Approximation of matrix functions via orthogonal matrix polynomials

Manuel Domínguez de la Iglesia[1]

Instituto de Matemáticas C.U., UNAM

First Joint International Meeting of the Israel Mathematical Union and the Mexican Mathematical Society

Oaxaca, September 11, 2015

[1]Supported by PAPIIT-DGAPA-UNAM grant IA100515

Outline

1. Orthogonal Polynomials
    - Definitions and basic properties
    - Differential equations
    - Applications
2. Orthogonal Matrix Polynomials
    - Definitions and basic properties
    - Matrix differential equations
    - Applications

Outline

1. Orthogonal Polynomials
   - Definitions and basic properties
   - Differential equations
   - Applications
2. Orthogonal Matrix Polynomials
   - Definitions and basic properties
   - Matrix differential equations
   - Applications

Definitions and basic properties

Let ω be a positive measure on $\mathbb{R}$ and denote $(p_n)_n$ a system of polynomials with $\deg p_n = n$ such that they are orthogonal w.r.t. ω, i.e.

$$\int_{\mathbb{R}} p_n(x)p_m(x)d\omega(x) = 0, \quad n \neq m$$

Set $(f,g)_\omega = \int_{\mathbb{R}} f(x)g(x)d\omega(x)$ and $\|f\|_\omega^2 = (f,f)_\omega$. These are the scalar product and the norm for the weighted function space

$$L^2_\omega(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} f^2(x)d\omega(x) < \infty \right\}$$

For any function $f \in L^2_\omega(\mathbb{R})$ the series

$$(Sf)(x) = \sum_{k=0}^{\infty} \hat{f}_k p_k(x), \quad \hat{f}_k = \frac{(f,p_k)_\omega}{\|p_k\|_\omega^2}$$

is called the *generalized Fourier series of f*.

Definitions and basic properties

Let ω be a positive measure on $\mathbb{R}$ and denote $(p_n)_n$ a system of polynomials with $\deg p_n = n$ such that they are orthogonal w.r.t. ω, i.e.

$$\int_{\mathbb{R}} p_n(x)p_m(x)d\omega(x) = 0, \quad n \neq m$$

Set $(f,g)_\omega = \int_{\mathbb{R}} f(x)g(x)d\omega(x)$ and $\|f\|_\omega^2 = (f,f)_\omega$. These are the scalar product and the norm for the weighted function space

$$L^2_\omega(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} f^2(x)d\omega(x) < \infty \right\}$$

For any function $f \in L^2_\omega(\mathbb{R})$ the series

$$(Sf)(x) = \sum_{k=0}^{\infty} \hat{f}_k p_k(x), \quad \hat{f}_k = \frac{(f,p_k)_\omega}{\|p_k\|_\omega^2}$$

is called the *generalized Fourier series of f*.

Definitions and basic properties

Let ω be a positive measure on $\mathbb{R}$ and denote $(p_n)_n$ a system of polynomials with $\deg p_n = n$ such that they are orthogonal w.r.t. ω, i.e.

$$\int_{\mathbb{R}} p_n(x)p_m(x)d\omega(x) = 0, \quad n \neq m$$

Set $(f,g)_\omega = \int_{\mathbb{R}} f(x)g(x)d\omega(x)$ and $\|f\|_\omega^2 = (f,f)_\omega$. These are the scalar product and the norm for the weighted function space

$$L^2_\omega(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} f^2(x)d\omega(x) < \infty \right\}$$

For any function $f \in L^2_\omega(\mathbb{R})$ the series

$$(Sf)(x) = \sum_{k=0}^{\infty} \hat{f}_k p_k(x), \quad \hat{f}_k = \frac{(f,p_k)_\omega}{\|p_k\|_\omega^2}$$

is called the *generalized Fourier series of f.*

Definitions and basic properties

The series $(Sf)(x)$ converges *in average* or *in the sense of* $L^2_\omega(\mathbb{R})$, meaning that if $f_n(x) = \sum_{k=0}^{n} \hat{f}_k p_k(x)$, then the following holds

$$\lim_{n\to\infty} \|f - f_n\|_\omega = 0$$

The polynomial $f_n \in \mathbb{P}_n$ satisfies the following minimization property

$$\|f - f_n\|_\omega = \min_{q\in\mathbb{P}_n} \|f - q\|_\omega$$

So it is important to compute the coefficients $\hat{f}_k$. This can be done approximating the integrals using Gaussian quadratures, so the discrete truncation of f is the best approximation of f *in the least-squares sense*.

Additionally all families of orthogonal polynomials (OPs) $(p_n)_n$ satisfy a three term recurrence relation (Jacobi operator)

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad p_{-1} = 0, \quad p_0 \in \mathbb{R}$$

where

$$a_n = \frac{(xp_n, p_{n+1})_\omega}{(p_{n+1}, p_{n+1})_\omega}, \quad b_n = \frac{(xp_n, p_n)_\omega}{(p_n, p_n)_\omega}, \quad c_n = \frac{(xp_n, p_{n-1})_\omega}{(p_{n-1}, p_{n-1})_\omega}$$

Definitions and basic properties

The series $(Sf)(x)$ converges *in average* or *in the sense of* $L^2_\omega(\mathbb{R})$, meaning that if $f_n(x) = \sum_{k=0}^n \hat{f}_k p_k(x)$, then the following holds

$$\lim_{n\to\infty} \|f - f_n\|_\omega = 0$$

The polynomial $f_n \in \mathbb{P}_n$ satisfies the following minimization property

$$\|f - f_n\|_\omega = \min_{q\in\mathbb{P}_n} \|f - q\|_\omega$$

So it is important to compute the coefficients $\hat{f}_k$. This can be done approximating the integrals using Gaussian quadratures, so the discrete truncation of f is the best approximation of f *in the least-squares sense*.

Additionally all families of orthogonal polynomials (OPs) $(p_n)_n$ satisfy a three term recurrence relation (Jacobi operator)

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad p_{-1} = 0, \quad p_0 \in \mathbb{R}$$

where

$$a_n = \frac{(xp_n, p_{n+1})_\omega}{(p_{n+1}, p_{n+1})_\omega}, \quad b_n = \frac{(xp_n, p_n)_\omega}{(p_n, p_n)_\omega}, \quad c_n = \frac{(xp_n, p_{n-1})_\omega}{(p_{n-1}, p_{n-1})_\omega}$$

Definitions and basic properties

The series $(Sf)(x)$ converges *in average* or *in the sense of* $L^2_\omega(\mathbb{R})$, meaning that if $f_n(x) = \sum_{k=0}^{n} \hat{f}_k p_k(x)$, then the following holds

$$\lim_{n\to\infty} \|f - f_n\|_\omega = 0$$

The polynomial $f_n \in \mathbb{P}_n$ satisfies the following minimization property

$$\|f - f_n\|_\omega = \min_{q\in\mathbb{P}_n} \|f - q\|_\omega$$

So it is important to compute the coefficients $\hat{f}_k$. This can be done approximating the integrals using Gaussian quadratures, so the discrete truncation of f is the best approximation of f *in the least-squares sense*.

Additionally all families of orthogonal polynomials (OPs) $(p_n)_n$ satisfy a three term recurrence relation (Jacobi operator)

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad p_{-1} = 0, \quad p_0 \in \mathbb{R}$$

where

$$a_n = \frac{(xp_n, p_{n+1})_\omega}{(p_{n+1}, p_{n+1})_\omega}, \quad b_n = \frac{(xp_n, p_n)_\omega}{(p_n, p_n)_\omega}, \quad c_n = \frac{(xp_n, p_{n-1})_\omega}{(p_{n-1}, p_{n-1})_\omega}$$

Definitions and basic properties

The series $(Sf)(x)$ converges *in average* or *in the sense of* $L^2_\omega(\mathbb{R})$, meaning that if $f_n(x) = \sum_{k=0}^{n} \hat{f}_k p_k(x)$, then the following holds

$$\lim_{n\to\infty} \|f - f_n\|_\omega = 0$$

The polynomial $f_n \in \mathbb{P}_n$ satisfies the following minimization property

$$\|f - f_n\|_\omega = \min_{q\in\mathbb{P}_n} \|f - q\|_\omega$$

So it is important to compute the coefficients $\hat{f}_k$. This can be done approximating the integrals using Gaussian quadratures, so the discrete truncation of f is the best approximation of f *in the least-squares sense*.

Additionally all families of orthogonal polynomials (OPs) $(p_n)_n$ satisfy a three term recurrence relation (Jacobi operator)

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad p_{-1} = 0, \quad p_0 \in \mathbb{R}$$

where

$$a_n = \frac{(xp_n, p_{n+1})_\omega}{(p_{n+1}, p_{n+1})_\omega}, \quad b_n = \frac{(xp_n, p_n)_\omega}{(p_n, p_n)_\omega}, \quad c_n = \frac{(xp_n, p_{n-1})_\omega}{(p_{n-1}, p_{n-1})_\omega}$$

Differential equations

Among all families of OPs there are some special instances that have special importance. These families usually satisfy an additional property.

Bochner (1929) (Routh (1884)): characterize $(p_n)_n$ satisfying

$$dp_n \equiv f_2(x)p_n''(x) + f_1(x)p_n'(x) = \lambda_n p_n(x)$$

where $\deg f_2 \leq 2$ and $\deg f_1 = 1$.

This is equivalent to the symmetry (or self-adjointness) of the second-order differential operator

$$d = f_2(x)\partial_x^2 + f_1(x)\partial_x^1$$

with respect to $(\cdot,\cdot)_\omega$,i.e.

$$(dp_n, p_m)_\omega = (p_n, dp_m)_\omega, \quad n, m \geq 0$$

The relation between the coefficients of the differential operator d and the weight function ω is what is called the Pearson equation

$$(f_2(x)\omega(x))' = f_1(x)\omega(x)$$

plus boundary conditions.

Differential equations

Among all families of OPs there are some special instances that have special importance. These families usually satisfy an additional property.

Bochner (1929) (Routh (1884)): characterize $(p_n)_n$ satisfying

$$dp_n \equiv f_2(x)p_n''(x) + f_1(x)p_n'(x) = \lambda_n p_n(x)$$

where $\deg f_2 \leq 2$ and $\deg f_1 = 1$.

This is equivalent to the symmetry (or self-adjointness) of the second-order differential operator

$$d = f_2(x)\partial_x^2 + f_1(x)\partial_x^1$$

with respect to $(\cdot,\cdot)_\omega$,i.e.

$$(dp_n, p_m)_\omega = (p_n, dp_m)_\omega, \quad n, m \geq 0$$

The relation between the coefficients of the differential operator d and the weight function ω is what is called the Pearson equation

$$(f_2(x)\omega(x))' = f_1(x)\omega(x)$$

plus boundary conditions.

Differential equations

Among all families of OPs there are some special instances that have special importance. These families usually satisfy an additional property.

Bochner (1929) (Routh (1884)): characterize $(p_n)_n$ satisfying

$$dp_n \equiv f_2(x)p_n''(x) + f_1(x)p_n'(x) = \lambda_n p_n(x)$$

where $\deg f_2 \leq 2$ and $\deg f_1 = 1$.

This is equivalent to the symmetry (or self-adjointness) of the second-order differential operator

$$d = f_2(x)\partial_x^2 + f_1(x)\partial_x^1$$

with respect to $(\cdot,\cdot)_\omega$,i.e.

$$(dp_n, p_m)_\omega = (p_n, dp_m)_\omega, \quad n, m \geq 0$$

The relation between the coefficients of the differential operator d and the weight function ω is what is called the Pearson equation

$$(f_2(x)\omega(x))' = f_1(x)\omega(x)$$

plus boundary conditions.

Classical families

Hermite: $f_2(x) = 1, d\omega(x) = e^{-x^2} dx,\ x \in \mathbb{R}$:

$$H_n(x)'' - 2xH_n(x)' = -2nH_n(x)$$

Laguerre: $f_2(x) = x, d\omega(x) = x^\alpha e^{-x} dx,\ \alpha > -1,\ x \in \mathbb{R}_+$:

$$xL_n^\alpha(x)'' + (\alpha + 1 - x)L_n^\alpha(x)' = -nL_n^\alpha(x)$$

Jacobi: $f_2(x) = 1 - x^2, d\omega(x) = (1-x)^\alpha(1-x)^\beta dx,\ x \in (-1,1)$:

$$(1-x^2)P_n^{(\alpha,\beta)}(x)'' + (\beta - \alpha - (\alpha+\beta+2)x)P_n^{(\alpha,\beta)}(x)' =$$

$$-n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$$

Among the Jacobi polynomials we have the well known

- Chebychev polynomials: when $\alpha = \beta = -1/2$.
- Legendre or spherical polynomials: when $\alpha = \beta = 0$.
- Gegenbauer or ultraspherical polynomials: when $\alpha = \beta = \lambda - 1/2$.

Classical families

Hermite: $f_2(x) = 1, d\omega(x) = e^{-x^2}dx,\ x \in \mathbb{R}$:

$$H_n(x)'' - 2xH_n(x)' = -2nH_n(x)$$

Laguerre: $f_2(x) = x, d\omega(x) = x^{\alpha}e^{-x}dx,\ \alpha > -1,\ x \in \mathbb{R}_+$:

$$xL_n^{\alpha}(x)'' + (\alpha + 1 - x)L_n^{\alpha}(x)' = -nL_n^{\alpha}(x)$$

Jacobi: $f_2(x) = 1 - x^2, d\omega(x) = (1-x)^{\alpha}(1-x)^{\beta}dx,\ x \in (-1,1)$:

$$(1-x^2)P_n^{(\alpha,\beta)}(x)'' + (\beta - \alpha - (\alpha+\beta+2)x)P_n^{(\alpha,\beta)}(x)' =$$

$$-n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x), \quad \alpha,\beta > -1$$

Among the Jacobi polynomials we have the well known

- Chebychev polynomials: when $\alpha = \beta = -1/2$.
- Legendre or spherical polynomials: when $\alpha = \beta = 0$.
- Gegenbauer or ultraspherical polynomials: when $\alpha = \beta = \lambda - 1/2$.

Classical families

Hermite: $f_2(x) = 1, d\omega(x) = e^{-x^2}dx$, $x \in \mathbb{R}$:

$$H_n(x)'' - 2xH_n(x)' = -2nH_n(x)$$

Laguerre: $f_2(x) = x, d\omega(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, $x \in \mathbb{R}_+$:

$$xL_n^\alpha(x)'' + (\alpha + 1 - x)L_n^\alpha(x)' = -nL_n^\alpha(x)$$

Jacobi: $f_2(x) = 1 - x^2, d\omega(x) = (1-x)^\alpha(1-x)^\beta dx$, $x \in (-1, 1)$:

$$(1-x^2)P_n^{(\alpha,\beta)}(x)'' + (\beta - \alpha - (\alpha + \beta + 2)x)P_n^{(\alpha,\beta)}(x)' =$$

$$-n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$$

Among the Jacobi polynomials we have the well known

- Chebychev polynomials: when $\alpha = \beta = -1/2$.
- Legendre or spherical polynomials: when $\alpha = \beta = 0$.
- Gegenbauer or ultraspherical polynomials: when $\alpha = \beta = \lambda - 1/2$.

Classical families

Hermite: $f_2(x) = 1, d\omega(x) = e^{-x^2}dx,\ x \in \mathbb{R}$:

$$H_n(x)'' - 2xH_n(x)' = -2nH_n(x)$$

Laguerre: $f_2(x) = x, d\omega(x) = x^\alpha e^{-x}dx,\ \alpha > -1,\ x \in \mathbb{R}_+$:

$$xL_n^\alpha(x)'' + (\alpha + 1 - x)L_n^\alpha(x)' = -nL_n^\alpha(x)$$

Jacobi: $f_2(x) = 1 - x^2, d\omega(x) = (1-x)^\alpha(1-x)^\beta dx,\ x \in (-1,1)$:

$$(1 - x^2)P_n^{(\alpha,\beta)}(x)'' + (\beta - \alpha - (\alpha + \beta + 2)x)P_n^{(\alpha,\beta)}(x)' =$$

$$-n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(x), \quad \alpha, \beta > -1$$

Among the Jacobi polynomials we have the well known

- Chebychev polynomials: when $\alpha = \beta = -1/2$.
- Legendre or spherical polynomials: when $\alpha = \beta = 0$.
- Gegenbauer or ultraspherical polynomials: when $\alpha = \beta = \lambda - 1/2$.

Applications

- Approximation theory: Interpolation, Gaussian integrations, quadrature formulae, least-square approximation, numerical differentiation, etc.
- Operator theory: Jacobi, Hankel or Toeplitz operators.
- Continued fractions.
- Group representation theory: spherical functions.
- Harmonic analysis: Hermite functions are eigenfunctions of the Fourier transform.
- Stochastic processes: Markov chains and diffusion processes.
- Electrostatic equilibrium with logarithmic potential: using zeros of OPs.
- Quantum mechanics: quantum harmonic oscillator, hydrogen atom, etc.
- . . .

Outline

Definitions and basic properties

Matrix-valued polynomials on the real line:

$$P(x) = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_0, \quad C_i \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}$$

Krein (1949): orthogonal matrix polynomials (OMP).
Orthogonality: weight matrix W (positive definite with finite moments).
Let $(P_n)_n$ a system of matrix polynomials with deg P_n and nonsingular leading coefficient such that they are orthogonal in the following sense

$$\langle P_n, P_m \rangle_W = \int_{\mathbb{R}} P_n(x) dW(x) P_m^*(x) = 0_N, \quad n \neq m$$

This is a *sesquilinear form* and it is not a scalar product in the common sense, but it has properties similar to the usual Hilbert spaces.
Now we have the weighted matrix function space

$$L^2_W(\mathbb{R}; \mathbb{C}^{N\times N}) = \left\{ F : \mathbb{R} \to \mathbb{C}^{N\times N} : \int_{\mathbb{R}} F(x) dW(x) F^*(x) < \infty \right\}$$

which is a Hilbert space with the norm $\|F\|^2_W = \mathrm{Tr}\langle F, F\rangle_W$.

Definitions and basic properties

Matrix-valued polynomials on the real line:

$$\boldsymbol{P}(x) = \boldsymbol{C}_n x^n + \boldsymbol{C}_{n-1} x^{n-1} + \cdots + \boldsymbol{C}_0, \quad \boldsymbol{C}_i \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}$$

Krein (1949): orthogonal matrix polynomials (OMP).

Orthogonality: weight matrix $\boldsymbol{W}$ (positive definite with finite moments).

Let $(\boldsymbol{P}_n)_n$ a system of matrix polynomials with deg $\boldsymbol{P}_n$ and nonsingular leading coefficient such that they are orthogonal in the following sense

$$\langle \boldsymbol{P}_n, \boldsymbol{P}_m \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{P}_n(x) d\boldsymbol{W}(x) \boldsymbol{P}_m^*(x) = \boldsymbol{0}_N, \quad n \neq m$$

This is a *sesquilinear form* and it is not a scalar product in the common sense, but it has properties similar to the usual Hilbert spaces.

Now we have the weighted matrix function space

$$L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N\times N}) = \left\{ \boldsymbol{F} : \mathbb{R} \to \mathbb{C}^{N\times N} : \int_{\mathbb{R}} \boldsymbol{F}(x) d\boldsymbol{W}(x) \boldsymbol{F}^*(x) < \infty \right\}$$

which is a Hilbert space with the norm $\|\boldsymbol{F}\|^2_{\boldsymbol{W}} = \operatorname{Tr}\langle \boldsymbol{F}, \boldsymbol{F} \rangle_{\boldsymbol{W}}$.

Definitions and basic properties

Matrix-valued polynomials on the real line:

$$\boldsymbol{P}(x) = \boldsymbol{C}_n x^n + \boldsymbol{C}_{n-1} x^{n-1} + \cdots + \boldsymbol{C}_0, \quad \boldsymbol{C}_i \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}$$

Krein (1949): orthogonal matrix polynomials (OMP).
Orthogonality: weight matrix $\boldsymbol{W}$ (positive definite with finite moments).
Let $(\boldsymbol{P}_n)_n$ a system of matrix polynomials with deg $\boldsymbol{P}_n$ and nonsingular leading coefficient such that they are orthogonal in the following sense

$$\langle \boldsymbol{P}_n, \boldsymbol{P}_m \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{P}_n(x) d\boldsymbol{W}(x) \boldsymbol{P}_m^*(x) = \boldsymbol{0}_N, \quad n \neq m$$

This is a *sesquilinear form* and it is not a scalar product in the common sense, but it has properties similar to the usual Hilbert spaces.
Now we have the weighted matrix function space

$$L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N\times N}) = \left\{ \boldsymbol{F} : \mathbb{R} \to \mathbb{C}^{N\times N} : \int_{\mathbb{R}} \boldsymbol{F}(x) d\boldsymbol{W}(x) \boldsymbol{F}^*(x) < \infty \right\}$$

which is a Hilbert space with the norm $\|\boldsymbol{F}\|^2_{\boldsymbol{W}} = \mathrm{Tr}\langle \boldsymbol{F}, \boldsymbol{F} \rangle_{\boldsymbol{W}}$.

Definitions and basic properties

Matrix-valued polynomials on the real line:

$$\boldsymbol{P}(x) = \boldsymbol{C}_n x^n + \boldsymbol{C}_{n-1} x^{n-1} + \cdots + \boldsymbol{C}_0, \quad \boldsymbol{C}_i \in \mathbb{C}^{N\times N}, \quad x \in \mathbb{R}$$

Krein (1949): orthogonal matrix polynomials (OMP).
Orthogonality: weight matrix $\boldsymbol{W}$ (positive definite with finite moments).
Let $(\boldsymbol{P}_n)_n$ a system of matrix polynomials with deg $\boldsymbol{P}_n$ and nonsingular leading coefficient such that they are orthogonal in the following sense

$$\langle \boldsymbol{P}_n, \boldsymbol{P}_m \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{P}_n(x) d\boldsymbol{W}(x) \boldsymbol{P}_m^*(x) = \boldsymbol{0}_N, \quad n \neq m$$

This is a *sesquilinear form* and it is not a scalar product in the common sense, but it has properties similar to the usual Hilbert spaces.
Now we have the weighted matrix function space

$$L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N\times N}) = \left\{ \boldsymbol{F} : \mathbb{R} \to \mathbb{C}^{N\times N} : \int_{\mathbb{R}} \boldsymbol{F}(x) d\boldsymbol{W}(x) \boldsymbol{F}^*(x) < \infty \right\}$$

which is a Hilbert space with the norm $\|\boldsymbol{F}\|^2_{\boldsymbol{W}} = \mathrm{Tr}\langle \boldsymbol{F}, \boldsymbol{F} \rangle_{\boldsymbol{W}}$.

Definitions and basic properties

Matrix-valued polynomials on the real line:

$$\boldsymbol{P}(x) = \boldsymbol{C}_n x^n + \boldsymbol{C}_{n-1} x^{n-1} + \cdots + \boldsymbol{C}_0, \quad \boldsymbol{C}_i \in \mathbb{C}^{N \times N}, \quad x \in \mathbb{R}$$

Krein (1949): orthogonal matrix polynomials (OMP).
Orthogonality: weight matrix $\boldsymbol{W}$ (positive definite with finite moments).
Let $(\boldsymbol{P}_n)_n$ a system of matrix polynomials with deg $\boldsymbol{P}_n$ and nonsingular leading coefficient such that they are orthogonal in the following sense

$$\langle \boldsymbol{P}_n, \boldsymbol{P}_m \rangle_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{P}_n(x) d\boldsymbol{W}(x) \boldsymbol{P}_m^*(x) = \boldsymbol{0}_N, \quad n \neq m$$

This is a *sesquilinear form* and it is not a scalar product in the common sense, but it has properties similar to the usual Hilbert spaces.
Now we have the weighted matrix function space

$$L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N \times N}) = \left\{ \boldsymbol{F} : \mathbb{R} \to \mathbb{C}^{N \times N} : \int_{\mathbb{R}} \boldsymbol{F}(x) d\boldsymbol{W}(x) \boldsymbol{F}^*(x) < \infty \right\}$$

which is a Hilbert space with the norm $\|\boldsymbol{F}\|^2_{\boldsymbol{W}} = \mathrm{Tr}\langle \boldsymbol{F}, \boldsymbol{F} \rangle_{\boldsymbol{W}}$.

Definitions and basic properties

Similarly, for any $\boldsymbol{F} \in L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N\times N})$, the matrix-valued series

$$(S\boldsymbol{F})(x) = \sum_{k=0}^{\infty} \hat{\boldsymbol{F}}_k \boldsymbol{P}_k(x), \quad \hat{\boldsymbol{F}}_k = \langle \boldsymbol{F}, \boldsymbol{P}_k \rangle_{\boldsymbol{W}} \langle \boldsymbol{P}_k, \boldsymbol{P}_k \rangle_{\boldsymbol{W}}^{-1}$$

is called the *generalized matrix-valued Fourier series of* $\boldsymbol{F}$.

The orthogonality of $(\boldsymbol{P}_n)_n$ with respect to a weight matrix $\boldsymbol{W}$ is equivalent to a three term recurrence relation

$$x\boldsymbol{P}_n(x) = \boldsymbol{A}_n \boldsymbol{P}_{n+1}(x) + \boldsymbol{B}_n \boldsymbol{P}_n(x) + \boldsymbol{C}_n \boldsymbol{P}_{n-1}(x), \quad \boldsymbol{P}_{-1} = \boldsymbol{0}$$

where $\boldsymbol{A}_n$ and $\boldsymbol{C}_n$ are nonsingular matrices and $\boldsymbol{B}_n = \boldsymbol{B}_n^*$.
That means that the OMP are eigenfunctions of a block tridiagonal Jacobi operator:

$$x \begin{pmatrix} \boldsymbol{P}_0(x) \\ \boldsymbol{P}_1(x) \\ \boldsymbol{P}_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0 & & & \\ \boldsymbol{C}_1 & \boldsymbol{B}_1 & \boldsymbol{A}_1 & & \\ & \boldsymbol{C}_2 & \boldsymbol{B}_2 & \boldsymbol{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_0(x) \\ \boldsymbol{P}_1(x) \\ \boldsymbol{P}_2(x) \\ \vdots \end{pmatrix}$$

Definitions and basic properties

Similarly, for any $\boldsymbol{F} \in L^2_{\boldsymbol{W}}(\mathbb{R};\mathbb{C}^{N\times N})$, the matrix-valued series

$$(S\boldsymbol{F})(x) = \sum_{k=0}^{\infty} \hat{\boldsymbol{F}}_k \boldsymbol{P}_k(x), \quad \hat{\boldsymbol{F}}_k = \langle \boldsymbol{F}, \boldsymbol{P}_k \rangle_{\boldsymbol{W}} \langle \boldsymbol{P}_k, \boldsymbol{P}_k \rangle_{\boldsymbol{W}}^{-1}$$

is called the *generalized matrix-valued Fourier series of* $\boldsymbol{F}$.

The orthogonality of $(\boldsymbol{P}_n)_n$ with respect to a weight matrix $\boldsymbol{W}$ is equivalent to a three term recurrence relation

$$x\boldsymbol{P}_n(x) = \boldsymbol{A}_n\boldsymbol{P}_{n+1}(x) + \boldsymbol{B}_n\boldsymbol{P}_n(x) + \boldsymbol{C}_n\boldsymbol{P}_{n-1}(x), \quad \boldsymbol{P}_{-1} = \boldsymbol{0}$$

where $\boldsymbol{A}_n$ and $\boldsymbol{C}_n$ are nonsingular matrices and $\boldsymbol{B}_n = \boldsymbol{B}_n^*$.
That means that the OMP are eigenfunctions of a block tridiagonal Jacobi operator:

$$x\begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 & A_0 & & \\ C_1 & B_1 & A_1 & \\ & C_2 & B_2 & A_2 \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix}$$

Definitions and basic properties

Similarly, for any $\boldsymbol{F} \in L^2_{\boldsymbol{W}}(\mathbb{R};\mathbb{C}^{N\times N})$, the matrix-valued series

$$(S\boldsymbol{F})(x) = \sum_{k=0}^{\infty} \hat{\boldsymbol{F}}_k \boldsymbol{P}_k(x), \quad \hat{\boldsymbol{F}}_k = \langle \boldsymbol{F}, \boldsymbol{P}_k \rangle_{\boldsymbol{W}} \langle \boldsymbol{P}_k, \boldsymbol{P}_k \rangle_{\boldsymbol{W}}^{-1}$$

is called the *generalized matrix-valued Fourier series of* $\boldsymbol{F}$.

The orthogonality of $(\boldsymbol{P}_n)_n$ with respect to a weight matrix $\boldsymbol{W}$ is equivalent to a three term recurrence relation

$$x\boldsymbol{P}_n(x) = \boldsymbol{A}_n \boldsymbol{P}_{n+1}(x) + \boldsymbol{B}_n \boldsymbol{P}_n(x) + \boldsymbol{C}_n \boldsymbol{P}_{n-1}(x), \quad \boldsymbol{P}_{-1} = \boldsymbol{0}$$

where $\boldsymbol{A}_n$ and $\boldsymbol{C}_n$ are nonsingular matrices and $\boldsymbol{B}_n = \boldsymbol{B}_n^*$.
That means that the OMP are eigenfunctions of a block tridiagonal Jacobi operator:

$$x \begin{pmatrix} \boldsymbol{P}_0(x) \\ \boldsymbol{P}_1(x) \\ \boldsymbol{P}_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0 & & & \\ \boldsymbol{C}_1 & \boldsymbol{B}_1 & \boldsymbol{A}_1 & & \\ & \boldsymbol{C}_2 & \boldsymbol{B}_2 & \boldsymbol{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \boldsymbol{P}_0(x) \\ \boldsymbol{P}_1(x) \\ \boldsymbol{P}_2(x) \\ \vdots \end{pmatrix}$$

Differential properties

Durán (1997): characterize families of OMP $(\boldsymbol{P}_n)_n$ satisfying

$$\mathcal{D}\boldsymbol{P}_n(x) \equiv \boldsymbol{F}_2(x)\boldsymbol{P}_n''(x) + \boldsymbol{F}_1(x)\boldsymbol{P}_n'(x) + \boldsymbol{F}_0(x)\boldsymbol{P}_n(x) = \boldsymbol{P}_n(x)\boldsymbol{\Gamma}_n$$

where $\boldsymbol{F}_2(x), \boldsymbol{F}_1(x), \boldsymbol{F}_0(x)$ are matrix polynomials of degree at most 2, 1 and 0, respectively, and $\boldsymbol{\Gamma}_n$ is a Hermitian matrix.

In this section we will be using the inner product

$$(\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{Q}^*(x)\boldsymbol{W}(x)\boldsymbol{P}(x)\,dx, \quad (\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = \langle \boldsymbol{P}^*, \boldsymbol{Q}^* \rangle_{\boldsymbol{W}}^*$$

The existence of such families is equivalent to the symmetry (or self-adjointness) of the matrix second-order differential operator

$$\mathcal{D} = \boldsymbol{F}_2(x)\partial_x^2 + \boldsymbol{F}_1(x)\partial_x^1 + \boldsymbol{F}_0(x)\partial_x^0$$
$$\text{with} \quad \mathcal{D}\boldsymbol{P}_n = \boldsymbol{P}_n\boldsymbol{\Gamma}_n$$

$\mathcal{D}$ is symmetric w.r.t. $\boldsymbol{W}$ if $(\mathcal{D}\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = (\boldsymbol{P}, \mathcal{D}\boldsymbol{Q})_{\boldsymbol{W}}$

Differential properties

Durán (1997): characterize families of OMP $(\boldsymbol{P}_n)_n$ satisfying

$$\mathcal{D}\boldsymbol{P}_n(x) \equiv \boldsymbol{F}_2(x)\boldsymbol{P}_n''(x) + \boldsymbol{F}_1(x)\boldsymbol{P}_n'(x) + \boldsymbol{F}_0(x)\boldsymbol{P}_n(x) = \boldsymbol{P}_n(x)\boldsymbol{\Gamma}_n$$

where $\boldsymbol{F}_2(x), \boldsymbol{F}_1(x), \boldsymbol{F}_0(x)$ are matrix polynomials of degree at most 2, 1 and 0, respectively, and $\boldsymbol{\Gamma}_n$ is a Hermitian matrix.
In this section we will be using the inner product

$$(\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{Q}^*(x)\boldsymbol{W}(x)\boldsymbol{P}(x)\, dx, \quad (\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = \langle \boldsymbol{P}^*, \boldsymbol{Q}^* \rangle_{\boldsymbol{W}}^*$$

The existence of such families is equivalent to the symmetry (or self-adjointness) of the matrix second-order differential operator

$$\mathcal{D} = \boldsymbol{F}_2(x)\partial_x^2 + \boldsymbol{F}_1(x)\partial_x^1 + \boldsymbol{F}_0(x)\partial_x^0$$
$$\text{with} \quad \mathcal{D}\boldsymbol{P}_n = \boldsymbol{P}_n\boldsymbol{\Gamma}_n$$

$\mathcal{D}$ is symmetric w.r.t. $\boldsymbol{W}$ if $(\mathcal{D}\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = (\boldsymbol{P}, \mathcal{D}\boldsymbol{Q})_{\boldsymbol{W}}$

Differential properties

Durán (1997): characterize families of OMP $(\boldsymbol{P}_n)_n$ satisfying

$$\mathcal{D}\boldsymbol{P}_n(x) \equiv \boldsymbol{F}_2(x)\boldsymbol{P}_n''(x) + \boldsymbol{F}_1(x)\boldsymbol{P}_n'(x) + \boldsymbol{F}_0(x)\boldsymbol{P}_n(x) = \boldsymbol{P}_n(x)\boldsymbol{\Gamma}_n$$

where $\boldsymbol{F}_2(x), \boldsymbol{F}_1(x), \boldsymbol{F}_0(x)$ are matrix polynomials of degree at most 2, 1 and 0, respectively, and $\boldsymbol{\Gamma}_n$ is a Hermitian matrix.
In this section we will be using the inner product

$$(\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = \int_{\mathbb{R}} \boldsymbol{Q}^*(x)\boldsymbol{W}(x)\boldsymbol{P}(x)\, dx, \quad (\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = \langle \boldsymbol{P}^*, \boldsymbol{Q}^* \rangle_{\boldsymbol{W}}^*$$

The existence of such families is equivalent to the symmetry (or self-adjointness) of the matrix second-order differential operator

$$\mathcal{D} = \boldsymbol{F}_2(x)\partial_x^2 + \boldsymbol{F}_1(x)\partial_x^1 + \boldsymbol{F}_0(x)\partial_x^0$$
$$\text{with} \quad \mathcal{D}\boldsymbol{P}_n = \boldsymbol{P}_n\boldsymbol{\Gamma}_n$$

$\mathcal{D}$ is symmetric w.r.t. $\boldsymbol{W}$ if $(\mathcal{D}\boldsymbol{P}, \boldsymbol{Q})_{\boldsymbol{W}} = (\boldsymbol{P}, \mathcal{D}\boldsymbol{Q})_{\boldsymbol{W}}$

How to generate examples

- Group Representation Theory: matrix-valued spherical functions associated with different groups (Grünbaum, Pacharoni, Tirao, Román, Zurrián, Koelink).
- Moment equations: from the symmetry equations we solve the corresponding equations (Durán, Grünbaum, MdI). In Durán-MdI (2008) we used this method to generate examples of OMP w.r.t. a weight matrix plus a Dirac delta at one point.
- Matrix-valued bispectral problem: solving the so-called *ad-conditions*

$$\mathrm{ad}_{\boldsymbol{J}}^{k+1}(\boldsymbol{\Gamma}) = \boldsymbol{0}$$

where k is the order of the differential operator, $\boldsymbol{J}$ is the Jacobi matrix and $\boldsymbol{\Gamma}$ is a diagonal matrix with the eigenvalues of $\mathcal{D}$ (Castro, Grünbaum, Tirao). It was used to generate examples or order $k = 1$ in Castro-Grünbaum (2005, 2008).

How to generate examples

- Group Representation Theory: matrix-valued spherical functions associated with different groups (Grünbaum, Pacharoni, Tirao, Román, Zurrián, Koelink).
- Moment equations: from the symmetry equations we solve the corresponding equations (Durán, Grünbaum, MdI). In Durán-MdI (2008) we used this method to generate examples of OMP w.r.t. a weight matrix plus a Dirac delta at one point.
- Matrix-valued bispectral problem: solving the so-called *ad-conditions*

$$\mathrm{ad}_{\boldsymbol{J}}^{k+1}(\boldsymbol{\Gamma}) = \mathbf{0}$$

where k is the order of the differential operator, $\boldsymbol{J}$ is the Jacobi matrix and $\boldsymbol{\Gamma}$ is a diagonal matrix with the eigenvalues of $\mathcal{D}$ (Castro, Grünbaum, Tirao). It was used to generate examples or order $k = 1$ in Castro-Grünbaum (2005, 2008).

How to generate examples

- Group Representation Theory: matrix-valued spherical functions associated with different groups (Grünbaum, Pacharoni, Tirao, Román, Zurrián, Koelink).
- Moment equations: from the symmetry equations we solve the corresponding equations (Durán, Grünbaum, MdI). In Durán-MdI (2008) we used this method to generate examples of OMP w.r.t. a weight matrix plus a Dirac delta at one point.
- Matrix-valued bispectral problem: solving the so-called *ad-conditions*

$$\mathrm{ad}_{\boldsymbol{J}}^{k+1}(\boldsymbol{\Gamma}) = \mathbf{0}$$

where k is the order of the differential operator, $\boldsymbol{J}$ is the Jacobi matrix and $\boldsymbol{\Gamma}$ is a diagonal matrix with the eigenvalues of $\mathcal{D}$ (Castro, Grünbaum, Tirao). It was used to generate examples or order $k = 1$ in Castro-Grünbaum (2005, 2008).

- Durán-Grünbaum (2004):

Symmetry equations (Pearson equations)

$$F_2^*(x)W(x) = W(x)F_2(x)$$
$$F_1^*(x)W(x) = (W(x)F_2(x))' - W(x)F_1(x)$$
$$F_0^*(x)W(x) = \frac{1}{2}(W(x)F_2(x))'' - (W(x)F_1(x))' + W(x)F_0(x)$$

$$\lim_{x\to t} W(x)F_2(x) = \mathbf{0} = \lim_{x\to t}(W(x)F_1(x) - F_1^*(x)W(x)), \quad t = a, b$$

General method: Suppose $F_2(x) = f_2(x)I$. We factorize

$$W(x) = \omega(x)T(x)T^*(x),$$

where ω is a classical weight (Hermite, Laguerre o Jacobi) and T is a matrix function solution of the first-order differential equation

$$T'(x) = G(x)T(x), \quad T(c) = I, \quad c \in (a, b)$$

- Durán-Grünbaum (2004):

Symmetry equations (Pearson equations)

$$F_2^*(x)W(x) = W(x)F_2(x)$$
$$F_1^*(x)W(x) = (W(x)F_2(x))' - W(x)F_1(x)$$
$$F_0^*(x)W(x) = \frac{1}{2}(W(x)F_2(x))'' - (W(x)F_1(x))' + W(x)F_0(x)$$

$$\lim_{x\to t} W(x)F_2(x) = \mathbf{0} = \lim_{x\to t}(W(x)F_1(x) - F_1^*(x)W(x)), \quad t = a, b$$

General method: Suppose $F_2(x) = f_2(x)I$. We factorize

$$W(x) = \omega(x)T(x)T^*(x),$$

where ω is a classical weight (Hermite, Laguerre o Jacobi) and T is a matrix function solution of the first-order differential equation

$$T'(x) = G(x)T(x), \quad T(c) = I, \quad c \in (a, b)$$

1. The first symmetry equations is trivial.
2. Defining

$$F_1(x) = 2f_2(x)G(x) + \frac{(f_2(x)\omega(x))'}{\omega(x)} I$$

the second symmetry equation also holds.

3. Finally the third symmetry equation is equivalent to

$$(W(x)F_1(x) - F_1^*(x)W(x))' = 2(W(x)F_0 - F_0^*W(x))$$

Therefore, it is enough to find F_0 such that

$$\chi(x) = T^{-1}(x)\left(f_2(x)G(x) + f_2(x)G(x)^2 + \frac{(f_2(x)\omega(x))'}{\omega(x)}G(x) - F_0\right)T(x)$$

is Hermitian for all x.

This method has been generalized when F_2 is not necessarily a scalar function (Durán, 2008).

1. The first symmetry equations is trivial.
2. Defining

$$\boldsymbol{F}_1(x) = 2f_2(x)\boldsymbol{G}(x) + \frac{(f_2(x)\omega(x))'}{\omega(x)}\boldsymbol{I}$$

   the second symmetry equation also holds.
3. Finally the third symmetry equation is equivalent to

$$(\boldsymbol{W}(x)\boldsymbol{F}_1(x) - \boldsymbol{F}_1^*(x)\boldsymbol{W}(x))' = 2(\boldsymbol{W}(x)\boldsymbol{F}_0 - \boldsymbol{F}_0^*\boldsymbol{W}(x))$$

   Therefore, it is enough to find $\boldsymbol{F}_0$ such that

$$\chi(x) = \boldsymbol{T}^{-1}(x)\left(f_2(x)\boldsymbol{G}(x)+f_2(x)\boldsymbol{G}(x)^2+\frac{(f_2(x)\omega(x))'}{\omega(x)}\boldsymbol{G}(x)-\boldsymbol{F}_0\right)\boldsymbol{T}(x)$$

   is Hermitian for all x.

This method has been generalized when $\boldsymbol{F}_2$ is not necessarily a scalar function (Durán, 2008).

1. The first symmetry equations is trivial.
2. Defining

$$\boldsymbol{F}_1(x) = 2f_2(x)\boldsymbol{G}(x) + \frac{(f_2(x)\omega(x))'}{\omega(x)}\boldsymbol{I}$$

the second symmetry equation also holds.

3. Finally the third symmetry equation is equivalent to

$$(\boldsymbol{W}(x)\boldsymbol{F}_1(x) - \boldsymbol{F}_1^*(x)\boldsymbol{W}(x))' = 2(\boldsymbol{W}(x)\boldsymbol{F}_0 - \boldsymbol{F}_0^*\boldsymbol{W}(x))$$

Therefore, it is enough to find $\boldsymbol{F}_0$ such that

$$\chi(x) = \boldsymbol{T}^{-1}(x)\left(f_2(x)\boldsymbol{G}(x)+f_2(x)\boldsymbol{G}(x)^2+\frac{(f_2(x)\omega(x))'}{\omega(x)}\boldsymbol{G}(x)-\boldsymbol{F}_0\right)\boldsymbol{T}(x)$$

is Hermitian for all x.

This method has been generalized when $\boldsymbol{F}_2$ is not necessarily a scalar function (Durán, 2008).

1. The first symmetry equations is trivial.
2. Defining

$$\boldsymbol{F}_1(x) = 2f_2(x)\boldsymbol{G}(x) + \frac{(f_2(x)\omega(x))'}{\omega(x)}\boldsymbol{I}$$

   the second symmetry equation also holds.
3. Finally the third symmetry equation is equivalent to

$$(\boldsymbol{W}(x)\boldsymbol{F}_1(x) - \boldsymbol{F}_1^*(x)\boldsymbol{W}(x))' = 2(\boldsymbol{W}(x)\boldsymbol{F}_0 - \boldsymbol{F}_0^*\boldsymbol{W}(x))$$

   Therefore, it is enough to find $\boldsymbol{F}_0$ such that

$$\chi(x) = \boldsymbol{T}^{-1}(x)\left(f_2(x)\boldsymbol{G}(x)+f_2(x)\boldsymbol{G}(x)^2+\frac{(f_2(x)\omega(x))'}{\omega(x)}\boldsymbol{G}(x)-\boldsymbol{F}_0\right)\boldsymbol{T}(x)$$

   is Hermitian for all x.

This method has been generalized when $\boldsymbol{F}_2$ is not necessarily a scalar function (Durán, 2008).

Examples

Let $f_2 = \boldsymbol{I}$ and $\omega = e^{-x^2}$. Then $\boldsymbol{G}(x) = \boldsymbol{A} + 2\boldsymbol{B}x$

$$\begin{cases} \text{Si } \boldsymbol{B} = \boldsymbol{0} \Rightarrow \boldsymbol{W}(x) = e^{-x^2} e^{\boldsymbol{A}x} e^{\boldsymbol{A}^* x} \\ \text{Si } \boldsymbol{A} = \boldsymbol{0} \Rightarrow \boldsymbol{W}(x) = e^{-x^2} e^{\boldsymbol{B}x^2} e^{\boldsymbol{B}^* x^2} \end{cases}$$

In the first case, a possible solution for the matrix function

$$\chi(x) = \boldsymbol{A}^2 - 2\boldsymbol{A}x - e^{-\boldsymbol{A}x} \boldsymbol{F}_0 e^{\boldsymbol{A}x}$$

to be Hermitian is choosing $\boldsymbol{F}_0 = \boldsymbol{A}^2 - 2\boldsymbol{J}$, where

$$\boldsymbol{A} = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

In the second case, a possible solution for the matrix function

$$\chi(x) = 2\boldsymbol{B} + (4\boldsymbol{B}^2 - 4\boldsymbol{B})x^2 - e^{-\boldsymbol{B}x^2} \boldsymbol{F}_0 e^{\boldsymbol{B}x^2}$$

to be Hermitian is choosing $\boldsymbol{F}_0 = 2\boldsymbol{B} - 4\boldsymbol{J}$ with $\boldsymbol{B} = \sum_{j=1}^{N-1} (-1)^{j+1} \boldsymbol{A}^j$

Examples

Let $f_2 = \boldsymbol{I}$ and $\omega = e^{-x^2}$. Then $\boldsymbol{G}(x) = \boldsymbol{A} + 2\boldsymbol{B}x$

$$\begin{cases} \text{Si } \boldsymbol{B} = \boldsymbol{0} \Rightarrow \boldsymbol{W}(x) = e^{-x^2} e^{\boldsymbol{A}x} e^{\boldsymbol{A}^* x} \\ \text{Si } \boldsymbol{A} = \boldsymbol{0} \Rightarrow \boldsymbol{W}(x) = e^{-x^2} e^{\boldsymbol{B}x^2} e^{\boldsymbol{B}^* x^2} \end{cases}$$

In the first case, a possible solution for the matrix function

$$\chi(x) = \boldsymbol{A}^2 - 2\boldsymbol{A}x - e^{-\boldsymbol{A}x} \boldsymbol{F}_0 e^{\boldsymbol{A}x}$$

to be Hermitian is choosing $\boldsymbol{F}_0 = \boldsymbol{A}^2 - 2\boldsymbol{J}$, where

$$\boldsymbol{A} = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

In the second case, a possible solution for the matrix function

$$\chi(x) = 2\boldsymbol{B} + (4\boldsymbol{B}^2 - 4\boldsymbol{B})x^2 - e^{-\boldsymbol{B}x^2} \boldsymbol{F}_0 e^{\boldsymbol{B}x^2}$$

to be Hermitian is choosing $\boldsymbol{F}_0 = 2\boldsymbol{B} - 4\boldsymbol{J}$ with $\boldsymbol{B} = \sum_{i=1}^{N-1} (-1)^{i+1} \boldsymbol{A}^i$

EXAMPLES

Let $f_2 = \boldsymbol{I}$ and $\omega = e^{-x^2}$. Then $\boldsymbol{G}(x) = \boldsymbol{A} + 2\boldsymbol{B}x$

$$\begin{cases} \text{Si } \boldsymbol{B} = \boldsymbol{0} \Rightarrow \boldsymbol{W}(x) = e^{-x^2} e^{\boldsymbol{A}x} e^{\boldsymbol{A}^* x} \\ \text{Si } \boldsymbol{A} = \boldsymbol{0} \Rightarrow \boldsymbol{W}(x) = e^{-x^2} e^{\boldsymbol{B}x^2} e^{\boldsymbol{B}^* x^2} \end{cases}$$

In the first case, a possible solution for the matrix function

$$\chi(x) = \boldsymbol{A}^2 - 2\boldsymbol{A}x - e^{-\boldsymbol{A}x} \boldsymbol{F}_0 e^{\boldsymbol{A}x}$$

to be Hermitian is choosing $\boldsymbol{F}_0 = \boldsymbol{A}^2 - 2\boldsymbol{J}$, where

$$\boldsymbol{A} = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

In the second case, a possible solution for the matrix function

$$\chi(x) = 2\boldsymbol{B} + (4\boldsymbol{B}^2 - 4\boldsymbol{B})x^2 - e^{-\boldsymbol{B}x^2} \boldsymbol{F}_0 e^{\boldsymbol{B}x^2}$$

to be Hermitian is choosing $\boldsymbol{F}_0 = 2\boldsymbol{B} - 4\boldsymbol{J}$ with $\boldsymbol{B} = \sum_{j=1}^{N-1} (-1)^{j+1} \boldsymbol{A}^j$

Examples

The same can be done for $f_2 = x\boldsymbol{I}$ and $\omega = x^{\alpha}e^{-x}$. Then

$$\boldsymbol{G}(x) = \boldsymbol{A} + \frac{\boldsymbol{B}}{x}$$

$$\begin{cases} \text{If } \boldsymbol{B} = \boldsymbol{0} \Rightarrow x^{\alpha}e^{-x}e^{\boldsymbol{A}x}e^{\boldsymbol{A}^{*}x} \\ \text{If } \boldsymbol{A} = \boldsymbol{0} \Rightarrow x^{\alpha}e^{-x}x^{\boldsymbol{B}}x^{\boldsymbol{B}} \end{cases}$$

and for $f_2 = (1-x^2)\boldsymbol{I}$ and $\omega = (1-x)^{\alpha}(1+x)^{\beta}$. Then

$$\boldsymbol{G}(x) = \frac{\boldsymbol{A}}{1-x} + \frac{\boldsymbol{B}}{1+x}$$

$$\begin{cases} \text{If } \boldsymbol{B} = \boldsymbol{0} \Rightarrow (1-x)^{\alpha}(1+x)^{\beta}(1-x)^{\boldsymbol{A}}(1-x)^{\boldsymbol{A}} \\ \text{If } \boldsymbol{A} = \boldsymbol{0} \Rightarrow (1-x)^{\alpha}(1+x)^{\beta}(1+x)^{\boldsymbol{B}}(1+x)^{\boldsymbol{B}} \end{cases}$$

Examples have also been found where the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ do not commute (Durán-MdI and Grünbaum-Pacharoni-Tirao).

Examples

The same can be done for $f_2 = x\boldsymbol{I}$ and $\omega = x^{\alpha}e^{-x}$.Then

$$\boldsymbol{G}(x) = \boldsymbol{A} + \frac{\boldsymbol{B}}{x}$$

$$\begin{cases} \text{If } \boldsymbol{B} = \boldsymbol{0} \Rightarrow x^{\alpha}e^{-x}e^{\boldsymbol{A}x}e^{\boldsymbol{A}^{*}x} \\ \text{If } \boldsymbol{A} = \boldsymbol{0} \Rightarrow x^{\alpha}e^{-x}x^{\boldsymbol{B}}x^{\boldsymbol{B}} \end{cases}$$

and for $f_2 = (1-x^2)\boldsymbol{I}$ and $\omega = (1-x)^{\alpha}(1+x)^{\beta}$. Then

$$\boldsymbol{G}(x) = \frac{\boldsymbol{A}}{1-x} + \frac{\boldsymbol{B}}{1+x}$$

$$\begin{cases} \text{If } \boldsymbol{B} = \boldsymbol{0} \Rightarrow (1-x)^{\alpha}(1+x)^{\beta}(1-x)^{\boldsymbol{A}}(1-x)^{\boldsymbol{A}} \\ \text{If } \boldsymbol{A} = \boldsymbol{0} \Rightarrow (1-x)^{\alpha}(1+x)^{\beta}(1+x)^{\boldsymbol{B}}(1+x)^{\boldsymbol{B}} \end{cases}$$

Examples have also been found where the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ do not commute (Durán-MdI and Grünbaum-Pacharoni-Tirao).

EXAMPLES

The same can be done for $f_2 = x\boldsymbol{I}$ and $\omega = x^\alpha e^{-x}$. Then

$$\boldsymbol{G}(x) = \boldsymbol{A} + \frac{\boldsymbol{B}}{x}$$

$$\begin{cases} \text{If } \boldsymbol{B} = \boldsymbol{0} \Rightarrow x^\alpha e^{-x} e^{\boldsymbol{A}x} e^{\boldsymbol{A}^* x} \\ \text{If } \boldsymbol{A} = \boldsymbol{0} \Rightarrow x^\alpha e^{-x} x^{\boldsymbol{B}} x^{\boldsymbol{B}} \end{cases}$$

and for $f_2 = (1-x^2)\boldsymbol{I}$ and $\omega = (1-x)^\alpha (1+x)^\beta$. Then

$$\boldsymbol{G}(x) = \frac{\boldsymbol{A}}{1-x} + \frac{\boldsymbol{B}}{1+x}$$

$$\begin{cases} \text{If } \boldsymbol{B} = \boldsymbol{0} \Rightarrow (1-x)^\alpha (1+x)^\beta (1-x)^{\boldsymbol{A}} (1-x)^{\boldsymbol{A}} \\ \text{If } \boldsymbol{A} = \boldsymbol{0} \Rightarrow (1-x)^\alpha (1+x)^\beta (1+x)^{\boldsymbol{B}} (1+x)^{\boldsymbol{B}} \end{cases}$$

Examples have also been found where the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ do not commute (Durán-MdI and Grünbaum-Pacharoni-Tirao).

New phenomena

- For a fixed family of OMP, there may exist several linear independent matrix differential operators having the family of OMP as eigenfunctions (Castro, Durán, MdI, Grünbaum, Pacharoni, Tirao, Román).
- For a fixed matrix differential operator, there may exist infinite linear independent families of OMP that are eigenfunctions of the same fixed differential operator (Durán, MdI).
- There exist examples of OMP satisfying odd-order matrix differential equations (Castro, Grünbaum, Durán, MdI).
- There exist families of ladder operators (annihilation and creation or lowering and raising) for some examples of OMP, some of them of 0-th order (Grünbaum, MdI, Martínez-Finkelshtein).

Applications

- Gaussian quadrature formulae: for any $\boldsymbol{F}, \boldsymbol{G} \in L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N\times N})$ it is possible to approximate the integral

$$\int_{\mathbb{R}} \boldsymbol{F}(x) d\boldsymbol{W}(x) \boldsymbol{G}^*(x) \approx \sum_{k=1}^{m} \boldsymbol{F}(x_k) \boldsymbol{\Lambda}_k \boldsymbol{G}^*(x_k)$$

  where x_k are the zeros of the OMP $\boldsymbol{P}_k$ (i.e. $\det \boldsymbol{P}_k(x) = 0$) counting all multiplicities and $\boldsymbol{\Lambda}_k$ are certain matrices (van Assche-Sinap, Durán-Polo).

- Sobolev orthogonal polynomials: It has been found a relation between Sobolev OPs with the inner product

$$\langle p_n, p_m \rangle = \int p_n p_m d\mu + \sum_{i=1}^{M} \sum_{j=0}^{M_i} \lambda_{ij} p_n^{(j)}(c_i) p_m^{(j)}(c_i) = \delta_{nm}, \quad c_i \in \mathbb{R}$$

  and OMP. The reason is that the Sobolev OPs satisfy certain higher-order difference equation, so it is possible to rewrite them in terms of OMP (Durán-van Assche).

Applications

- Gaussian quadrature formulae: for any $\boldsymbol{F}, \boldsymbol{G} \in L^2_{\boldsymbol{W}}(\mathbb{R}; \mathbb{C}^{N\times N})$ it is possible to approximate the integral

$$\int_{\mathbb{R}} \boldsymbol{F}(x) d\boldsymbol{W}(x) \boldsymbol{G}^*(x) \approx \sum_{k=1}^{m} \boldsymbol{F}(x_k) \boldsymbol{\Lambda}_k \boldsymbol{G}^*(x_k)$$

  where x_k are the zeros of the OMP $\boldsymbol{P}_k$ (i.e. $\det \boldsymbol{P}_k(x) = 0$) counting all multiplicities and $\boldsymbol{\Lambda}_k$ are certain matrices (van Assche-Sinap, Durán-Polo).

- Sobolev orthogonal polynomials: It has been found a relation between Sobolev OPs with the inner product

$$\langle p_n, p_m \rangle = \int p_n p_m d\mu + \sum_{i=1}^{M} \sum_{j=0}^{M_i} \lambda_{ij} p_n^{(j)}(c_i) p_m^{(j)}(c_i) = \delta_{nm}, \quad c_i \in \mathbb{R}$$

  and OMP. The reason is that the Sobolev OPs satisfy certain higher-order difference equation, so it is possible to rewrite them in terms of OMP (Durán-van Assche).

Applications

- Bivariate Markov processes: The block tridiagonal structure of the Jacobi matrix for OMP is suitable for some processes called *quasi-birth-and-death processes*, defined on two-dimensional grids (Grünbaum, Dette, Reuther, Zygmunt, Clayton).
  Additionally, the examples of OMP satisfying second-order differential equations are suitable for some processes called *switching diffusion processes*. Here we have a collection of N phases where a different diffusion is performed in each phase and is jumping to other phases according to a continuous time Markov chain (MdI).
- Other applications: scattering theory (Geronimo), Lanczos method for block matrices (Golub, Underwood), time-and-band limiting problems (Durán, Grünbaum, Pacharoni, Zurrián), noncommutative integrable systems and Painlevé equations (Cafasso, MdI), etc.

Applications

- Bivariate Markov processes: The block tridiagonal structure of the Jacobi matrix for OMP is suitable for some processes called *quasi-birth-and-death processes*, defined on two-dimensional grids (Grünbaum, Dette, Reuther, Zygmunt, Clayton).
  Additionally, the examples of OMP satisfying second-order differential equations are suitable for some processes called *switching diffusion processes*. Here we have a collection of N phases where a different diffusion is performed in each phase and is jumping to other phases according to a continuous time Markov chain (MdI).
- Other applications: scattering theory (Geronimo), Lanczos method for block matrices (Golub, Underwood), time-and-band limiting problems (Durán, Grünbaum, Pacharoni, Zurrián), noncommutative integrable systems and Painlevé equations (Cafasso, MdI), etc.