbf{I}_2 + \mathbf{V_A} \\ \mathbf{z}'' &= 2\mathbf{u}'\mathbf{z} + 2\mathbf{u}\mathbf{z}' - 2s\mathbf{z}' \end{cases}$$

where $\mathbf{V_A} = 2[\mathbf{J}_2, \mathbf{y}]\mathbf{y}^{-1}$ ($[\mathbf{x}, \mathbf{y}] = \mathbf{xy} - \mathbf{yx}$) and

$$\mathbf{z} = -(\boldsymbol{\Gamma}_1)'_{11}, \quad \mathbf{y} = -2(\boldsymbol{\Gamma}_1)_{12}, \quad \mathbf{u} = (\boldsymbol{\Gamma}_1)'_{12}(\boldsymbol{\Gamma}_1)^{-1}_{12} + 2s\mathbf{I}_2$$

Combining these two equations we obtain a non-commutative version of the derived Painlevé IV equation ($\{\mathbf{x}, \mathbf{y}\} = \mathbf{xy} + \mathbf{yx}$)

$$\mathbf{u}''' + [\mathbf{u}'', \mathbf{u}] - 4(n + 1 + s^2)\mathbf{u}' - 2\left(\{\mathbf{u}', \mathbf{u}^2\} + \mathbf{u}\mathbf{u}'\mathbf{u}\right)$$
$$+ 6s\{\mathbf{u}', \mathbf{u}\} + 4\mathbf{u}(\mathbf{u} - s\mathbf{I}_2) + (\mathbf{V}'_\mathbf{A} - 2(\mathbf{u}\mathbf{V_A}))' + 2s\mathbf{V}'_\mathbf{A} = \mathbf{0}$$

The Painlevé IV equation

The reason why we claim that the previous equation is a non-commutative version of the derived Painlevé IV is that if we assume that all the variables commute, we get the equation

Derived PIV equation

$$u''' - 4u' - 6u^2u' + 12u'u - 4nu' + 4u^2 - 4su - 4s^2u' = 0$$

an this equation is the derivative of the Painlevé IV equation

PIV equation

$$u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 - 4su^2 + 2(s^2 + 1 + n)u - \frac{2n^2}{u}$$

THE SECOND EXAMPLE $(N = 2)$

THEOREM (CAFASSO-MDI, 2013)

Let $\boldsymbol{\Gamma}(\lambda)$ be the solution of the previous Riemann-Hilbert problem with $\boldsymbol{\mathcal{B}}_n(z) = z^{2\mathbf{J}_2}\mathbf{B}_n z^{-\mathbf{J}_3}$ and $\hat{\boldsymbol{\mathcal{B}}}_n(w) = w^{\mathbf{J}_3}\hat{\mathbf{B}}_n w^{-2\mathbf{J}_2}$, where

$$\mathbf{B}_n = \begin{pmatrix} \frac{1}{\delta_{n+1}^2} & \frac{n\nu^2}{2\delta_{n+1}^2\delta_n^2} & -\nu \\ \frac{\nu n(n+1)}{4\delta_{n+1}^2} & -\frac{n\nu}{2\delta_{n+1}^2\delta_n^2} & 1 \end{pmatrix}, \quad \hat{\mathbf{B}}_n = \begin{pmatrix} 1 & \nu \\ 1 & \nu \\ -\frac{\nu n(n-1)}{4\delta_n^2} & \frac{1}{\delta_n^2} \end{pmatrix}$$

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This transformation allows $\boldsymbol{\Psi}(\lambda)$ to satisfy a Riemann-Hilbert problem like the previous one but with constant jump.

THE FIRST EXAMPLE ($N = 2$)

THEOREM (CONTINUATION)

$\mathbf{\Psi}(\lambda)$ satisfies the Lax equations

$$\partial_\lambda \mathbf{\Psi} = \left(\lambda \boldsymbol{\mathcal{A}}_1 + \boldsymbol{\mathcal{A}}_0 + \lambda^{-1}\boldsymbol{\mathcal{A}}_{-1}\right)\mathbf{\Psi}, \quad \partial_s \mathbf{\Psi} = \left(\lambda \boldsymbol{\mathcal{U}}_1 + \boldsymbol{\mathcal{U}}_0\right)\mathbf{\Psi}$$

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where $\mathbf{V_B} = 4\mathbf{J}_2 - 2\mathbf{y}\mathbf{J}_3\mathbf{y}^\dagger$ ($\mathbf{y}^\dagger$ is a *right inverse* of $\mathbf{y}$, i.e. $\mathbf{y}\mathbf{y}^\dagger = \mathbf{I}$) and

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