Spectral methods for bivariate Markov processes

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OUTLINE

1. MARKOV PROCESSES
   - Preliminaries
   - Spectral methods

2. BIVARIATE MARKOV PROCESSES
   - Preliminaries
   - Spectral methods

3. AN EXAMPLE
   - A quasi-birth-and-death process
   - A variant of the Wright-Fisher model

4. FUTURE WORK
   - 3 years ahead
   - 5 years ahead
1. **Markov processes**
   - Preliminaries
   - Spectral methods

2. **Bivariate Markov processes**
   - Preliminaries
   - Spectral methods

3. **An example**
   - A quasi-birth-and-death process
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4. **Future work**
   - 3 years ahead
   - 5 years ahead
ORTHOGONAL POLYNOMIALS

Let \( \omega \) be a positive measure on \( S \subset \mathbb{R} \) and consider \( L^2_\omega(S) \).
A system of polynomials \( (q_n)_n \) is orthogonal if

\[
\langle q_n, q_m \rangle_\omega = \int_S q_n(x)q_m(x)d\omega(x) = \|q_n\|^2_\omega \delta_{nm}, \quad n, m \geq 0
\]

This is equivalent to a three-term recurrence relation \( (q_{-1} = 0, q_0 = 1) \)

\[
xq_n(x) = a_nq_{n+1}(x) + b_nq_n(x) + c_nq_{n-1}(x), \quad n \geq 1
\]

where \( a_n, c_n \neq 0, b_n \in \mathbb{R} \) and \( q_0(x) = 1, q_{-1}(x) = 0. \)

Jacobi operator (tridiagonal):

\[
Jq = \begin{pmatrix}
  b_0 & a_0 & & \\
  c_1 & b_1 & a_1 & \\
  c_2 & b_2 & a_2 & \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
  q_0(x) \\
  q_1(x) \\
  q_2(x) \\
  \vdots
\end{pmatrix}
= x
\begin{pmatrix}
  q_0(x) \\
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  q_2(x) \\
  \vdots
\end{pmatrix}
= xq, \quad x \in S
\]

The converse result is also true (Favard’s or spectral theorem).
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$x \in S$

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CLASSICAL FAMILIES

(Bochner, 1929) Characterize families \((q_n)_n\) satisfying

\[
\sigma(x)q''_n(x) + \tau(x)q'_n(x) = \lambda_n q_n(x), \quad \deg(\sigma) \leq 2, \quad \deg(\tau) = 1
\]

The positive measure \(\omega\) (symmetric) satisfies the Pearson equation

\[
(\sigma(x)\omega(x))' = \tau(x)\omega(x)
\]

- **Hermite**: \(\sigma(x) = 1, \tau(x) = -2x, \lambda_n = -2n\)
  \(\omega(x) = e^{-x^2}, x \in \mathbb{R}\), normal or Gaussian distribution.

- **Laguerre**: \(\sigma(x) = x, \tau(x) = -x + \alpha + 1, \lambda_n = -n\)
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Markov processes

A Markov process with state space $S \subset \mathbb{R}$ is a collection of random variables $\{X_t \in S : t \in \mathcal{T}\}$ indexed by time $\mathcal{T}$ (discrete or continuous) such that they have the Markov property: the future event only depends on the present, not on the past (no memory).

- **$S$ discrete (Markov chains)**
  The transition probabilities
  
  \[ P_{ij}(t) \equiv \Pr(X_t = j|X_0 = i), \quad i, j \in \{0, 1, \ldots\} \]

  come in terms of a stochastic matrix
  \[
  P(t) = \begin{pmatrix}
  P_{00}(t) & P_{01}(t) & \cdots \\
  P_{10}(t) & P_{11}(t) & \cdots \\
  \vdots & \vdots & \ddots
  \end{pmatrix}
  \]

- **$S$ continuous (Markov processes)**
  The probabilities are described in terms of a density
  
  \[ p(t; x, y) \equiv \frac{\partial}{\partial y} \Pr(X_t \leq y|X_0 = x), \quad x, y \in (a, b) \subset \mathbb{R} \]
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THREE IMPORTANT CASES

1. **Random walks**: $S = \{0, 1, 2, \ldots \}$, $T = \{0, 1, 2, \ldots \}$.

   
   \[
   P = \begin{pmatrix}
   b_0 & a_0 \\
   c_1 & b_1 & a_1 \\
   \vdots & \vdots & \vdots & \ddots & \ddots & \ddots 
\end{pmatrix}, \quad b_i \geq 0, \ a_i, \ c_i > 0, \quad a_i + b_i + c_i = 1
   \]

   \[P(n) = P^n\] is the \emph{n-step transition probability matrix}.

2. **Birth-and-death processes**: $S = \{0, 1, 2, \ldots \}$, $T = [0, \infty)$.

   \(P(t)\) satisfy the \emph{backward and forward equation}

   \[
P'(t) = AP(t), \quad P'(t) = P(t)A, \quad P(0) = I
   \]

   \[
   A = \begin{pmatrix}
   -\lambda_0 & \lambda_0 \\
   \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\
   \vdots & \vdots & \vdots & \ddots & \ddots & \ddots 
\end{pmatrix}, \quad \lambda_i, \mu_i > 0
   \]

3. **Diffusion processes**: $S = (a, b) \subseteq \mathbb{R}$, $T = [0, \infty)$.

   The density $p(t; x, y)$ satisfy the \emph{backward and forward equation}

   \[
   \frac{\partial}{\partial t} p(t; x, y) = Ap(t; x, y), \quad \frac{\partial}{\partial t} p(t; x, y) = A^* p(t; x, y)
   \]

   \[
   A = \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} + \tau(x) \frac{d}{dx}
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Random walks
Random walks
Markov processes

Bivariate Markov processes

An example

Future work

**RANDOM WALKS**

![Random Walk Diagram](image)
**Random walks**

![Diagram of a random walk](image)

- **Markov processes**
- **Bivariate Markov processes**
- **An example**
- **Future work**
**Random walks**

![Random Walk Diagram]

- **Markov processes**
- **Bivariate Markov processes**
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**Random walks**

![Random walks diagram]

- $S$
- $T$
- $b_0, b_1, b_2, b_3, b_4, b_5$
- $a_0, a_1, a_2, a_3, a_4, a_5$
- $c_1, c_2, c_3, c_4, c_5, c_6$
RANDOM WALKS
**Random walks**

\[
\begin{align*}
    &b_0 & a_0 & b_1 & a_1 & b_2 & a_2 & b_3 & a_3 & b_4 & a_4 & b_5 & a_5 & \ldots \\
    &0 & 1 & 2 & 3 & 4 & 5 & & & & & & & \\
\end{align*}
\]

\[S\]

\[T\]
Random walks
**Random walks**

A random walk is a stochastic process that involves a sequence of random steps. Each step is determined by a probability distribution. In the context of Markov processes, random walks can be used to model various phenomena, such as the movement of particles in a fluid or the evolution of a financial market.

The diagram illustrates a simple random walk on a finite state space. The states are labeled from 0 to 5, and the transitions between states are shown by arrows. Each transition is associated with a probability, represented by the labels on the arrows.

The graph also shows a cumulative sum of the random walk, indicating the progress over time. This can be useful in understanding the long-term behavior of the process.
Random walks

Markov processes
Bivariate Markov processes
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Future work
Random walks
**Random Walks**

![Random Walk Diagram]

- **States**: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

- **Transitions**:
  - $b_0$ to $a_0$, $c_1$
  - $1$ to $a_1$, $b_1$, $c_2$
  - $2$ to $a_2$, $b_2$, $c_3$
  - $3$ to $a_3$, $b_3$, $c_4$
  - $4$ to $a_4$, $b_4$, $c_5$
  - $5$ to $a_5$, $b_5$

- **Graph S**

- **Timeline $\mathcal{T}$**
BIRTH-AND-DEATH PROCESSES
Birth-and-death processes
BIRTH-AND-DEATH PROCESSES

\[
\begin{array}{cccccc}
0 & \lambda_0 & \mu_1 & 1 & \lambda_1 & \mu_2 \\
& & & & & \\
\mu_1 & & \lambda_2 & \mu_2 & \lambda_3 & \mu_3 \\
\mu_2 & & & \mu_2 & \lambda_4 & \mu_4 \\
& & & & \mu_3 & \lambda_5 & \mu_5 \\
& & & & & \mu_4 & \lambda_6 & \mu_6 \\
& & & & & & \cdots
\end{array}
\]
BIRTH-AND-DEATH PROCESSES

\[
\begin{align*}
0 & \xrightarrow{\lambda_0} 1 \\
1 & \xrightarrow{\lambda_1} 2 \\
2 & \xrightarrow{\lambda_2} 3 \\
3 & \xrightarrow{\lambda_3} 4 \\
4 & \xrightarrow{\lambda_4} 5 \\
5 & \xrightarrow{\lambda_5} \ldots
\end{align*}
\]

\[
\begin{align*}
0 & \xrightarrow{\mu_1} 1 \\
1 & \xrightarrow{\mu_2} 2 \\
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\end{align*}
\]
Birth-and-death processes

\[
\begin{align*}
0 & \xrightarrow{\lambda_0} 1 & 1 & \xrightarrow{\lambda_1} 2 \\
& \xleftarrow{\mu_1} 0 & 2 & \xleftarrow{\mu_2} 1 & 3 & \xleftarrow{\mu_3} 2 \\
& 3 & \xleftarrow{\mu_4} 4 & 4 & \xleftarrow{\mu_5} 5 & \ldots \\
& & & 5 & \xleftarrow{\mu_6} & \\
\end{align*}
\]
Birth-and-death processes

\[ \begin{array}{cccc}
0 & \xrightarrow{\lambda_0} & 1 & \xrightarrow{\lambda_1} 2 \\
\mu_1 & \xleftarrow{\mu_2} & \xrightarrow{\mu_3} & \xleftarrow{\mu_4} 4 \\
\lambda_2 & \xrightarrow{\lambda_3} 5 & \xleftarrow{\lambda_4} & \xrightarrow{\lambda_5} \ldots \\
\mu_5 & \xleftarrow{\mu_6} & \xrightarrow{\mu_4} & \xleftarrow{\mu_3} 3 \\
\end{array} \]
Birth-and-death processes

\[ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \ldots \]

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\[ S \]

\[ T \]
BIRTH-AND-DEATH PROCESSES

\[
\begin{array}{cccccc}
0 & \xrightarrow{\lambda_0} & 1 & \xrightarrow{\lambda_1} & 2 & \xrightarrow{\lambda_2} & 3 & \xrightarrow{\lambda_3} & 4 & \xrightarrow{\lambda_4} & 5 & \xrightarrow{\lambda_5} & \ldots \\
\mu_1 & \xleftarrow{} & \mu_2 & \xleftarrow{} & \mu_3 & \xleftarrow{} & \mu_4 & \xleftarrow{} & \mu_5 & \xleftarrow{} & \mu_6 & \\
\end{array}
\]
Birth-and-death processes

\[ \begin{align*}
0 & \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} 3 \xrightarrow{\lambda_3} 5 \xrightarrow{\lambda_4} \cdots \\
\mu_1 & \xleftarrow{} 1 \xleftarrow{} 2 \xleftarrow{} 3 \xleftarrow{} 5 \xleftarrow{} mu_6
\end{align*} \]
**Diffusion processes**

**Ornstein-Uhlenbeck diffusion process:** $S = \mathbb{R}$ and $\sigma^2(x) = 1$, $\tau(x) = -x$

It describes the velocity of a massive Brownian particle under the influence of friction. It is the only nontrivial process which is stationary, Gaussian and Markovian.

![Graphs showing the Ornstein-Uhlenbeck process for different initial conditions](image)

- $X_0 = 0$
- $X_0 = 3$
- $X_0 = -3$
- $X_0 = 10$
Spectral methods

Given a infinitesimal operator $\mathcal{A}$, if we can find a measure $\omega(x)$ associated with $\mathcal{A}$, and a set of orthogonal eigenfunctions $f(i, x)$ such that

$$\mathcal{A}f(i, x) = \lambda(i, x)f(i, x)$$

then it is possible to find spectral representations of

- **Transition probabilities**
  - Discrete case: transition probability matrix $P_{ij}(t)$.
  - Continuous case: transition density $p(t; x, y)$.

- **Invariant measure or distribution**
  - Discrete case: $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ with
    $$\pi_j = \lim_{t \to \infty} P_{ij}(t)$$
  - Continuous case: $\psi(y)$ with
    $$\psi(y) = \lim_{t \to \infty} p(t; x, y)$$
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**Random walks**

\[ S = \mathcal{T} = \{0, 1, 2, \ldots\}. \]

**Spectral theorem** (Karlin-MacGregor, 1959): there exists a measure \( \omega \) associated with \( P \) which orthogonal polynomials \( (q_n)_n \) satisfy \( (q_{-1} = 0, q_0 = 1) \)

\[
Pq = \begin{pmatrix} b_0 & a_0 & \cdots \\ c_1 & b_1 & a_1 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix}, \quad x \in [-1, 1]
\]

**Transition probabilities**

\[
P^n_{ij} = \Pr(X_n = j | X_0 = i) = \frac{1}{\|q_i\|_2^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)
\]

**Invariant measure**

Non-null vector \( \pi = (\pi_0, \pi_1, \ldots) \geq 0 \) such that

\[
\pi P = \pi \quad \Rightarrow \quad \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\|q_i\|_2^2}
\]

Examples: Urn models related with Jacobi polynomials (Legendre, Gegenbauer, Chebyshev).
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    & \ddots & \ddots & \ddots
\end{pmatrix}
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\]

Examples: Urn models related with Jacobi polynomials (Legendre, Gegenbauer, Chebyshev).
**BIRTH-AND-DEATH PROCESSES**

\[ S = \{0, 1, 2, \ldots\}, \; \mathcal{T} = [0, \infty). \]

**Spectral theorem** (Karlin-MacGregor, 1959): there exists a measure \( \omega \) associated with \( \mathcal{A} \) which orthogonal polynomials \( (q_n)_n \) satisfy \( (q_{n-1} = 0, q_0 = 1) \)

\[
\mathcal{A}q = \begin{pmatrix}
-l_0 & \lambda_0 \\
\mu_1 & -\left(\lambda_1 + \mu_1\right) & \lambda_1 \\
& \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots 
\end{pmatrix}
\begin{pmatrix}
q_0(x) \\
q_1(x) \\
\vdots \\
\vdots \\
\vdots 
\end{pmatrix} = -x \begin{pmatrix}
q_0(x) \\
q_1(x) \\
\vdots \\
\vdots \\
\vdots 
\end{pmatrix}
\]

**Transition probabilities**

\[
P_{ij}(t) = \Pr(X_t = j | X_0 = i) = \frac{1}{\|q_i\|^2_\omega} \int_0^\infty e^{-xt} q_i(x) q_j(x) d\omega(x)
\]

**Invariant measure**

Non-null vector \( \pi = (\pi_0, \pi_1, \ldots) \geq 0 \) such that

\[
\pi \mathcal{A} = 0 \quad \Rightarrow \quad \pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} = \frac{1}{\|q_i\|^2_\omega}
\]

Examples: Laguerre, Meixner (linear growth models), Charlier \((M/M/\infty \text{ queue})\), Krawtchouk (Ehrenfest model) polynomials.
Birth-and-death processes

\[ S = \{0, 1, 2, \ldots\}, \ T = [0, \infty). \]

Spectral theorem (Karlin-MacGregor, 1959): there exists a measure \( \omega \) associated with \( A \) which orthogonal polynomials \((q_n)_n\) satisfy \( (q_{-1} = 0, q_0 = 1)\)

\[
Aq = \begin{pmatrix}
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& \ddots & \ddots & \ddots
\end{pmatrix}
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q_0(x) \\
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\vdots
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P_{ij}(t) = \Pr(X_t = j|X_0 = i) = \frac{1}{\|q_i\|^2_\omega} \int_0^\infty e^{-xt} q_i(x)q_j(x) d\omega(x)
\]

Invariant measure

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\[
\pi A = 0 \quad \Rightarrow \quad \pi_i = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} = \frac{1}{\|q_i\|^2_\omega}
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$S = \{0, 1, 2, \ldots \}$, $\mathcal{T} = [0, \infty)$.

**Spectral theorem** (Karlin-MacGregor, 1959): there exists a measure $\omega$ associated with $\mathcal{A}$ which orthogonal polynomials $(q_n)_n$ satisfy ($q_{-1} = 0$, $q_0 = 1$)

$$\mathcal{A}q = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \\ \vdots \end{pmatrix} = -x \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \\ \vdots \end{pmatrix}$$

**Transition probabilities**

$$P_{ij}(t) = \Pr(X_t = j|X_0 = i) = \frac{1}{\|q_i\|_2\omega} \int_0^\infty e^{-xt} q_i(x) q_j(x) d\omega(x)$$

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\end{pmatrix}
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q_0(x) \\
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\vdots \\
\end{pmatrix} = -x \begin{pmatrix}
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Examples: Laguerre, Meixner (linear growth models), Charlier \((M/M/\infty \text{ queue})\), Krawtchouk (Ehrenfest model) polynomials.
**DIFFUSION PROCESSES**

\[ S = (a, b) \subseteq \mathbb{R}, \quad T = [0, \infty). \]

If there exists a positive measure \( \omega \) symmetric with respect to \( \mathcal{A} \) and the corresponding family of **orthonormal functions** \((\phi_n)_n\) satisfy

\[
\mathcal{A}\phi_n(x) = \frac{1}{2}\sigma^2(x)\phi''_n(x) + \tau(x)\phi'_n(x) = \lambda_n\phi_n(x)
\]

**Transition Probability Density**

\[
p(t; x, y) = \sum_{n=0}^{\infty} e^{\lambda_nt}\phi_n(x)\phi_n(y)\omega(y)
\]

**Invariant Measure**

\[
\psi(y) \quad \text{such that} \quad \mathcal{A}^*\psi(y) = 0 \Rightarrow \psi(y) = \frac{1}{\int_S \omega(x)dx}\omega(y)
\]

Examples: Hermite (Orstein-Uhlenbeck process), Laguerre (squared Bessel process), Jacobi polynomials (Wright-Fisher model).
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A\phi_n(x) = \frac{1}{2} \sigma^2(x) \phi''_n(x) + \tau(x) \phi'_n(x) = \lambda_n \phi_n(x)
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\[
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\]

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\]

**Transition probability density**

\[
p(t; x, y) = \sum_{n=0}^{\infty} e^{\lambda_n t} \phi_n(x) \phi_n(y) \omega(y)
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MARKOV PROCESSES
- Preliminaries
- Spectral methods

BIVARIATE MARKOV PROCESSES
- Preliminaries
- Spectral methods

AN EXAMPLE
- A quasi-birth-and-death process
- A variant of the Wright-Fisher model

FUTURE WORK
- 3 years ahead
- 5 years ahead
Matrix valued polynomials on the real line:

\[ A_n x^n + \cdots + A_1 x + A_0, \quad A_i \in \mathbb{C}^{N \times N} \]

Krein (1949): matrix-valued orthogonal polynomials (MOP)
Orthogonality: weight matrix \( W \) supported on \( S \subset \mathbb{R} \) (positive definite with finite moments) and a matrix valued inner product \( (L^2_\omega(S; \mathbb{C}^{N \times N})) \):

\[
\langle P, Q \rangle_W = \int_S P(x)W(x)Q^*(x) \, dx
\]

A system of MOP \( (Q_n)_n \) \( \langle Q_n, Q_m \rangle_W = \|Q_n\|_W^2 \delta_{nm} \) satisfies a three-term recurrence relation \( (Q_{-1} = 0, Q_0 = I) \)

\[
xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x), \quad \text{det}(C_n) \neq 0
\]

Jacobi operator (block tridiagonal)

\[
JQ = \begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
C_2 & B_2 & A_2 \\
\vdots & \ddots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
Q_0(x) \\
Q_1(x) \\
Q_2(x) \\
\vdots \\
\end{pmatrix}
= x
\begin{pmatrix}
Q_0(x) \\
Q_1(x) \\
Q_2(x) \\
\vdots \\
\end{pmatrix}
= xQ, \quad x \in S
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& & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & & & & & & & & & & & & & & & & \ddots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
Q_0(x) \\
Q_1(x) \\
Q_2(x) \\
\vdots
\end{pmatrix} = x \begin{pmatrix}
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\[
x Q_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x), \quad \det(C_n) \neq 0
\]

Jacobi operator (block tridiagonal)

\[
JQ = \begin{pmatrix}
B_0 & A_0 & & \\
C_1 & B_1 & A_1 & \\
& C_2 & B_2 & A_2 \\
& & & \ddots
\end{pmatrix}
\begin{pmatrix}
Q_0(x) \\
Q_1(x) \\
Q_2(x) \\
\vdots
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= xQ, \quad x \in S
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The converse result is also true (Favard’s or spectral theorem**).
Matrix-valued orthogonal polynomials on the real line:

\[ A_n x^n + \cdots + A_1 x + A_0, \quad A_i \in \mathbb{C}^{N \times N} \]

Krein (1949): matrix-valued orthogonal polynomials (MOP)

Orthogonality: weight matrix \( W \) supported on \( S \subset \mathbb{R} \) (positive definite with finite moments) and a matrix valued inner product (\( L_2^W(S; \mathbb{C}^{N \times N}) \)):

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where \(\text{deg}(F_j(x) \leq j)\) and \(\Gamma_n \in \mathbb{R}^{N \times N}\).
Equivalent to the symmetry of \(\mathcal{D}\) with respect to the inner product, i.e.
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**Symmetry equations or matrix Pearson equations**

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Bivariate Markov processes

Now we have a bivariate or 2-component Markov process of the form

\[ \{(X_t, Y_t) : t \in T\} \]

indexed by time \( T \) (time) and with state space \( \mathcal{S} \times \{1, 2, \ldots, N\} \), \( \mathcal{S} \subset \mathbb{R} \). The first component is the level while the second component is the phase. Now the transition probabilities are matrix-valued.

- \( \mathcal{S} \) discrete: block transition probabilities

  \[ (P_{ij})_{i'j'}(t) \equiv \Pr(X_t = j, Y_t = j' | X_0 = i, Y_0 = i') \]

  The block matrix is stochastic.

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  \[ P_{ij}(t; x, y) = \frac{\partial}{\partial y} \Pr(X_t \leq y, Y_t = j | X_0 = x, Y_0 = i) \]

  Every entry must be nonnegative and

  \[ P(t; x, A)e_N \leq e_N, \quad e_N = (1, 1, \ldots, 1)^T \]

Ideas behind: random evolutions

(Griego-Hersh-Papanicolaou-Pinsky-Kurtz...60’s and 70’s).
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QUASI-BIRTH-AND-DEATH PROCESSES

- **Discrete time**: State space \( \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\} \), time \( T = \{0, 1, 2, \ldots\} \) and

\[
(P_{ij})_{i'j'} = \Pr(X_{n+1} = j, Y_{n+1} = j' | X_n = i, Y_n = i') = 0 \quad \text{for} \quad |i - j| > 1.
\]
i.e. a \( N \times N \) block tridiagonal transition probability matrix (stochastic)

\[
P = \begin{pmatrix} B_0 & A_0 \\ C_1 & B_1 & A_1 \\ & \ddots & \ddots & \ddots \end{pmatrix}
\]

- **Continuous time**: State space \( \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\} \), time \( T = [0, +\infty) \). The matrix \( P \) is now given by

\[
(P_{ij})_{i'j'}(t) = \Pr(X_t = j, Y_t = j' | X_0 = i, Y_0 = i')
\]
and will satisfy the so-called **backward and forward equations**

\[
P'(t) = AP(t), \quad P'(t) = P(t)A
\]

In both cases, the **invariant distribution** \( (n, t \to \infty) \) is

\[
\pi = (\pi_0; \pi_1; \cdots) \geq 0, \quad \pi_i \in \mathbb{R}^N
\]
such that \( \pi P = \pi \) (discrete case) or \( \pi A = 0 \) (continuous case).
**Quasi-birth-and-death processes**

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i.e. a $N \times N$ block tridiagonal transition probability matrix (stochastic)

$$P = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_0 \\ \mathbf{C}_1 & \mathbf{B}_1 & \mathbf{A}_1 \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

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**STOCHASTIC REPRESENTATION** \( (N = 4 \text{ PHASES}) \)

Special case of \( A_n, B_n, C_n \) tridiagonal:
STOCHASTIC REPRESENTATION ($N = 4$ PHASES)

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Special case of $A_n, B_n, C_n$ tridiagonal:

![Graph diagram with nodes and arrows representing transitions between phases.]

- Node labels: 1, 5, 9, 13, 17, 21, 2, 6, 10, 14, 18, 22, 3, 7, 11, 15, 19, 23, 4, 8, 12, 16, 20, 24, and a red arrow indicating a special transition.
Stochastic representation \((N = 4\) phases\)

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Switching diffusion processes

The state space is now \((a, b) \times \{1, 2, \ldots, N\}\) and time \(\mathcal{T} = [0, \infty)\).

The density matrix \(P(t; x, y)\) satisfies the \textit{backward and forward equations}

\[
\frac{\partial}{\partial t} P(t; x, y) = \mathcal{A} P(t; x, y), \quad \frac{\partial}{\partial t} P(t; x, y) = P(t; x, y) A^*
\]

where \(\mathcal{A}\) is a matrix-valued differential operator

\[
\mathcal{A} = \frac{1}{2} A(x) \frac{d^2}{dx^2} + B(x) \frac{d^1}{dx^1} + Q(x) \frac{d^0}{dx^0}
\]

We have that \(A(x)\) and \(B(x)\) are diagonal matrices and \(Q(x)\) is the infinitesimal operator of a continuous time Markov chain, i.e.

\[
Q_{ii}(x) \leq 0, \quad Q_{ij}(x) \geq 0, \quad i \neq j, \quad Q(x)e_N = 0
\]

The \textit{row vector-valued invariant distribution} \((t \to \infty)\)

\[
\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y)), \quad 0 \leq \psi_j(y) \leq 1, \quad \left(\int_a^b \psi(y)dy\right) e_N = 1
\]

satisfies

\[
\psi(y) A^* \equiv \frac{1}{2} (\psi(y)A(y))'' - (\psi(y)B(y))' + \psi(y)Q(y) = 0
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A = \frac{1}{2} A(x) \frac{d^2}{dx^2} + B(x) \frac{d^1}{dx^1} + Q(x) \frac{d^0}{dx^0}
\]

We have that \(A(x)\) and \(B(x)\) are **diagonal** matrices and \(Q(x)\) is the **infinitesimal operator** of a continuous time Markov chain, i.e.

\[
Q_{ii}(x) \leq 0, \quad Q_{ij}(x) \geq 0, \quad i \neq j, \quad Q(x) e_N = 0
\]

The **row vector-valued invariant distribution** \((t \to \infty)\)

\[
\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y)), \quad 0 \leq \psi_j(y) \leq 1, \quad \left(\int_a^b \psi(y) dy\right) e_N = 1
\]

satisfies

\[
\psi(y) A^* = \frac{1}{2} (\psi(y) A(y))'' - (\psi(y) B(y))' + \psi(y) Q(y) = 0
\]
Stochastic representation \((N = 3 \text{ phases})\)

\(N = 3\) phases and \(S = \mathbb{R}\) with

\[
A_{ii}(x) = i^2, \quad B_{ii}(x) = -ix, \quad i = 1, 2, 3
\]
Stochastic representation \((N = 3 \text{ phases})\)

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\[
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\]
Stochastic representation ($N = 3$ phases)

$N = 3$ phases and $S = \mathbb{R}$ with

$$A_{ii}(x) = i^2, \quad B_{ii}(x) = -ix, \quad i = 1, 2, 3$$
STOCHASTIC REPRESENTATION \( (N = 3 \text{ phases}) \)

\[ N = 3 \text{ phases and } S = \mathbb{R} \text{ with} \]

\[
\begin{align*}
A_{ii}(x) &= i^2, & B_{ii}(x) &= -ix, & i = 1, 2, 3
\end{align*}
\]
Stochastic representation ($N = 3$ phases)

$N = 3$ phases and $S = \mathbb{R}$ with

$$A_{ii}(x) = i^2, \quad B_{ii}(x) = -ix, \quad i = 1, 2, 3$$
Spectral methods

Now, given a matrix-valued infinitesimal operator $\mathcal{A}$, if we can find a weight matrix $\mathbf{W}(x)$ associated with $\mathcal{A}$, and a set of orthogonal matrix eigenfunctions $\mathbf{F}(i, x)$ such that

$$\mathcal{A}\mathbf{F}(i, x) = \mathbf{F}(i, x)\Lambda(i, x)$$

then it is possible to find spectral representations of

- Transition probabilities
  - Discrete case: transition probability matrix $\mathbf{P}(t)$
  - Continuous case: transition density $\mathbf{P}(t; x, y)$.

- Invariant measure or distribution
  - Discrete case: $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ with
    $$\pi_j = \lim_{t \to \infty} \mathbf{P}_{ij}(t) \in \mathbb{R}^N$$
  - Continuous case: $\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y))$ with
    $$\psi_j(y) = \lim_{t \to \infty} \mathbf{P}_{ij}(t; x, y)$$
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    $$\psi_j(y) = \lim_{t \to \infty} P_j(t; x, y)$$
QUASI-BIRTH-AND-DEATH PROCESSES

**Discrete time:** \(\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}, \ T = \{0, 1, 2, \ldots\}\).

**Spectral theorem** (Grünbaum, Dette et al., 2006): \(\exists^* W\) on \([-1, 1]\) associated with \(P\) which MOP \((Q_n)_n\) satisfy \(PQ = xQ\) \((Q_{-1} = 0, Q_0 = I)\)

\[
P_{ij}^n = \left(\int_{-1}^{1} x^n Q_i(x)W(x)Q_j^*(x)dx\right)\left(\int_{-1}^{1} Q_j(x)W(x)Q_j^*(x)dx\right)^{-1}
\]

**Continuous time:** \(\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}, \ T = [0, \infty)\)

**Spectral theorem** (Detter-Reuther, 2010): \(\exists^* W\) on \([0, \infty)\) associated with \(A\) which MOP \((Q_n)_n\) satisfy \(AQ = -xQ\) \((Q_{-1} = 0, Q_0 = I)\)

\[
P_{ij}(t) = \left(\int_{0}^{\infty} e^{-xt} Q_i(x)W(x)Q_j^*(x)dx\right)\left(\int_{0}^{\infty} Q_j(x)W(x)Q_j^*(x)dx\right)^{-1}
\]

**Invariant measure** (MdI, 2011)

\(\pi = (\pi_0; \pi_1; \cdots) \equiv (\Pi_0 e_N; \Pi_1 e_N; \cdots)\) such that \(\pi P = \pi\) (discrete time) or \(\pi A = 0\) (continuous time)

\[
\Pi_n = (C_n^T \cdots C_1^T)^{-1} \Pi_0 (A_0 \cdots A_{n-1}) = \left(\int_{\text{supp}(w)} Q_n(x)W(x)Q_n^*(x)dx\right)^{-1}
\]
**Quasi-birth-and-death processes**

- **Discrete time**: \(\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}, \mathcal{T} = \{0, 1, 2, \ldots\}\).  
  **Spectral theorem** (Grünbaum, Dette et al., 2006): \(\exists^* \mathbf{W}\) on \([-1, 1]\) associated with \(\mathbf{P}\) which MOP \((\mathbf{Q}_n)_n\) satisfy \(\mathbf{P}\mathbf{Q} = \mathbf{x}\mathbf{Q}\ (\mathbf{Q}_{-1} = 0, \mathbf{Q}_0 = \mathbf{I})\)

\[
P_{ij}^n = \left(\int_{-1}^{1} x^n Q_i(x)W(x)Q_j^*(x)dx\right)\left(\int_{-1}^{1} Q_j(x)W(x)Q_j^*(x)dx\right)^{-1}
\]

- **Continuous time**: \(\{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}, \mathcal{T} = [0, \infty)\)  
  **Spectral theorem** (Detter-Reuther, 2010): \(\exists^* \mathbf{W}\) on \([0, \infty)\) associated with \(\mathcal{A}\) which MOP \((\mathbf{Q}_n)_n\) satisfy \(\mathcal{A}\mathbf{Q} = -\mathbf{x}\mathbf{Q}\ (\mathbf{Q}_{-1} = 0, \mathbf{Q}_0 = \mathbf{I})\)

\[
P_{ij}(t) = \left(\int_{0}^{\infty} e^{-xt} Q_i(x)W(x)Q_j^*(x)dx\right)\left(\int_{0}^{\infty} Q_j(x)W(x)Q_j^*(x)dx\right)^{-1}
\]

**Invariant measure (MdI, 2011)**

\[
\pi = (\pi_0; \pi_1; \cdots) \equiv (\Pi_0 e_N; \Pi_1 e_N; \cdots) \text{ such that } \pi\mathbf{P} = \pi \text{ (discrete time)} \text{ or } \pi\mathcal{A} = 0 \text{ (continuous time)}
\]

\[
\Pi_n = (\mathbf{C}_1^T \cdots \mathbf{C}_n^T)^{-1} \Pi_0 (\mathbf{A}_0 \cdots \mathbf{A}_{n-1}) = \left(\int_{\text{supp}(W)} Q_n(x)W(x)Q_n^*(x)dx\right)^{-1}
\]
QUASI-BIRTH-AND-DEATH PROCESSES

- **Discrete time**: \( \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}, \mathcal{T} = \{0, 1, 2, \ldots\} \).

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  \[
P^n_{ij} = \left( \int_{-1}^{1} x^n Q_i(x) W(x) Q^*_j(x) \, dx \right) \left( \int_{-1}^{1} Q_j(x) W(x) Q^*_j(x) \, dx \right)^{-1}
  \]

- **Continuous time**: \( \{0, 1, 2, \ldots\} \times \{1, 2, \ldots, N\}, \mathcal{T} = [0, \infty) \).

  **Spectral theorem** (Detter-Reuther, 2010): \( \exists^* W \) on \([0, \infty)\) associated with \( A \) which MOP \( (Q_n)_n \) satisfy \( AQ = -xQ \) \( (Q_{-1} = 0, Q_0 = I) \)

  \[
P_{ij}(t) = \left( \int_{0}^{\infty} e^{-xt} Q_i(x) W(x) Q^*_j(x) \, dx \right) \left( \int_{0}^{\infty} Q_j(x) W(x) Q^*_j(x) \, dx \right)^{-1}
  \]

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\( \pi = (\pi_0; \pi_1; \cdots) \equiv (\Pi_0 e_N; \Pi_1 e_N; \cdots) \) such that \( \pi P = \pi \) (discrete time) or \( \pi A = 0 \) (continuous time)

\[
\Pi_n = (C_1^T \cdots C_n^T)^{-1} \Pi_0 (A_0 \cdots A_{n-1}) = \left( \int_{\text{supp}(W)} Q_n(x) W(x) Q^*_n(x) \, dx \right)^{-1}
\]
SWITCHING DIFFUSION MODELS

State space: \((a, b) \times \{1, 2, \ldots, N\}\). Time: \(T = [0, \infty)\)
If there exists a weight matrix \(W\) symmetric with respect to \(A\) which matrix-valued orthonormal functions \((\Phi_n)_n\) satisfies

\[
A\Phi_n(x) = \frac{1}{2}A(x)\Phi''_n(x) + B(x)\Phi'_n(x) + Q(x)\Phi_n(x) = \Phi_n(x)\Gamma_n
\]

**Transition probability density matrix (MdI, 2012)**

\[
P(t; x, y) = \sum_{n=0}^{\infty} \Phi_n(x)e^{\Gamma_nt}\Phi^*_n(y)W(y)
\]

**Invariant distribution (MdI, 2012)**

\[
\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y)) \text{ such that } \psi(y)A^* = 0
\]

\[
\Rightarrow \psi(y) = \left(\int_a^b e_N^T W(x)e_N dx\right)^{-1} e_N^T W(y)
\]
Switching diffusion models

State space: \((a, b) \times \{1, 2, \ldots, N\}\). Time: \(\mathcal{T} = [0, \infty)\)

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\[
\mathcal{A}\Phi_n(x) = \frac{1}{2}A(x)\Phi''_n(x) + B(x)\Phi'_n(x) + Q(x)\Phi_n(x) = \Phi_n(x)\Gamma_n
\]

Transition probability density matrix (MdI, 2012)

\[
P(t; x, y) = \sum_{n=0}^{\infty} \Phi_n(x)e^{\Gamma_n t}\Phi^*_n(y)\mathbf{W}(y)
\]

Invariant distribution (MdI, 2012)

\[
\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y)) \text{ such that } \psi(y)A^* = 0
\]

\[
\Rightarrow \psi(y) = \left(\int_a^b e^T_N\mathbf{W}(x)e_N dx\right)^{-1}e^T_N\mathbf{W}(y)
\]
Switching diffusion models

State space: \((a, b) \times \{1, 2, \ldots, N\}\). Time: \(\mathcal{T} = [0, \infty)\)
If there exists a weight matrix \(\mathbf{W}\) symmetric with respect to \(\mathcal{A}\) which matrix-valued orthonormal functions \((\Phi_n)_n\) satisfies

\[
\mathcal{A}\Phi_n(x) = \frac{1}{2} A(x) \Phi_n''(x) + B(x) \Phi_n'(x) + Q(x) \Phi_n(x) = \Phi_n(x) \Gamma_n
\]

Transition probability density matrix (MdI, 2012)

\[
\mathbf{P}(t; x, y) = \sum_{n=0}^{\infty} \Phi_n(x) e^{\Gamma_n t} \Phi_n^*(y) \mathbf{W}(y)
\]

Invariant distribution (MdI, 2012)

\[
\psi(y) = (\psi_1(y), \psi_2(y), \ldots, \psi_N(y)) \text{ such that } \psi(y) \mathcal{A}^* = 0
\]

\[
\Rightarrow \psi(y) = \left( \int_a^b \mathbf{e}_N^T \mathbf{W}(x) \mathbf{e}_N \, dx \right)^{-1} \mathbf{e}_N^T \mathbf{W}(y)
\]
Outline

1. Markov processes
   - Preliminaries
   - Spectral methods

2. Bivariate Markov processes
   - Preliminaries
   - Spectral methods

3. An example
   - A quasi-birth-and-death process
   - A variant of the Wright-Fisher model

4. Future work
   - 3 years ahead
   - 5 years ahead
An example coming from group representation

Let \( N \in \{1, 2, \ldots\} \), \( \alpha, \beta > -1 \), \( 0 < k < \beta + 1 \) and \( E_{ij} \) will denote the matrix with 1 at entry \((i, j)\) and 0 otherwise.

For \( x \in (0, 1) \), we have a symmetric pair \( \{W, A\} \) (Gr"unbaum-Pacharoni-Tirao, 2002) where

\[
W(x) = x^\alpha (1 - x)\beta \sum_{i=1}^{N} \binom{\beta - k + i - 1}{i - 1} \binom{N + k - i - 1}{N - i} x^{N-i} E_{ii}
\]

\[
A = \frac{1}{2} A(x) \frac{d^2}{dx^2} + B(x) \frac{d}{dx} + Q(x) \frac{d^0}{dx^0}
\]

\[
A(x) = 2x(1-x)I, \quad B(x) = \sum_{i=1}^{N} [\alpha + 1 + N - i - x(\alpha + \beta + 2 + N - i)] E_{ii}
\]

\[
Q(x) = \sum_{i=2}^{N} \mu_i(x) E_{i,i-1} - \sum_{i=1}^{N} (\lambda_i(x) + \mu_i(x)) E_{ii} + \sum_{i=1}^{N-1} \lambda_i(x) E_{i,i+1},
\]

\[
\lambda_i(x) = \frac{1}{1-x} (N - i)(i + \beta - k), \quad \mu_i(x) = \frac{x}{1-x} (i - 1)(N - i + k).
\]
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$$A = \frac{1}{2} A(x) \frac{d^2}{dx^2} + B(x) \frac{d}{dx} + Q(x) \frac{d^0}{dx^0}$$

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$$Q(x) = \sum_{i=2}^{N} \mu_i(x) E_{i,i-1} - \sum_{i=1}^{N} (\lambda_i(x) + \mu_i(x)) E_{ii} + \sum_{i=1}^{N-1} \lambda_i(x) E_{i,i+1},$$

$$\lambda_i(x) = \frac{1}{1-x} (N-i)(i+\beta-k), \quad \mu_i(x) = \frac{x}{1-x} (i-1)(N-i+k).$$
An example coming from group representation

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$$\lambda_i(x) = \frac{1}{1-x} (N - i)(i + \beta - k), \quad \mu_i(x) = \frac{x}{1-x} (i - 1)(N - i + k).$$
A QUASI-BIRTH-AND-DEATH PROCESS

It is possible to get (Grünbaum-Mdl, 2008) an equivalent symmetric pair \( \{ \tilde{W}, \tilde{A} \} \) and a family of MOP \((Q_n)_n\) such that \( Q_{-1} = 0, Q_0 = I \) and

\[
Q_n(1)e_N = e_N, \quad e_N = (1, 1, \ldots, 1)^T
\]

The family \((Q_n)_n\) satisfies a three-term recurrence relation

\[
xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x), \quad n = 0, 1, \ldots,
\]

where \(B_n\) is tridiagonal, \(A_n\) is lower bidiagonal and \(C_n\) is upper bidiagonal. The corresponding Jacobi matrix \(P\) is stochastic

\[
P = \begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
C_2 & B_2 & A_2 \\
& \ddots & \ddots & \ddots
\end{pmatrix}
\]

\(n\)-step transition probability matrix \(P^n\) is terms of the Karlin-McGregor representation and the invariant measure \(\pi\).

We studied recurrence, the shape of the invariant distribution and other probabilistic aspects.
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C_1 & B_1 & A_1 & & \\
C_2 & B_2 & A_2 & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{pmatrix}
\]

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C_2 & B_2 & A_2 \\
& & & \ddots & \ddots & \ddots
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\[
P = \begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
C_2 & B_2 & A_2 \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

- \(n\)-step transition probability matrix \(P^n\) is terms of the Karlin-McGregor representation and the invariant measure \(\pi\).
- We studied recurrence, the shape of the invariant distribution and other probabilistic aspects.
PARTicular case $\alpha = \beta = 0$, $k = 1/2$, $N = 2$

Pentadiagonal transition probability matrix:

$$
P = \begin{pmatrix}
\frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\
\frac{2}{9} & \frac{7}{9} & \frac{4}{3} & \frac{3}{10} \\
\frac{9}{2} & \frac{18}{45} & \frac{45}{100} & \frac{3}{27} & \frac{2}{100} \\
\frac{5}{36} & \frac{18}{225} & \frac{107}{50} & \frac{3}{100} & \frac{27}{100} \\
\frac{1}{36} & \frac{4}{225} & \frac{23}{50} & \frac{6}{175} & \frac{2}{175} \\
\frac{6}{36} & \frac{75}{225} & \frac{50}{175} & \frac{175}{7} & \frac{3}{40} \\
\frac{14}{36} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{147}{147} \\
\frac{75}{36} & \frac{75}{75} & \frac{1225}{147} & \frac{147}{8} & \frac{5}{18} \\
\frac{1}{36} & \frac{6}{225} & \frac{47}{50} & \frac{8}{175} & \frac{5}{175} \\
\frac{5}{36} & \frac{245}{225} & \frac{98}{50} & \frac{441}{175} & \frac{18}{175} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$

Invariant measure such that $\pi P = \pi$

$$
\pi = \begin{pmatrix}
\frac{2}{3} & \frac{2}{3} & \frac{16}{15} & \frac{6}{5} & \frac{54}{35} & \frac{12}{7} & \frac{128}{63} & \frac{20}{9} & \frac{250}{99} & \frac{30}{11} & \frac{432}{143} & \frac{42}{13} & \frac{686}{195} & \frac{56}{15} & \vdots \\
\end{pmatrix}
$$
**PARTICULAR CASE** \( \alpha = \beta = 0, \ k = 1/2, \ N = 2 \)

Pentadiagonal transition probability matrix:

\[
P = \begin{pmatrix}
5 & 2 & 2 \\
9 & 9 & 9 \\
2 & 7 & 4 & 3 \\
9 & 18 & 45 & 10 \\
5 & 1 & 107 & 3 & 27 \\
36 & 18 & 225 & 50 & 100 \\
1 & 4 & 23 & 6 & 2 \\
6 & 75 & 50 & 175 & 7 \\
14 & 2 & 597 & 4 & 40 \\
75 & 75 & 1225 & 147 & 147 \\
1 & 6 & 47 & 8 & 5 \\
5 & 245 & 98 & 441 & 18 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

Invariant measure such that \( \pi P = \pi \)

\[
\pi = \left( \frac{2}{3}, \frac{2}{3}, \frac{16}{15}, \frac{6}{5}, \frac{54}{35}, \frac{12}{7}, \frac{128}{63}, \frac{20}{9}, \frac{250}{99}, \frac{30}{11}, \frac{432}{143}, \frac{42}{13}, \frac{686}{195}, \frac{56}{15}, \ldots \right)
\]
ASSOCIATED NETWORK
A variant of the Wright-Fisher model

The Wright-Fisher diffusion model involving only mutation effects considers a big population of constant size $M$ of two types $A$ and $B$

\[
A \overset{1+\beta}{\underset{2}{\rightarrow}} B, \quad B \overset{1+\alpha}{\underset{2}{\rightarrow}} A, \quad \alpha, \beta > -1
\]

As $M \to \infty$, this model can be described by a diffusion process whose state space is $S = [0, 1]$ with drift and diffusion coefficient

\[
\tau(x) = \alpha + 1 - x(\alpha + \beta + 2), \quad \sigma^2(x) = 2x(1 - x), \quad \alpha, \beta > -1
\]

The $N$ phases of our bivariate Markov process are variations of the Wright-Fisher model in the drift coefficients:

\[
B_{ii}(x) = \alpha + 1 + N - i - x(\alpha + \beta + 2 + N - i), \quad A_{ii}(x) = 2x(1 - x)
\]

Now there is an extra parameter $k \in (0, \beta + 1)$ in $Q(x)$, which measures how the process moves through all the phases.
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SOME SAMPLE PATHS

\( \alpha = 1, \beta = 2, k = 0.2, 7 \text{ changes} \)

\( \alpha = 0.5, \beta = 2, k = 2.8, 10 \text{ changes} \)
SPECTRAL ANALYSIS

There exists a family of orthonormal MOP \((\Phi_n)_n\) (matrix-valued spherical functions) such that

\[
\frac{1}{2}A(x)\Phi''_n(x) + B(x)\Phi'_n(x) + Q(x)\Phi_n(x) = \Phi_n(x)\Gamma_n
\]

**Transition probability matrix (MdI, 2012)**

\[
P(t; x, y) = \sum_{n=0}^{\infty} \Phi_n(x)e^{\Gamma_n t}\Phi_n^*(y)W(y)
\]

**Invariant distribution (MdI, 2012)**

\[
\Rightarrow \psi(y) = \left(\int_0^1 e_N^T W(x)e_N dx\right)^{-1} e_N^T W(y), \quad e^T = (1, 1, \ldots, 1)
\]

\[
\psi_j(y) = y^{\alpha+N-j}(1 - y)^{\beta \binom{N-1}{j-1}} \left(\frac{(\beta+N)(k)_{N-j}^j(\beta-k+1)_{j-1}}{((\alpha+\beta-k+2))_{N-1}}\right)
\]

In (MdI, 2012) I have studied the probabilistic aspects of this process in terms of the parameters, like waiting times, tendency, the invariant distribution or the probabilistic meaning of the new parameter.
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1 Markov processes
   • Preliminaries
   • Spectral methods

2 Bivariate Markov processes
   • Preliminaries
   • Spectral methods

3 An example
   • A quasi-birth-and-death process
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4 Future work
   • 3 years ahead
   • 5 years ahead
Accommodate examples of MOP to bivariate Markov processes

- **Discrete case:** Find appropriate families of MOP for which the corresponding Jacobi matrix is either stochastic or the infinitesimal operator of a continuous-time quasi-birth-and-death process.
- **Continuous case:** Given a symmetric pair \( \{ \tilde{W}, \tilde{A} \} \), find an appropriate transformation (depending on \( x \)) \( \{ \tilde{W}, \tilde{A} \} \rightarrow \{ W, A \} \) such that the new \( A \) is the infinitesimal operator of a switching diffusion process.

Apply spectral methods to examples of real world bivariate Markov processes in the literature.

**Main problem:** Find eigenfunctions and the corresponding weight matrix \( W \) for a given infinitesimal operator \( A \).

**Possible approach:** Find weight matrices such that they are variations of continuous and scalar weights, or matrix-valued eigenfunctions combinations of scalar special functions.
3 YEARS AHEAD I

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3 YEARS AHEAD II

**Noncommutative Integrable Systems**

Some of the families of MOP are related to noncommutative Painlevé equations via Riemann-Hilbert problems (Cafasso-Mdi, 2013).

**Main tool:** Integral representations of some families of MOP. We plan to continue this work by considering connections of other families with some nonlinear differential equations, or with random matrices problems.

**Other Open Problems**

- Finding new examples and phenomena of some of these MOP.
- Electrostatic interpretation of the zeros of MOP (these are given by $\det(Q_n(x)) = 0$)
- Asymptotic analysis of some of these families.
- Principal dynamical components (MdI-Tabak, 2013).
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Signal processing

Related to commuting integral and differential operators (for example prolate spheroidal wave functions).

From the discrete setting (Grünbaum, 1983), given a full matrix $M$ (integral operator) one tries to find a tridiagonal matrix $T$ (differential operator) with simple spectrum such that $MT = TM$. Something similar can be done in the matrix case (Durán-Grünbaum, 2005), but now with a $2 \times 2$ block matrix $M$. The reproducing kernel of $M$ is given by

$$M_{ij} = \int_0^\Omega Q_i(x) \tilde{W}(x) Q_j^*(x) dx, \quad i, j = 0, 1, \ldots, T$$

Here the band limiting is the restriction to the interval $(0, \Omega)$ and the time limiting the restriction to the range $0, 1, \ldots, T$. They found a block tridiagonal $T$ such that $MT = TM$.

Main goal: Explore this example in depth or find other examples.
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5 YEARS AHEAD II

2 Noncommutative harmonic oscillators

It is very well known that the quantum harmonic oscillator can be studied using Hermite or wave functions \( \psi_n(x) \). These functions are eigenfunctions of the Schrödinger equation

\[
\psi''(x) - x^2 \psi_n(x) = -(2n + 1) \psi_n(x)
\]

In the matrix case Parmeggiani-Wakayama, 2002 consider 2 \( \times \) 2 differential operators of the form

\[
A = \begin{pmatrix} A \left( -\frac{d^2}{dx^2} + \frac{x^2}{2} \right) + B \left( x \frac{d}{dx} + \frac{1}{2} \right) \end{pmatrix}, \quad x \in \mathbb{R}
\]

where \( A, B \in \mathbb{R}^{2 \times 2} \), \( A > 0 \), \( B = -B^T \) and \( A + iB > 0 \).

Main goal: relate these differential operators with MOP or switching diffusion models on \( \mathbb{R} \), or extend to \( N \times N \) examples.
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