Spectral methods for bivariate Markov processes

Manuel Domínguez de la Iglesia

Departamento de Análisis Matemático, Universidad de Sevilla

Instituto de Matemáticas, UNAM
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Outline

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Orthogonal polynomials

Let ω be a positive measure on $\mathcal{S} \subset \mathbb{R}$ and consider $L^2_\omega(\mathcal{S})$.
A system of polynomials $(q_n)_n$ is orthogonal if

$$\langle q_n, q_m \rangle_\omega = \int_{\mathcal{S}} q_n(x) q_m(x) d\omega(x) = \|q_n\|^2_\omega \delta_{nm}, \quad n, m \geq 0$$

This is equivalent to a three-term recurrence relation $(q_{-1} = 0, q_0 = 1)$

$$x q_n(x) = a_n q_{n+1}(x) + b_n q_n(x) + c_n q_{n-1}(x), \quad n \geq 1$$

where $a_n, c_n \neq 0$, $b_n \in \mathbb{R}$ and $q_0(x) = 1, q_{-1}(x) = 0$.
Jacobi operator (tridiagonal):

$$Jq = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & c_2 & b_2 & a_2 \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ q_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} q_0(x) \\ q_1(x) \\ q_2(x) \\ \vdots \end{pmatrix} = xq, \quad x \in \mathcal{S}$$

The converse result is also true (Favard's or spectral theorem)

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Classical families

(Bochner, 1929) Characterize families $(q_n)_n$ satisfying

$$\sigma(x)q_n''(x)+\tau(x)q_n'(x)=\lambda_n q_n(x), \quad \deg(\sigma)\leq 2, \quad \deg(\tau)=1$$

The *positive* measure ω (symmetric) satisfies the Pearson equation

$$(\sigma(x)\omega(x))'=\tau(x)\omega(x)$$

1. Hermite: $\sigma(x)=1$, $\tau(x)=-2x$, $\lambda_n=-2n$
   $\omega(x)=e^{-x^2}, x\in\mathbb{R}$, normal or Gaussian distribution.
2. Laguerre: $\sigma(x)=x$, $\tau(x)=-x+\alpha+1$, $\lambda_n=-n$
   $\omega(x)=x^{\alpha}e^{-x}, x\in[0,+\infty), \alpha>-1$, Gamma distribution.
3. Jacobi: $\sigma(x)=1-x^2$, $\tau(x)=-(\alpha+\beta+2)x+\beta-\alpha$,
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A *Markov process* with state space $\mathcal{S} \subset \mathbb{R}$ is a collection of random variables $\{X_t \in \mathcal{S} : t \in \mathcal{T}\}$ indexed by time $\mathcal{T}$ (discrete or continuous) such that they have the Markov property: the future event only depends on the present, not on the past (no memory).

- $\mathcal{S}$ discrete (Markov chains)
  The transition probabilities

$$P_{ij}(t) = \Pr(X_t = j | X_0 = i), \quad i, j \in \{0, 1, \ldots\}$$

come in terms of a stochastic matrix

$$P(t) = \begin{pmatrix} P_{00}(t) & P_{01}(t) & \cdots \\ P_{10}(t) & P_{11}(t) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

- $\mathcal{S}$ continuous (Markov processes)
  The probabilities are described in terms of a density

$$p(t; x, y) = \frac{\partial}{\partial y} \Pr(X_t \leq y | X_0 = x), \quad x, y \in (a, b) \subset \mathbb{R}$$

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Three important cases

1. Random walks: $\mathcal{S}=\{0,1,2,\dots\}$, $\mathcal{T}=\{0,1,2,\dots\}$.

$$P=\begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_i \geq 0, a_i, c_i > 0, \quad a_i+b_i+c_i=1$$

   $P(n)=P^n$ is the *n-step transition probability matrix*.

2. Birth-and-death processes: $\mathcal{S}=\{0,1,2,\dots\}$, $\mathcal{T}=[0,\infty)$.
   $P(t)$ satisfy the *backward and forward equation*

$$P'(t)=\mathcal{A}P(t), \quad P'(t)=P(t)\mathcal{A}, \quad P(0)=I$$

$$\mathcal{A}=\begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad \lambda_i, \mu_i > 0$$

3. Diffusion processes: $\mathcal{S}=(a,b)\subseteq\mathbb{R}$, $\mathcal{T}=[0,\infty)$.
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$$\frac{\partial}{\partial t}p(t;x,y)=\mathcal{A}p(t;x,y), \quad \frac{\partial}{\partial t}p(t;x,y)=\mathcal{A}^*p(t;x,y)$$

$$\mathcal{A}=\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}+\tau(x)\frac{d}{dx}$$

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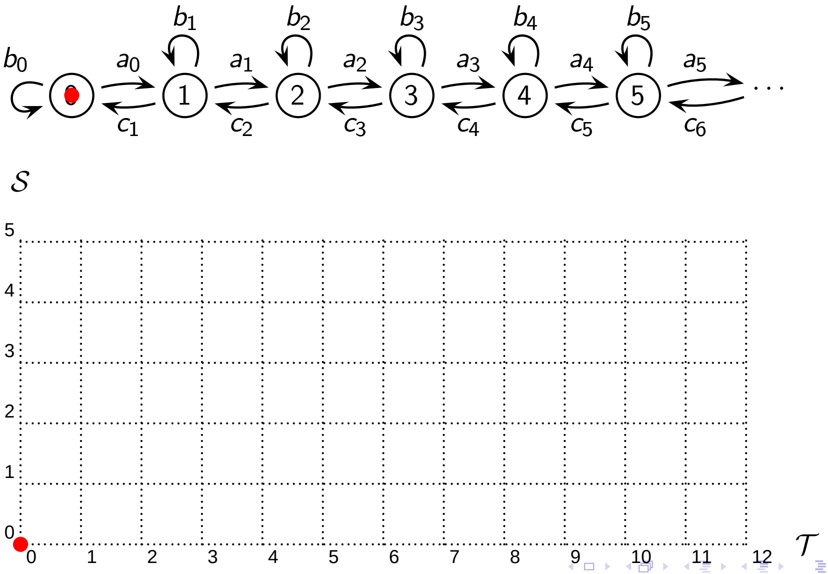

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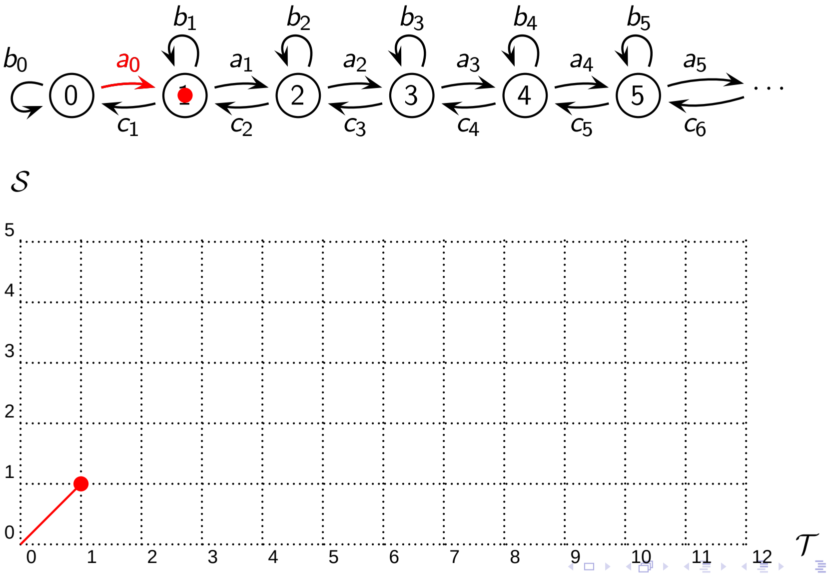

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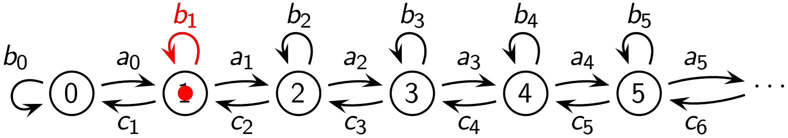

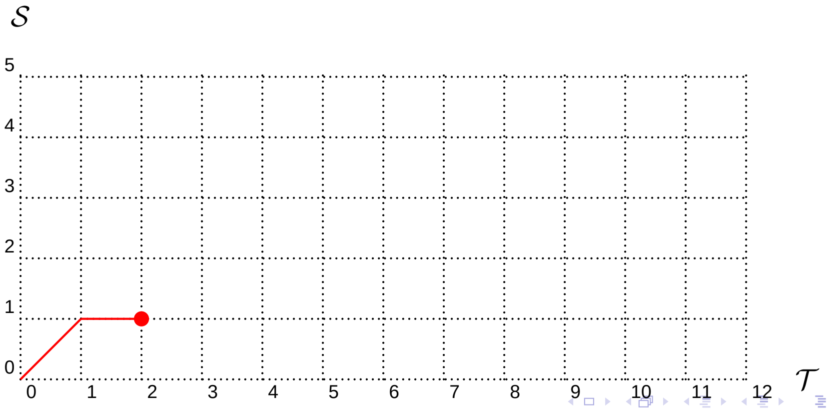

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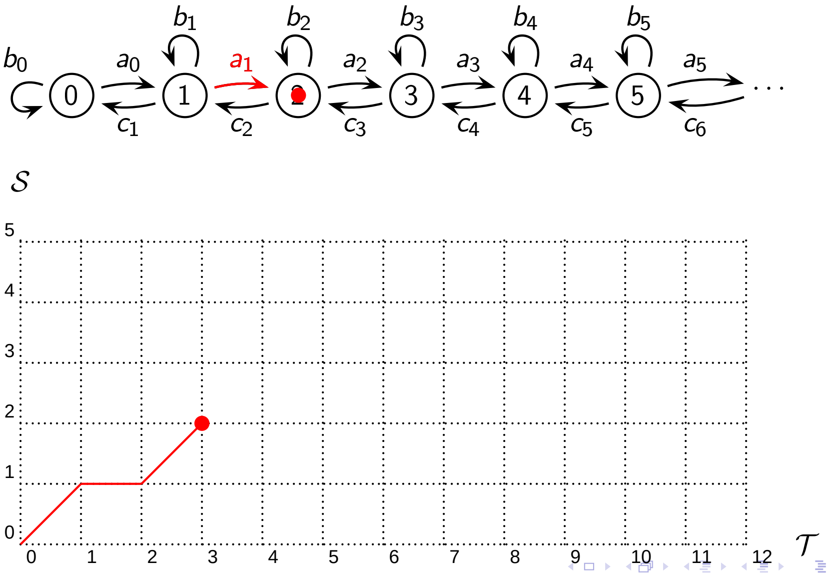

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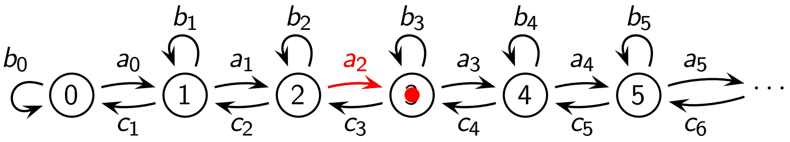

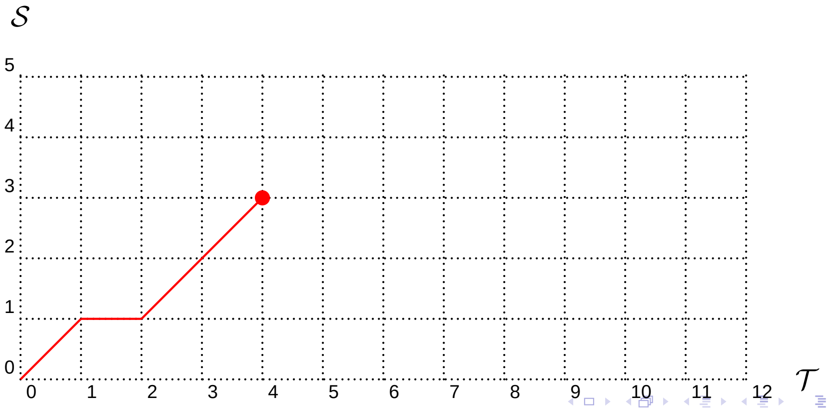

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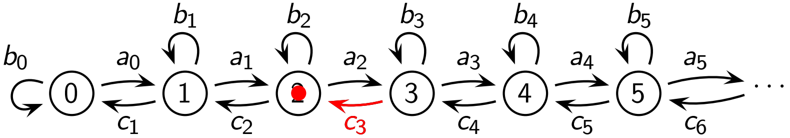

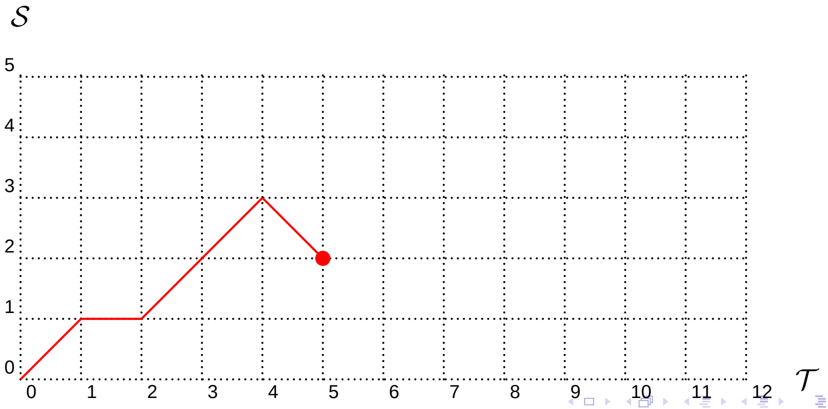

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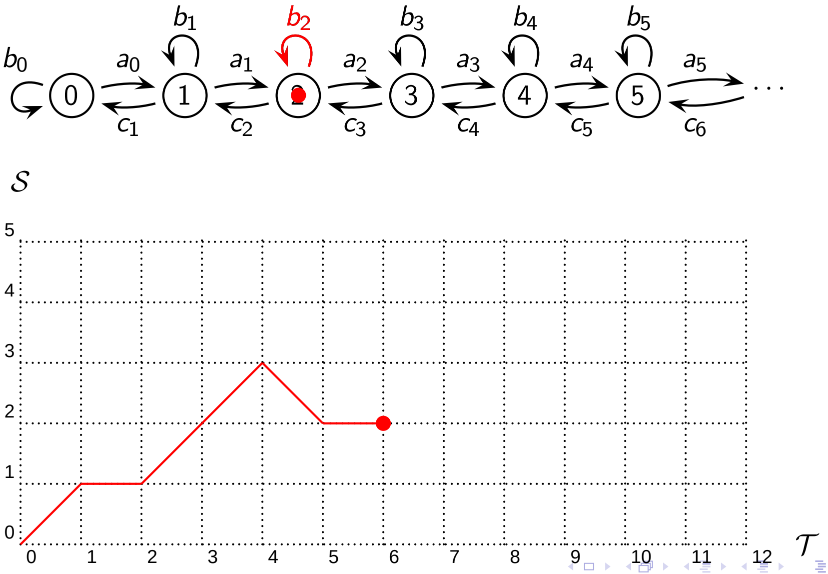

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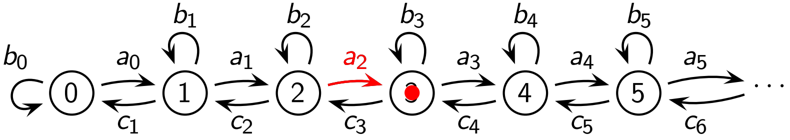

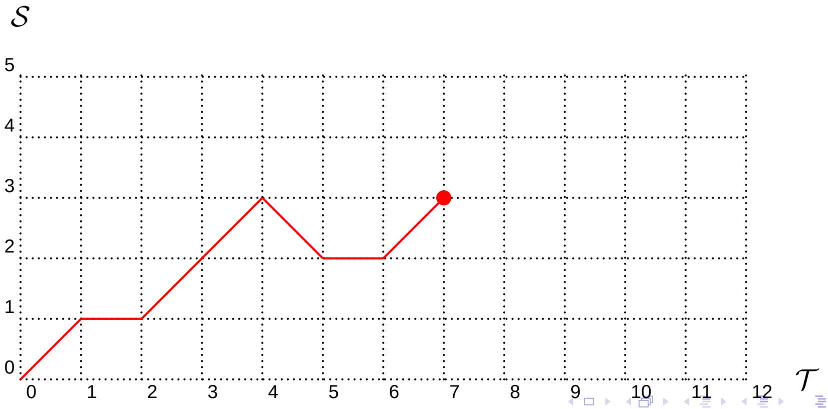

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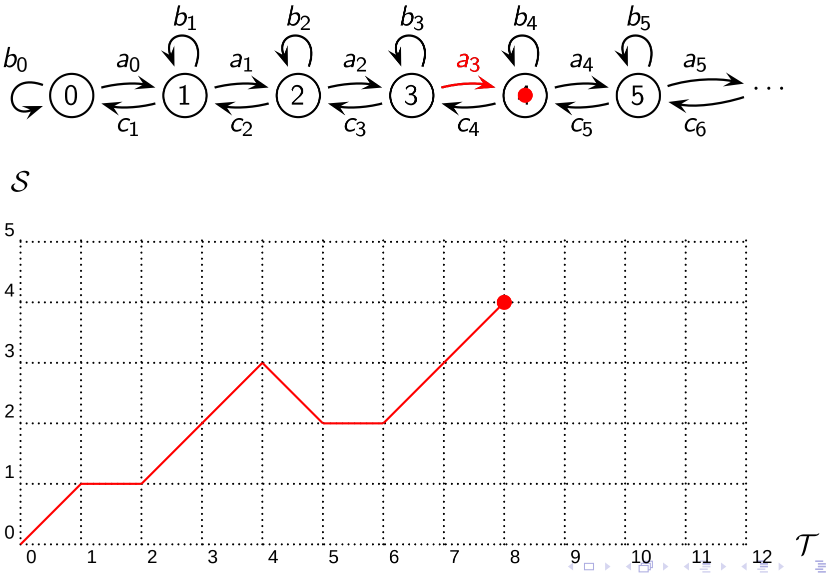

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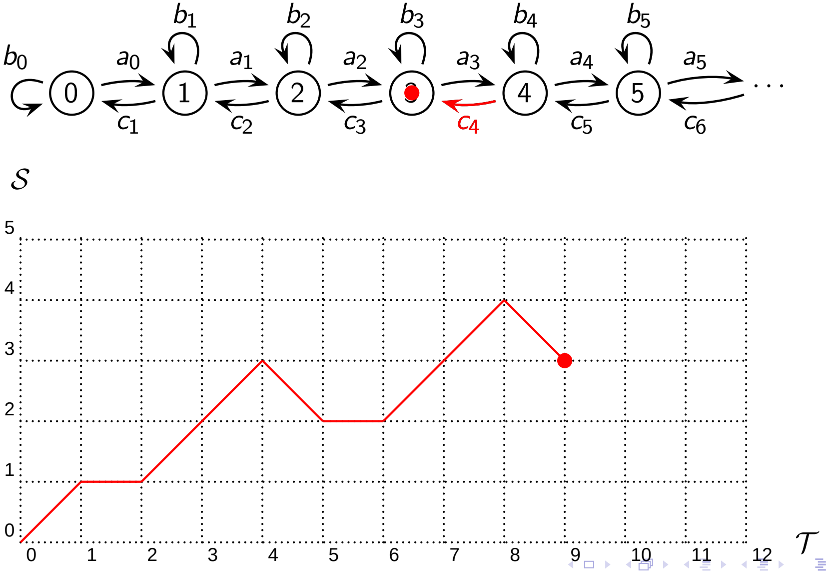

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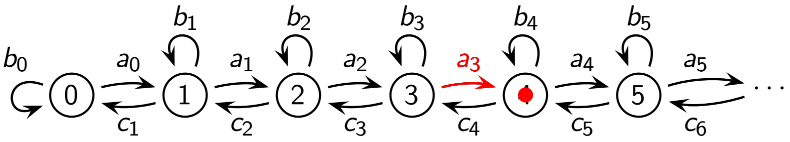

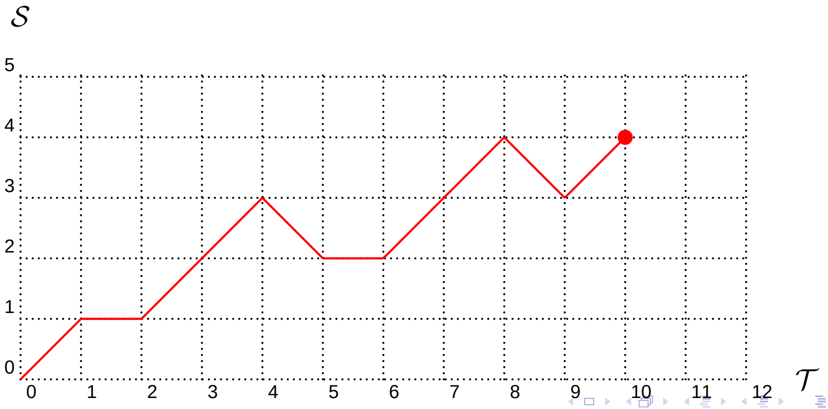

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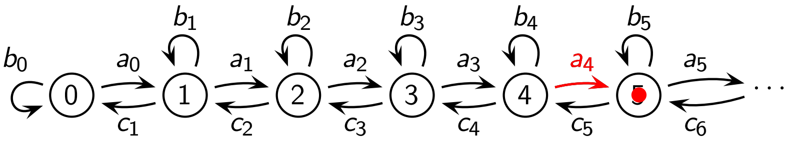

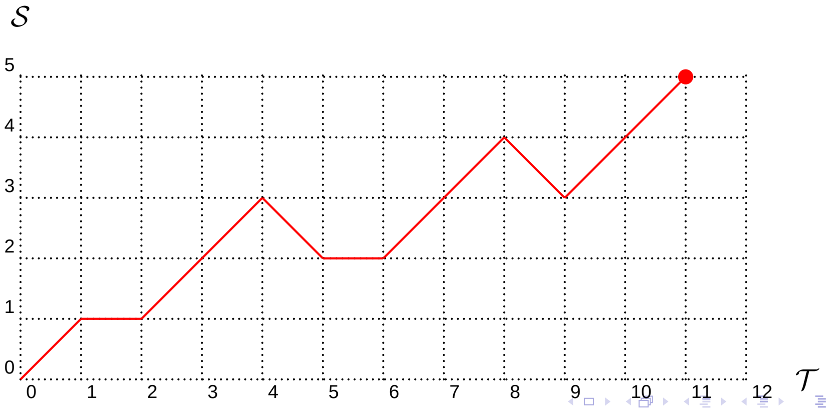

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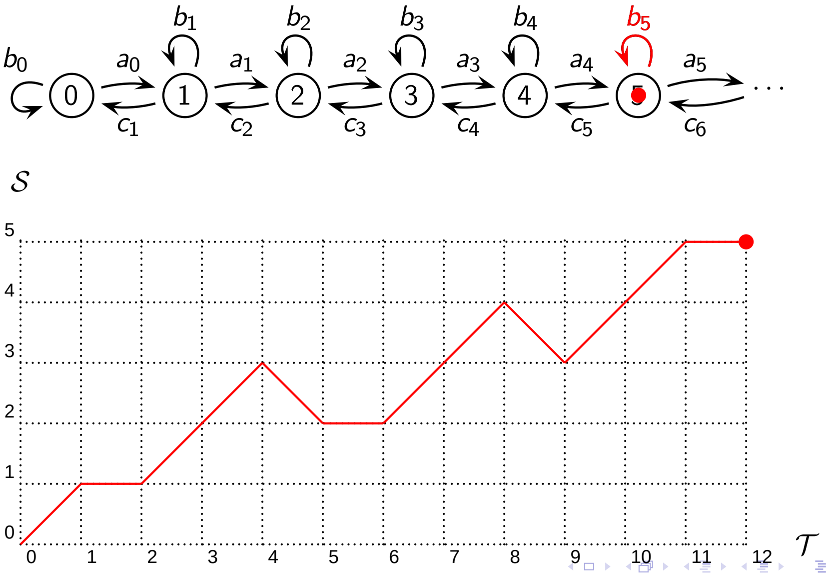

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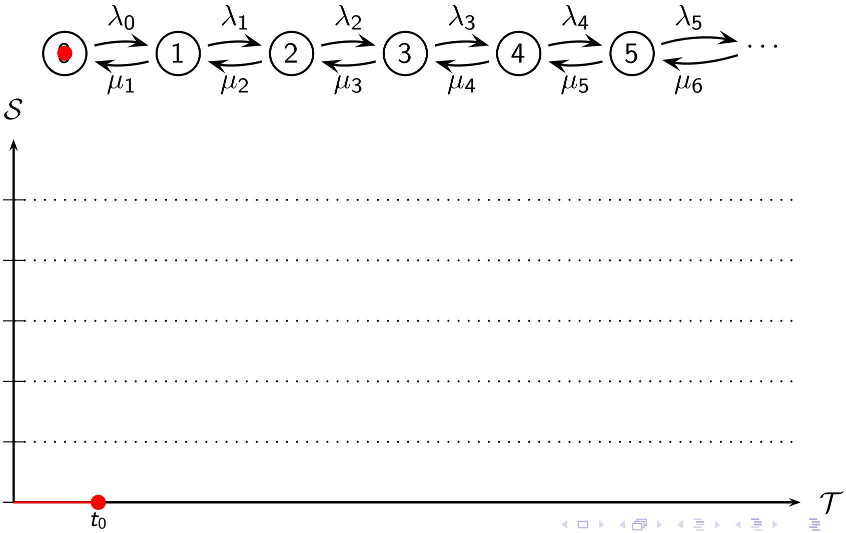

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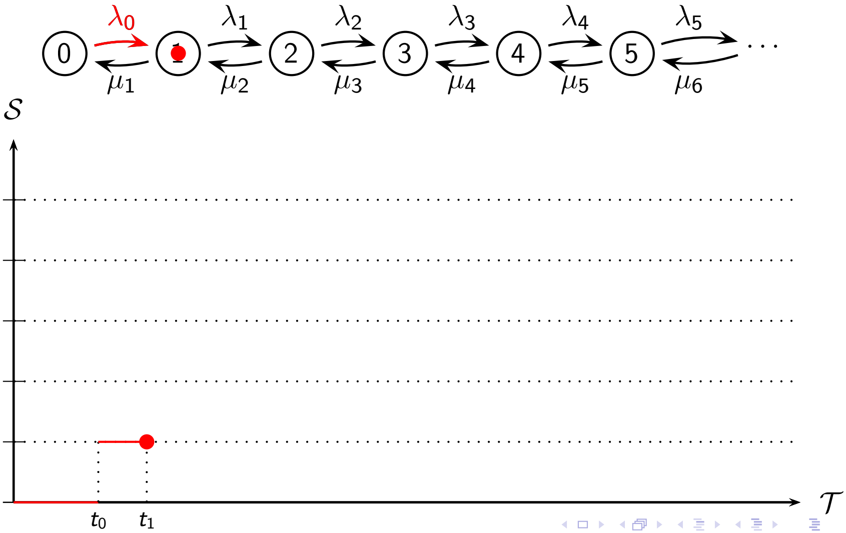

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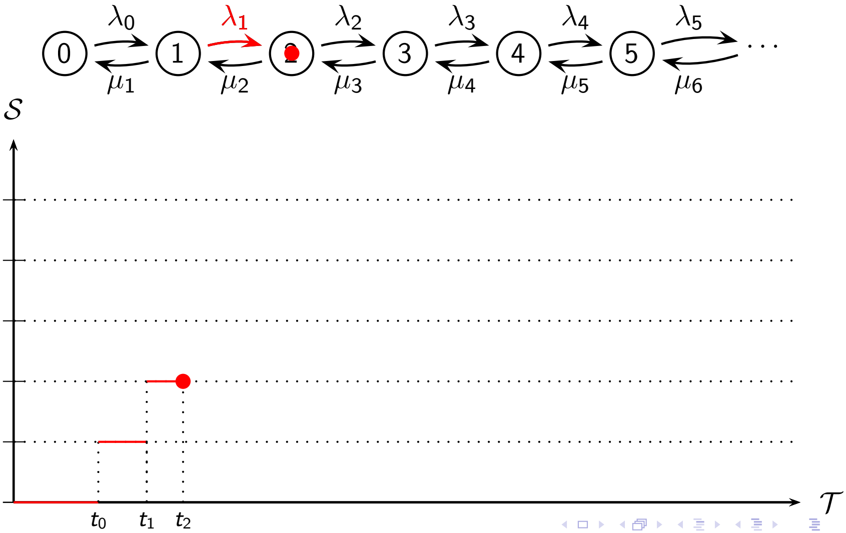

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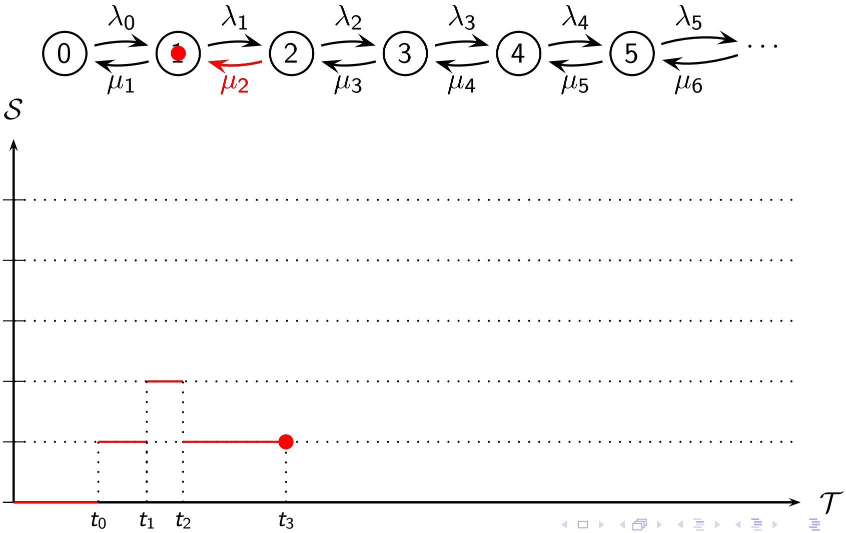

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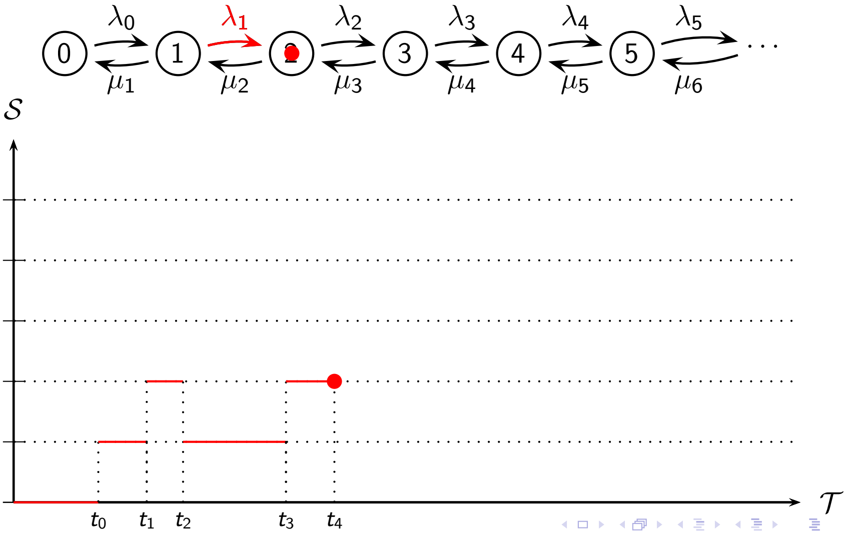

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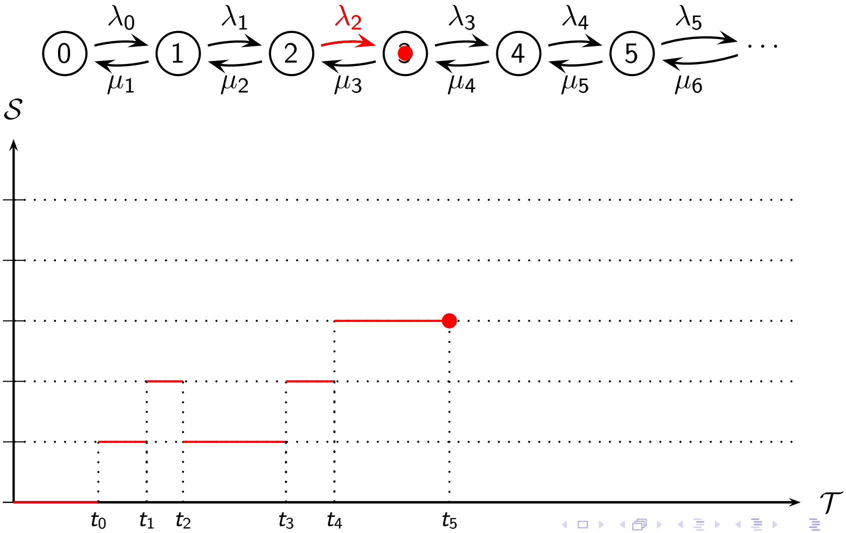

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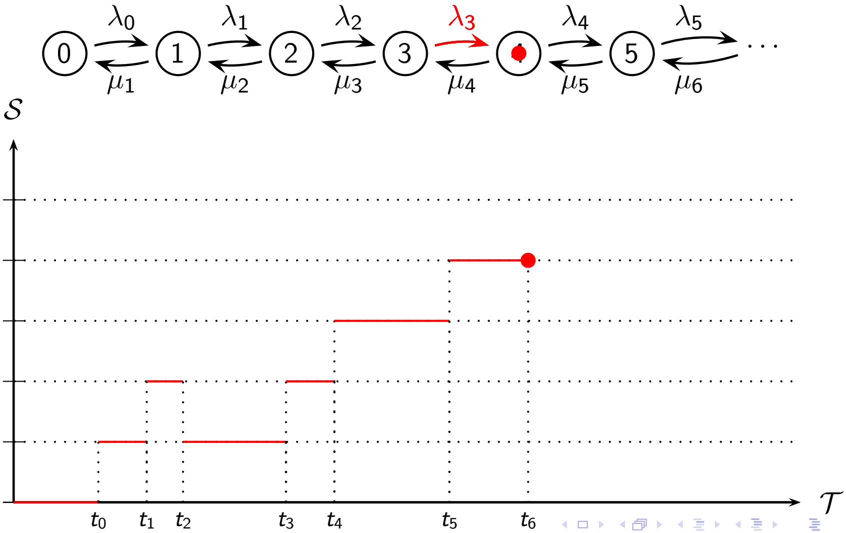

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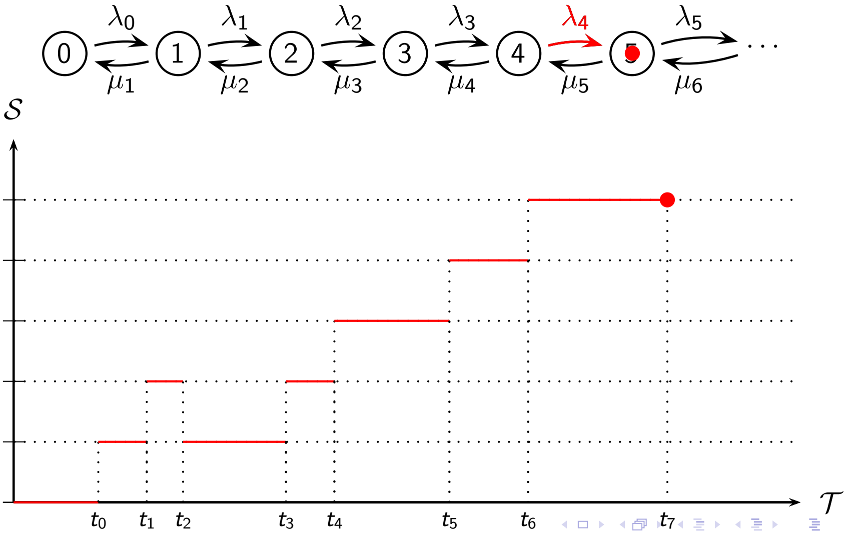

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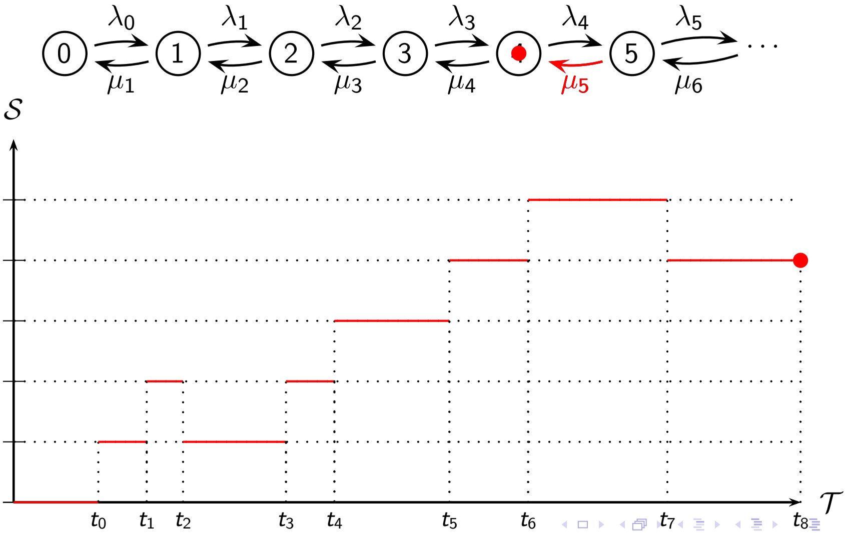
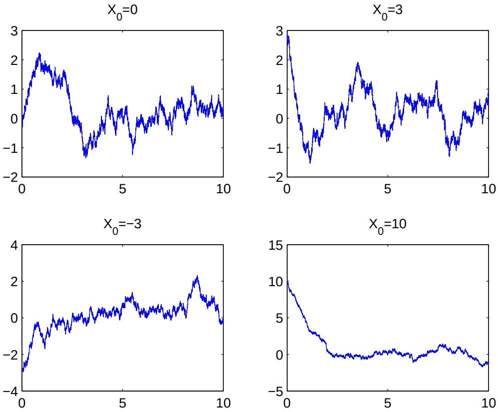


Diffusion processes

Ornstein-Uhlenbeck diffusion process: $\mathcal{S} = \mathbb{R}$ and $\sigma^2(x) = 1$, $\tau(x) = -x$
It describes the velocity of a massive Brownian particle under the influence of friction. It is the only nontrivial process which is stationary, Gaussian and Markovian.



Spectral methods

Given a infinitesimal operator $\mathcal{A}$, if we can find a measure $\omega(x)$ associated with $\mathcal{A}$, and a set of orthogonal eigenfunctions $f(i,x)$ such that

$$\mathcal{A}f(i,x) = \lambda(i,x)f(i,x)$$

then it is possible to find spectral representations of

- Transition probabilities
  - Discrete case: transition probability matrix $P_{ij}(t)$.
  - Continuous case: transition density $p(t;x,y)$.
- Invariant measure or distribution
  - Discrete case: $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ with

$$\pi_j = \lim_{t\to\infty} P_{ij}(t)$$

  - Continuous case: $\psi(y)$ with

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Random walks

$\mathcal{S} = \mathcal{T} = \{0, 1, 2, \ldots\}$.

Spectral theorem (Karlin-MacGregor, 1959): there exists a measure ω associated with P which orthogonal polynomials $(q_n)_n$ satisfy $(q_{-1} = 0, q_0 = 1)$

$$Pq = \begin{pmatrix} b_0 & a_0 & & \\ c_1 & b_1 & a_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix}, \quad x \in [-1, 1]$$

Transition probabilities

$$P_{ij}^n = \Pr(X_n = j | X_0 = i) = \frac{1}{\|q_i\|_\omega^2} \int_{-1}^{1} x^n q_i(x) q_j(x) d\omega(x)$$

Invariant measure

Non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots) \geq 0$ such that

$$\boldsymbol{\pi} P = \boldsymbol{\pi} \quad \Rightarrow \pi_i = \frac{a_0 a_1 \cdots a_{i-1}}{c_1 c_2 \cdots c_i} = \frac{1}{\|q_i\|_\omega^2}$$

Examples: Urn models related with Jacobi polynomials (Legendre, Gegenbauer, Chebyshev).

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Birth-and-death processes

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$$\mathcal{A}q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix} = -x \begin{pmatrix} q_0(x) \\ q_1(x) \\ \vdots \end{pmatrix}$$

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$$P_{ij}(t) = \Pr(X_t = j | X_0 = i) = \frac{1}{\|q_i\|_\omega^2} \int_0^\infty e^{-xt} q_i(x) q_j(x) d\omega(x)$$

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Diffusion processes

$\mathcal{S} = (a, b) \subseteq \mathbb{R}$, $\mathcal{T} = [0, \infty)$.

If there exists a positive measure ω symmetric with respect to $\mathcal{A}$ and the corresponding family of orthonormal functions $(\phi_n)_n$ satisfy

$$\mathcal{A}\phi_n(x) = \frac{1}{2}\sigma^2(x)\phi_n''(x) + \tau(x)\phi_n'(x) = \lambda_n\phi_n(x)$$

Transition probability density

$$p(t; x, y) = \sum_{n=0}^{\infty} e^{\lambda_n t}\phi_n(x)\phi_n(y)\omega(y)$$

Invariant measure

$$\psi(y) \quad \text{such that} \quad \mathcal{A}^*\psi(y) = 0 \Rightarrow \psi(y) = \frac{1}{\int_S \omega(x)dx}\omega(y)$$

Examples: Hermite (Orstein-Uhlenbeck process), Laguerre (squared Bessel process), Jacobi polynomials (Wright-Fisher model).

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Outline

Matrix-valued orthogonal polynomials

Matrix valued polynomials on the real line:

$$\mathbf{A}_n x^n + \cdots + \mathbf{A}_1 x + \mathbf{A}_0, \quad \mathbf{A}_i \in \mathbb{C}^{N\times N}$$

Krein (1949): matrix-valued orthogonal polynomials (MOP)

Orthogonality: weight matrix $\mathbf{W}$ supported on $\mathcal{S} \subset \mathbb{R}$ (positive definite with finite moments) and a matrix valued inner product ($L^2_{\mathbf{W}}(\mathcal{S};\mathbb{C}^{N\times N})$):

$$\langle \mathbf{P}, \mathbf{Q}\rangle_{\mathbf{W}} = \int_{\mathcal{S}} \mathbf{P}(x)\mathbf{W}(x)\mathbf{Q}^*(x)\,dx$$

A system of MOP $(\mathbf{Q}_n)_n$ ($\langle \mathbf{Q}_n, \mathbf{Q}_m\rangle_{\mathbf{W}} = \|\mathbf{Q}_n\|^2_{\mathbf{W}}\delta_{nm}$) satisfies a three-term recurrence relation ($\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I}$)

$$x\mathbf{Q}_n(x) = \mathbf{A}_n\mathbf{Q}_{n+1}(x) + \mathbf{B}_n\mathbf{Q}_n(x) + \mathbf{C}_n\mathbf{Q}_{n-1}(x), \quad \det(\mathbf{C}_n) \neq 0$$

Jacobi operator (block tridiagonal)

$$\mathbf{J}\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & \\ \mathbf{C}_1 & \mathbf{B}_1 & \mathbf{A}_1 & & \\ & \mathbf{C}_2 & \mathbf{B}_2 & \mathbf{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{Q}_0(x) \\ \mathbf{Q}_1(x) \\ \mathbf{Q}_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} \mathbf{Q}_0(x) \\ \mathbf{Q}_1(x) \\ \mathbf{Q}_2(x) \\ \vdots \end{pmatrix} = x\mathbf{Q}, \quad x \in \mathcal{S}$$

The converse result is also true (Favard's or spectral theorem**)

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$$\mathbf{J}\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & \\ \mathbf{C}_1 & \mathbf{B}_1 & \mathbf{A}_1 & \\ & \mathbf{C}_2 & \mathbf{B}_2 & \mathbf{A}_2 \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{Q}_0(x) \\ \mathbf{Q}_1(x) \\ \mathbf{Q}_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} \mathbf{Q}_0(x) \\ \mathbf{Q}_1(x) \\ \mathbf{Q}_2(x) \\ \vdots \end{pmatrix} = x\mathbf{Q}, \quad x \in \mathcal{S}$$

The converse result is also true (Favard's or spectral theorem**)

Matrix-valued orthogonal polynomials

Matrix valued polynomials on the real line:

$$\mathbf{A}_n x^n + \cdots + \mathbf{A}_1 x + \mathbf{A}_0, \quad \mathbf{A}_i \in \mathbb{C}^{N\times N}$$

Krein (1949): matrix-valued orthogonal polynomials (MOP)
Orthogonality: weight matrix $\mathbf{W}$ supported on $\mathcal{S} \subset \mathbb{R}$ (positive definite with finite moments) and a matrix valued inner product ($L^2_{\mathbf{W}}(\mathcal{S};\mathbb{C}^{N\times N})$):

$$\langle \mathbf{P}, \mathbf{Q}\rangle_{\mathbf{W}} = \int_{\mathcal{S}} \mathbf{P}(x)\mathbf{W}(x)\mathbf{Q}^*(x)\,dx$$

A system of MOP $(\mathbf{Q}_n)_n$ ($\langle \mathbf{Q}_n, \mathbf{Q}_m\rangle_{\mathbf{W}} = \|\mathbf{Q}_n\|^2_{\mathbf{W}}\delta_{nm}$) satisfies a three-term recurrence relation ($\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I}$)

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Equivalent to the symmetry of $\mathcal{D}$ with respect to the inner product, i.e. $(\mathcal{D}\mathbf{P},\mathbf{Q})_{\mathbf{W}} = (\mathbf{P},\mathcal{D}\mathbf{Q})_{\mathbf{W}}$, for all matrix polynomials $\mathbf{P},\mathbf{Q}$.

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Bivariate Markov processes

Now we have a bivariate or 2-component Markov process of the form

$$\{(X_t, Y_t) : t \in \mathcal{T}\}$$

indexed by time $\mathcal{T}$ (time) and with state space $\mathcal{S} \times \{1, 2, \ldots, N\}$, $\mathcal{S} \subset \mathbb{R}$. The first component is the level while the second component is the phase. Now the transition probabilities are matrix-valued.

- $\mathcal{S}$ discrete: block transition probabilities

$$(\mathbf{P}_{ij})_{i'j'}(t) \equiv \Pr(X_t = j, Y_t = j' | X_0 = i, Y_0 = i')$$

  The block matrix is stochastic.

- $\mathcal{S}$ continuous: matrix transition density

$$\mathbf{P}_{ij}(t; x, y) \equiv \frac{\partial}{\partial y} \Pr(X_t \leq y, Y_t = j | X_0 = x, Y_0 = i)$$

  Every entry must be nonnegative and

$$\mathbf{P}(t; x, A)\mathbf{e}_N \leq \mathbf{e}_N, \quad \mathbf{e}_N = (1, 1, \ldots, 1)^T$$

Ideas behind: *random evolutions*

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Quasi-birth-and-death processes

- Discrete time: State space $\{0,1,2,\ldots\}\times\{1,2,\ldots,N\}$, time $\mathcal{T}=\{0,1,2,\ldots\}$ and

$$(\mathbf{P}_{ij})_{i'j'} = \Pr(X_{n+1}=j, Y_{n+1}=j' | X_n=i, Y_n=i') = 0 \quad \text{for} \quad |i-j|>1.$$

i.e. a $N\times N$ block tridiagonal transition probability matrix (stochastic)

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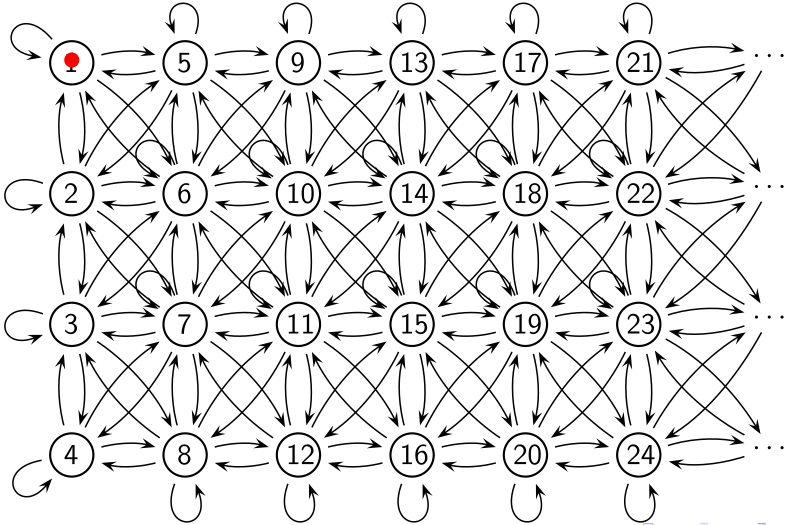
Stochastic representation ($N = 4$ phases)

Special case of $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ tridiagonal:



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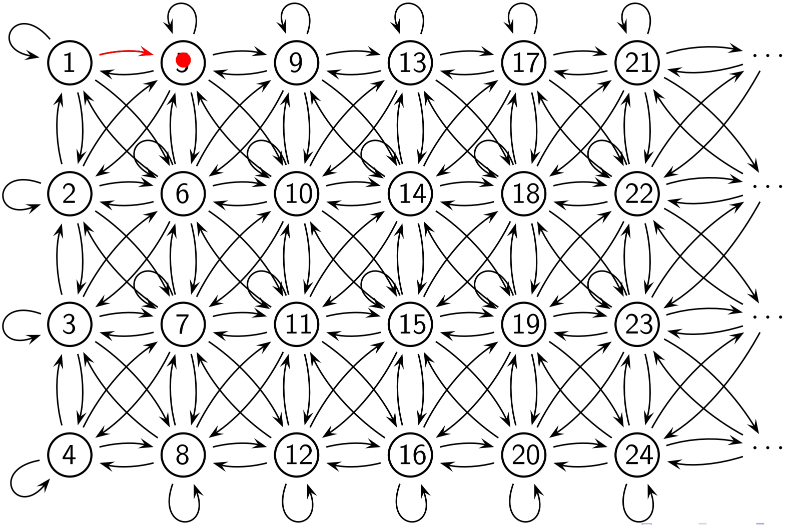

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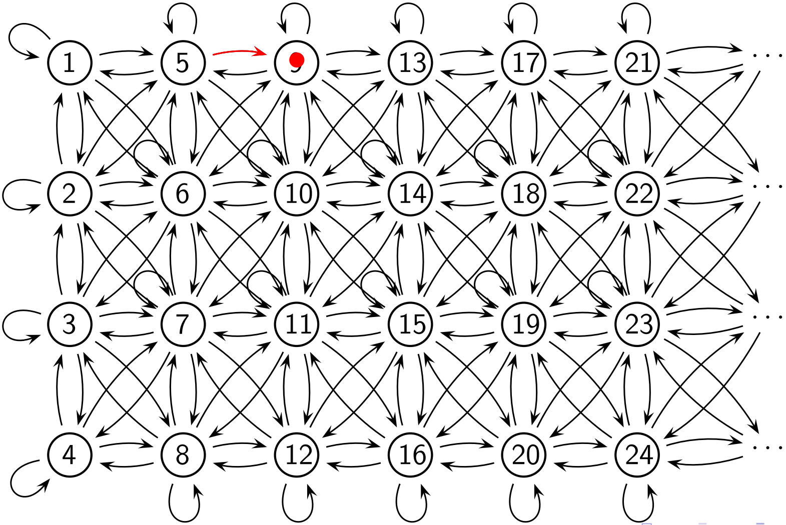

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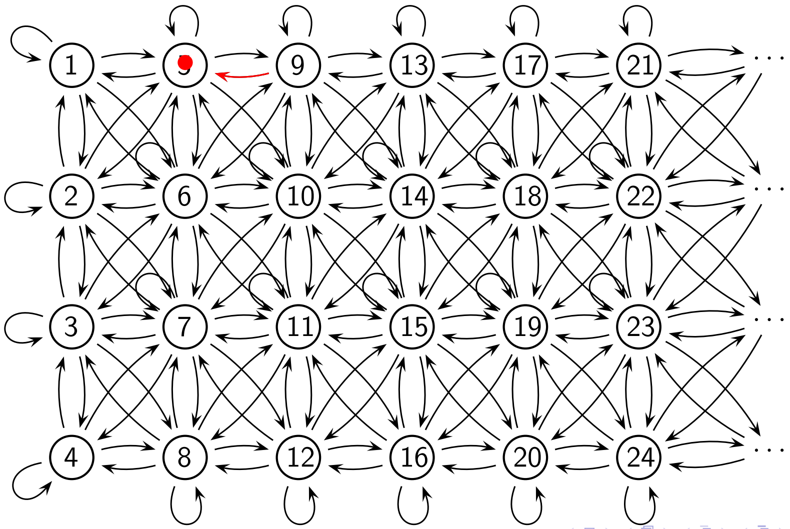

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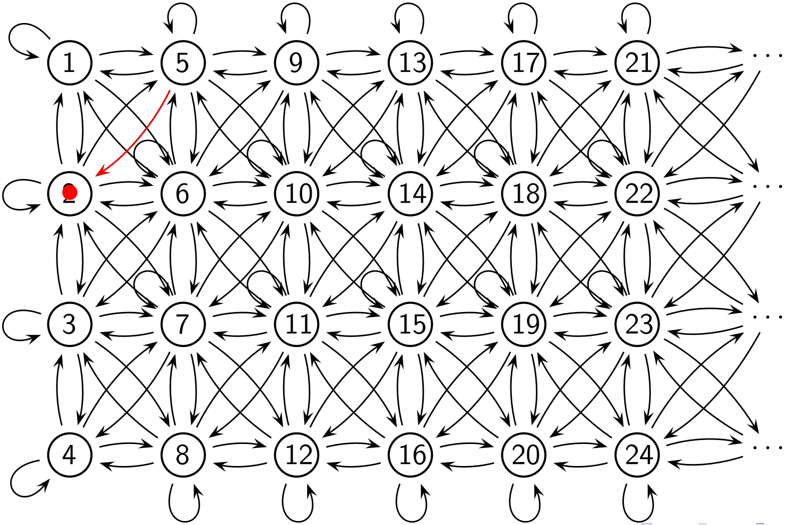

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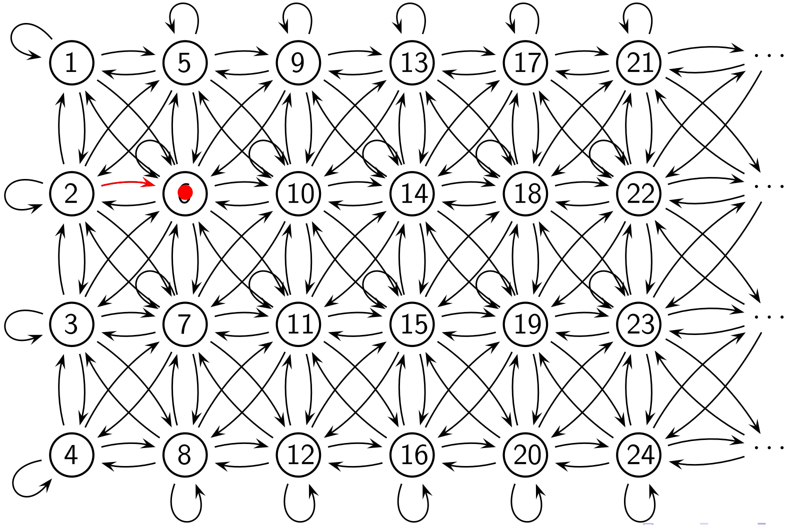

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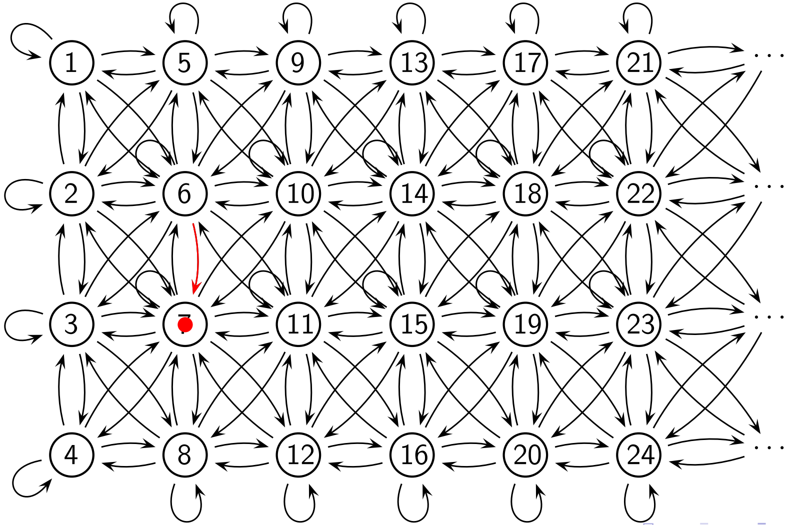

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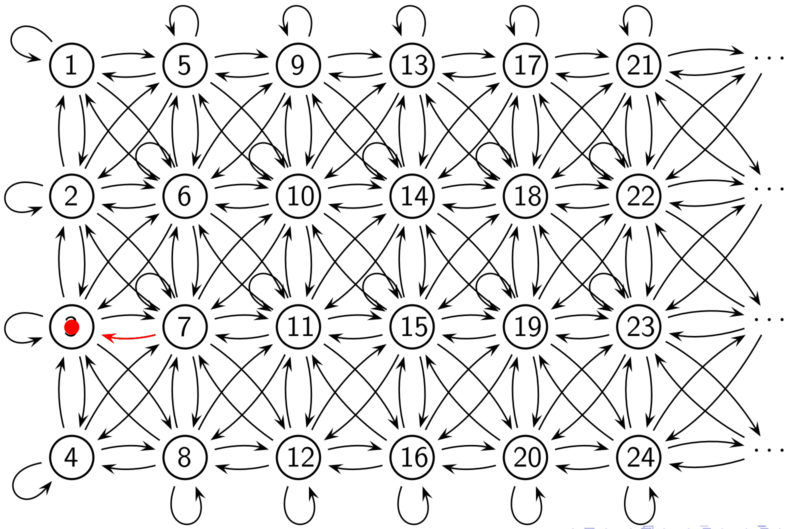

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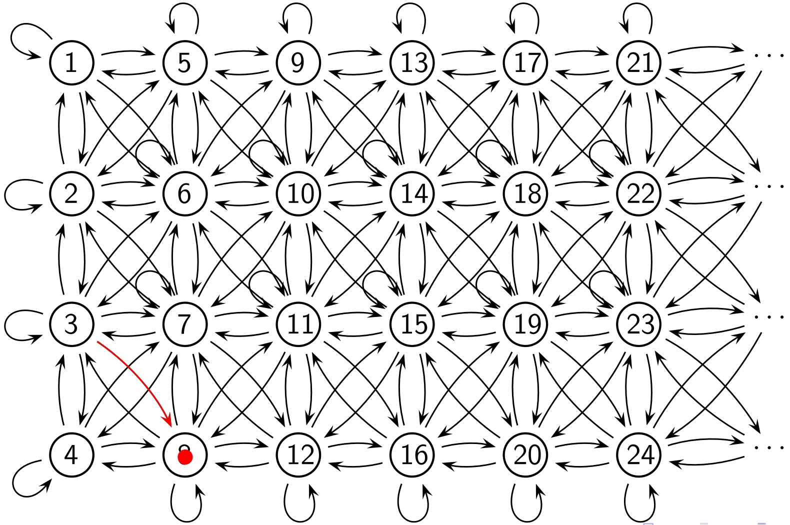
Special case of $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ tridiagonal:



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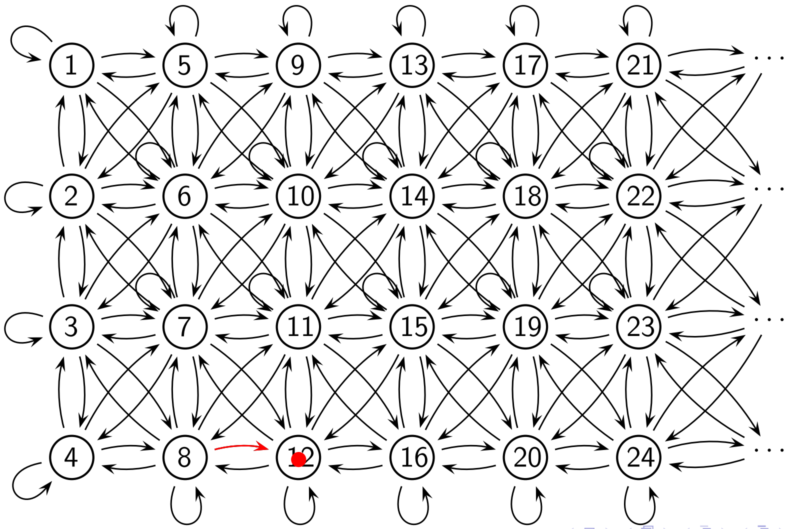

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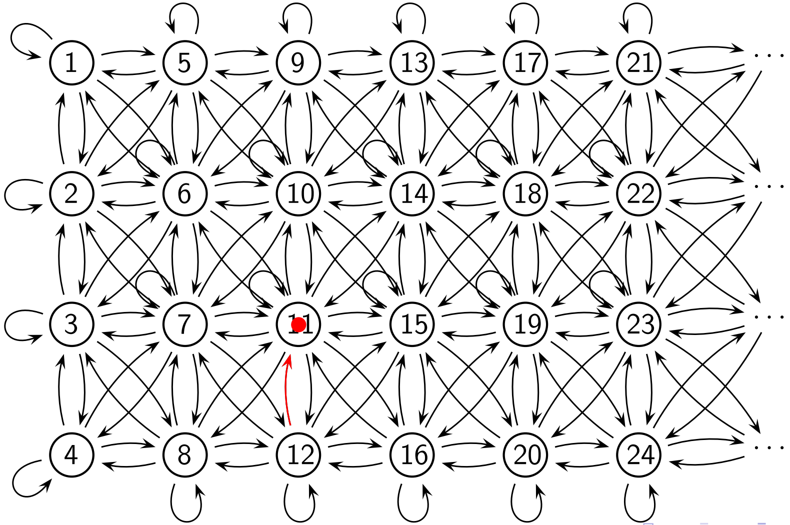

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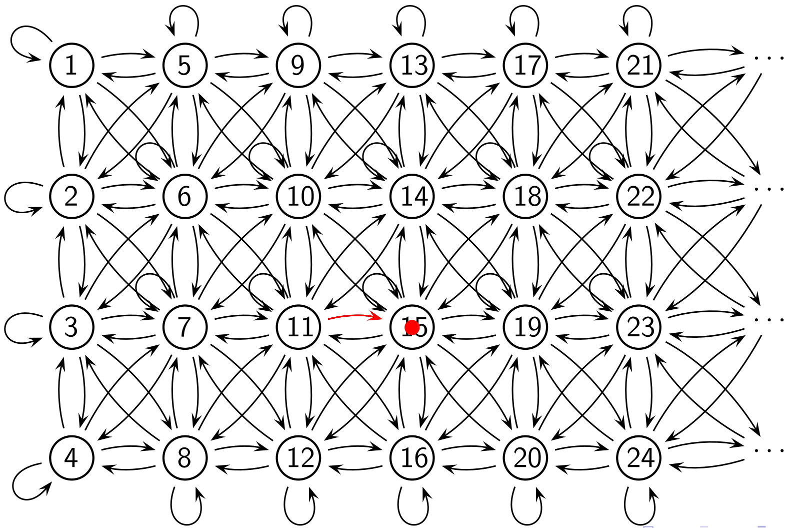

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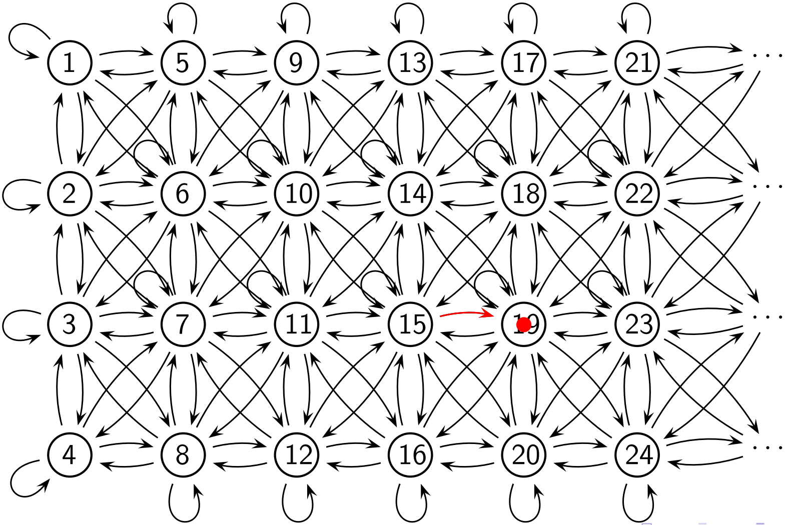

Switching diffusion processes

The state space is now $(a, b) \times \{1, 2, \dots, N\}$ and time $\mathcal{T} = [0, \infty)$.
The density matrix $\mathbf{P}(t; x, y)$ satisfies the *backward and forward equations*

$$\frac{\partial}{\partial t}\mathbf{P}(t; x, y) = \mathcal{A}\mathbf{P}(t; x, y), \quad \frac{\partial}{\partial t}\mathbf{P}(t; x, y) = \mathbf{P}(t; x, y)\mathcal{A}^*$$

where $\mathcal{A}$ is a matrix-valued differential operator

$$\mathcal{A} = \frac{1}{2}\mathbf{A}(x)\frac{d^2}{dx^2} + \mathbf{B}(x)\frac{d^1}{dx^1} + \mathbf{Q}(x)\frac{d^0}{dx^0}$$

We have that $\mathbf{A}(x)$ and $\mathbf{B}(x)$ are diagonal matrices and $\mathbf{Q}(x)$ is the infinitesimal operator of a continuous time Markov chain, i.e.

$$\mathbf{Q}_{ii}(x) \leq 0, \quad \mathbf{Q}_{ij}(x) \geq 0, i \neq j, \quad \mathbf{Q}(x)\mathbf{e}_N = \mathbf{0}$$

The *row vector-valued* invariant distribution $(t \to \infty)$

$$\psi(y) = (\psi_1(y), \psi_2(y), \dots, \psi_N(y)), \quad 0 \leq \psi_j(y) \leq 1, \quad \left(\int_a^b \psi(y)dy\right)\mathbf{e}_N = 1$$

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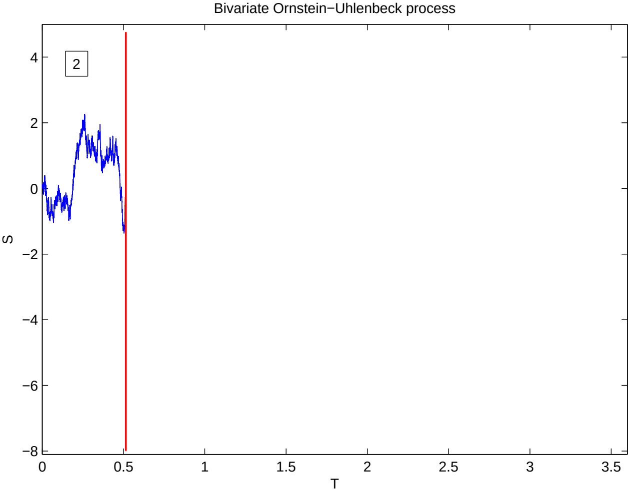
STOCHASTIC REPRESENTATION ($N = 3$ PHASES)

$N = 3$ phases and $\mathcal{S} = \mathbb{R}$ with

$$\mathbf{A}_{ii}(x) = i^2, \quad \mathbf{B}_{ii}(x) = -ix, \quad i = 1, 2, 3$$



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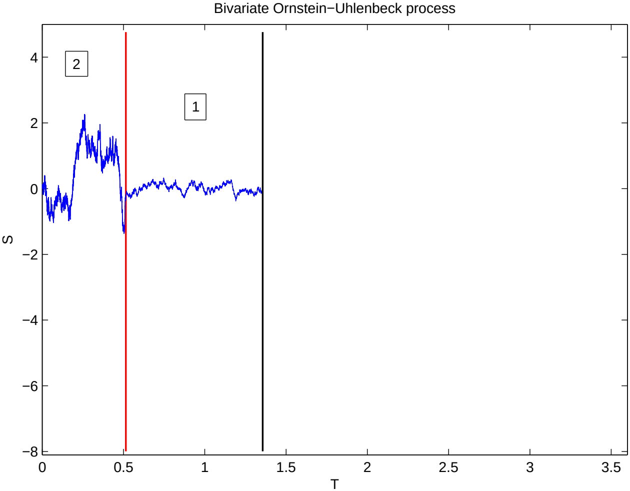

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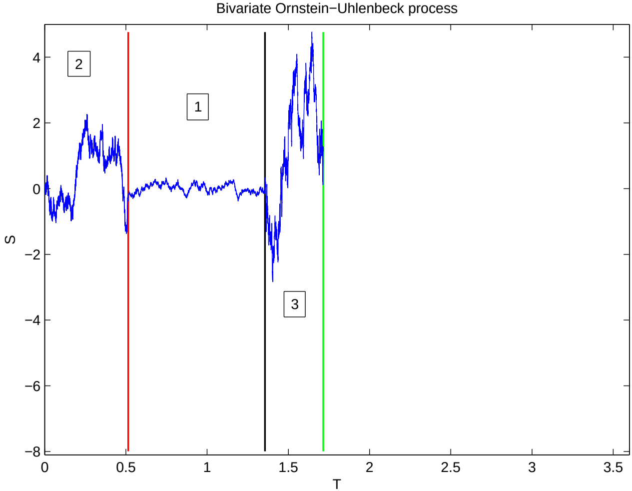

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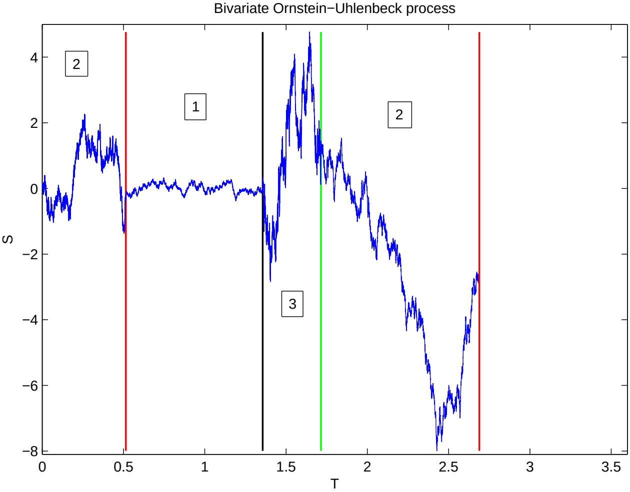

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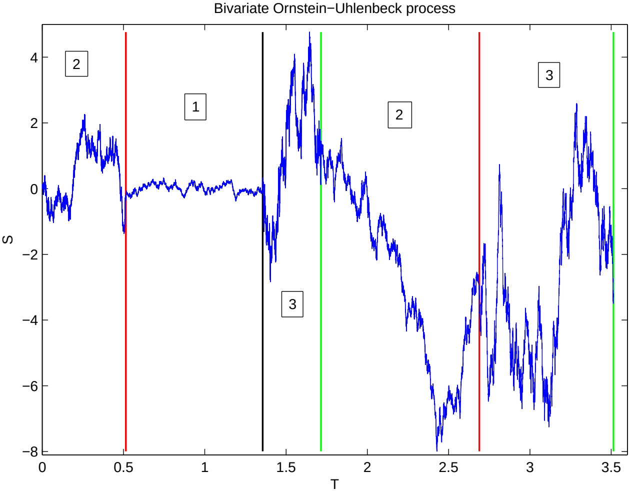

Spectral methods

Now, given a matrix-valued infinitesimal operator $\mathcal{A}$, if we can find a weight matrix $\mathbf{W}(x)$ associated with $\mathcal{A}$, and a set of orthogonal matrix eigenfunctions $\mathbf{F}(i,x)$ such that

$$\mathcal{A}\mathbf{F}(i,x) = \mathbf{F}(i,x)\boldsymbol{\Lambda}(i,x)$$

then it is possible to find spectral representations of

- Transition probabilities
  - Discrete case: transition probability matrix $\mathbf{P}(t)$
  - Continuous case: transition density $\mathbf{P}(t;x,y)$.
- Invariant measure or distribution
  - Discrete case: $\pi = (\pi_0, \pi_1, \ldots) \geq 0$ with

$$\pi_j = \lim_{t\to\infty} \mathbf{P}_{\cdot j}(t) \in \mathbb{R}^N$$

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QUASI-BIRTH-AND-DEATH PROCESSES

- DISCRETE TIME: $\{0,1,2,\ldots\} \times \{1,2,\ldots,N\}$, $\mathcal{T} = \{0,1,2,\ldots\}$.
  Spectral theorem (Grünbaum, Dette et al., 2006): $\exists^*$ $\mathbf{W}$ on $[-1,1]$ associated with $\mathbf{P}$ which MOP $(\mathbf{Q}_n)_n$ satisfy $\mathbf{P}\mathbf{Q} = x\mathbf{Q}$ $(\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I})$

$$\mathbf{P}_{ij}^n = \left(\int_{-1}^{1} x^n \mathbf{Q}_i(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)\left(\int_{-1}^{1} \mathbf{Q}_j(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)^{-1}$$

- CONTINUOUS TIME: $\{0,1,2,\ldots\} \times \{1,2,\ldots,N\}$, $\mathcal{T} = [0,\infty)$
  Spectral theorem (Detter-Reuther, 2010): $\exists^*$ $\mathbf{W}$ on $[0,\infty)$ associated with $\mathcal{A}$ which MOP $(\mathbf{Q}_n)_n$ satisfy $\mathcal{A}\mathbf{Q} = -x\mathbf{Q}$ $(\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I})$

$$\mathbf{P}_{ij}(t) = \left(\int_{0}^{\infty} e^{-xt} \mathbf{Q}_i(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)\left(\int_{0}^{\infty} \mathbf{Q}_j(x)\mathbf{W}(x)\mathbf{Q}_j^*(x)dx\right)^{-1}$$

INVARIANT MEASURE (MDI, 2011)

$\boldsymbol{\pi} = (\boldsymbol{\pi}_0; \boldsymbol{\pi}_1; \cdots) \equiv (\boldsymbol{\Pi}_0 \mathbf{e}_N; \boldsymbol{\Pi}_1 \mathbf{e}_N; \cdots)$ such that $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ (discrete time) or $\boldsymbol{\pi}\mathcal{A} = \mathbf{0}$ (continuous time)

$$\boldsymbol{\Pi}_n = (\mathbf{C}_1^T \cdots \mathbf{C}_n^T)^{-1}\boldsymbol{\Pi}_0(\mathbf{A}_0 \cdots \mathbf{A}_{n-1}) = \left(\int_{\text{supp}(\mathbf{W})} \mathbf{Q}_n(x)\mathbf{W}(x)\mathbf{Q}_n^*(x)dx\right)^{-1}$$

Quasi-birth-and-death processes

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State space: $(a, b) \times \{1, 2, \ldots, N\}$. Time: $\mathcal{T} = [0, \infty)$
If there exists a weight matrix $\mathbf{W}$ symmetric with respect to $\mathcal{A}$ which matrix-valued orthonormal functions $(\mathbf{\Phi}_n)_n$ satisfies

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Transition probability density matrix (MdI, 2012)

$$\mathbf{P}(t; x, y) = \sum_{n=0}^{\infty} \mathbf{\Phi}_n(x) e^{\mathbf{\Gamma}_n t} \mathbf{\Phi}_n^*(y)\mathbf{W}(y)$$

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Outline

1. Markov processes
   - Preliminaries
   - Spectral methods
2. Bivariate Markov processes
   - Preliminaries
   - Spectral methods
3. An example
   - A quasi-birth-and-death process
   - A variant of the Wright-Fisher model
4. Future work
   - 3 years ahead
   - 5 years ahead

An example coming from group representation

Let $N \in \{1,2,\ldots\}$, $\alpha,\beta > -1$, $0 < k < \beta+1$ and $\mathbf{E}_{ij}$ will denote the matrix with 1 at entry (i,j) and 0 otherwise.
For $x \in (0,1)$, we have a symmetric pair $\{\mathbf{W}, \mathcal{A}\}$ (Grünbaum-Pacharoni-Tirao, 2002) where

$$\mathbf{W}(x) = x^{\alpha}(1-x)^{\beta}\sum_{i=1}^{N}\binom{\beta-k+i-1}{i-1}\binom{N+k-i-1}{N-i}x^{N-i}\mathbf{E}_{ii}$$

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$$\mathbf{W}(x) = x^{\alpha}(1-x)^{\beta} \sum_{i=1}^{N} \binom{\beta - k + i - 1}{i - 1} \binom{N + k - i - 1}{N - i} x^{N-i} \mathbf{E}_{ii}$$

$$\mathcal{A} = \frac{1}{2}\mathbf{A}(x)\frac{d^2}{dx^2} + \mathbf{B}(x)\frac{d}{dx} + \mathbf{Q}(x)\frac{d^0}{dx^0}$$

$$\mathbf{A}(x) = 2x(1-x)\mathbf{I}, \quad \mathbf{B}(x) = \sum_{i=1}^{N} [\alpha + 1 + N - i - x(\alpha + \beta + 2 + N - i)]\mathbf{E}_{ii}$$

$$\mathbf{Q}(x) = \sum_{i=2}^{N} \mu_i(x)\mathbf{E}_{i,i-1} - \sum_{i=1}^{N} (\lambda_i(x) + \mu_i(x))\mathbf{E}_{ii} + \sum_{i=1}^{N-1} \lambda_i(x)\mathbf{E}_{i,i+1},$$

$$\lambda_i(x) = \frac{1}{1-x}(N-i)(i+\beta-k), \quad \mu_i(x) = \frac{x}{1-x}(i-1)(N-i+k).$$

A QUASI-BIRTH-AND-DEATH PROCESS

It is possible to get (Grünbaum-MdI, 2008) an equivalent symmetric pair $\{\widetilde{\mathbf{W}}, \widetilde{\mathcal{A}}\}$ and a family of MOP $(\mathbf{Q}_n)_n$ such that $\mathbf{Q}_{-1} = \mathbf{0}, \mathbf{Q}_0 = \mathbf{I}$ and

$$\mathbf{Q}_n(1)\mathbf{e}_N = \mathbf{e}_N, \quad \mathbf{e}_N = (1, 1, \cdots, 1)^T$$

The family $(\mathbf{Q}_n)_n$ satisfies a three-term recurrence relation

$$x\mathbf{Q}_n(x) = \mathbf{A}_n\mathbf{Q}_{n+1}(x) + \mathbf{B}_n\mathbf{Q}_n(x) + \mathbf{C}_n\mathbf{Q}_{n-1}(x), \quad n = 0, 1, \ldots,$$

where $\mathbf{B}_n$ is tridiagonal, $\mathbf{A}_n$ is lower bidiagonal and $\mathbf{C}_n$ is upper bidiagonal. The corresponding Jacobi matrix $\mathbf{P}$ is stochastic

$$\mathbf{P} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & \\ \mathbf{C}_1 & \mathbf{B}_1 & \mathbf{A}_1 & & \\ & \mathbf{C}_2 & \mathbf{B}_2 & \mathbf{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

- n-step transition probability matrix $\mathbf{P}^n$ is terms of the Karlin-McGregor representation and the invariant measure π.
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Particular case $\alpha = \beta = 0,\ k = 1/2,\ N = 2$

Pentadiagonal transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{100} & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Invariant measure such that $\pi\mathbf{P} = \pi$

$$\pi = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \ldots\right)$$

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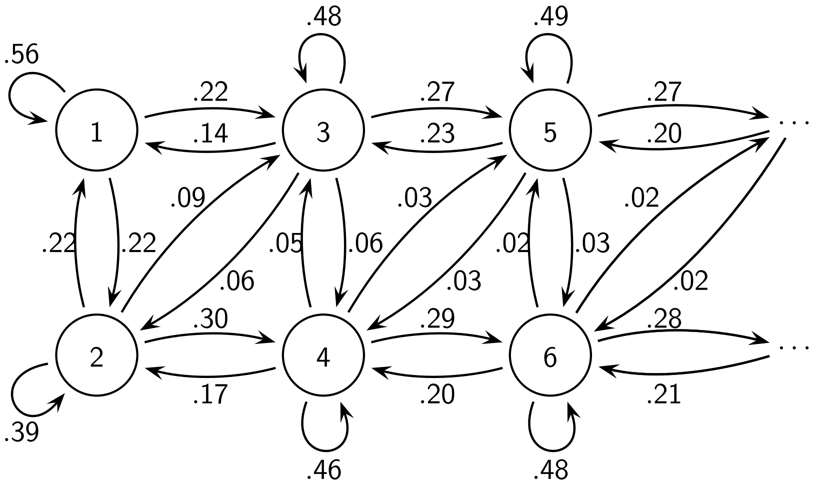
Associated network



A variant of the Wright-Fisher model

The Wright-Fisher diffusion model involving only mutation effects considers a big population of constant size M of two types A and B

$$A \xrightarrow{\frac{1+\beta}{2}} B, \quad B \xrightarrow{\frac{1+\alpha}{2}} A, \quad \alpha, \beta > -1$$

As $M \to \infty$, this model can be described by a diffusion process whose state space is $\mathcal{S} = [0,1]$ with drift and diffusion coefficient

$$\tau(x) = \alpha + 1 - x(\alpha + \beta + 2), \quad \sigma^2(x) = 2x(1-x), \quad \alpha, \beta > -1$$

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Now there is an extra parameter $k \in (0, \beta + 1)$ in $\mathbf{Q}(x)$, which measures how the process moves through all the phases.

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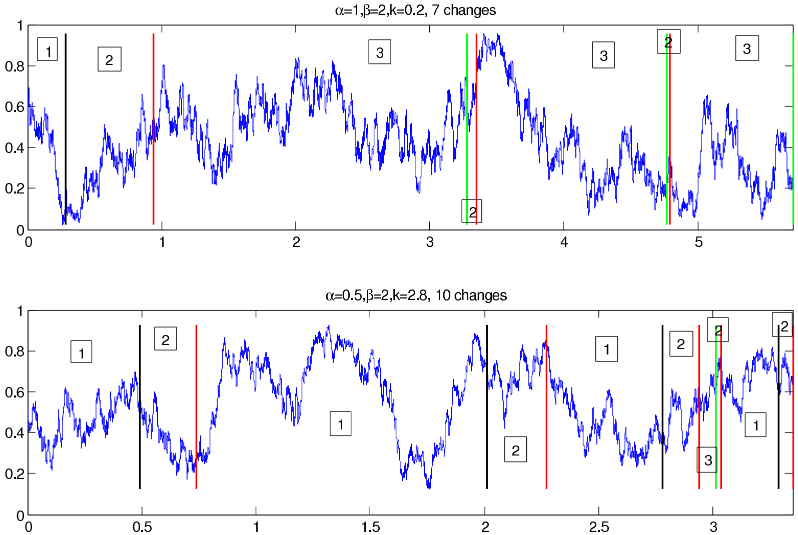
Some sample paths



Spectral analysis

There exists a family of orthonormal MOP $(\mathbf{\Phi}_n)_n$ (matrix-valued spherical functions) such that

$$\frac{1}{2}\mathbf{A}(x)\mathbf{\Phi}_n''(x) + \mathbf{B}(x)\mathbf{\Phi}_n'(x) + \mathbf{Q}(x)\mathbf{\Phi}_n(x) = \mathbf{\Phi}_n(x)\mathbf{\Gamma}_n$$

Transition probability matrix (MdI, 2012)

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$$\Rightarrow \psi(y) = \left(\int_0^1 \mathbf{e}_N^T \mathbf{W}(x)\mathbf{e}_N dx\right)^{-1} \mathbf{e}_N^T \mathbf{W}(y), \quad \mathbf{e}^T = (1,1,\ldots,1)$$

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Outline

3 YEARS AHEAD I

1. Accommodate examples of MOP to bivariate Markov processes
   - DISCRETE CASE: Find appropriate families of MOP for which the corresponding Jacobi matrix is either stochastic or the infinitesimal operator of a continuous-time quasi-birth-and-death process.
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2. Apply spectral methods to examples of real world bivariate Markov processes in the literature.

   Main problem: Find eigenfunctions and the corresponding weight matrix $\mathbf{W}$ for a given infinitesimal operator $\mathcal{A}$.

   Possible approach: Find weight matrices such that they are variations of continuous and scalar weights, or matrix-valued eigenfunctions combinations of scalar special functions.

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3. Noncommutative integrable systems

   Some of the families of MOP are related to noncommutative Painlevé equations via Riemann-Hilbert problems (Cafasso-MdI, 2013).

   Main tool: Integral representations of some families of MOP. We plan to in continue this work by considering connections of other families with some nonlinear differential equations, or with random matrices problems.

4. Other open problems

   - Finding new examples and phenomena of some of these MOP.
   - Electrostatic interpretation of the zeros of MOP (these are given by $\det(Q_n(x)) = 0$)
   - Asymptotic analysis of some of these families.
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5 years ahead I

1. Signal processing

Related to commuting integral and differential operators (for example prolate spheroidal wave functions).

From the discrete setting (Grünbaum, 1983), given a full matrix M (integral operator) one tries to find a tridiagonal matrix T (differential operator) with simple spectrum such that $MT = TM$.

Something similar can be done in the matrix case (Durán-Grünbaum, 2005), but now with a 2×2 block matrix $\mathbf{M}$. The reproducing kernel of $\mathbf{M}$ is given by

$$\mathbf{M}_{ij} = \int_0^{\Omega} \mathbf{Q}_i(x)\widetilde{\mathbf{W}}(x)\mathbf{Q}_j^*(x)dx, \quad i,j = 0,1,\ldots,T$$

Here the *band limiting* is the restriction to the interval $(0,\Omega)$ and the *time limiting* the restriction to the range $0,1,\ldots,T$.

They found a block tridiagonal $\mathbf{T}$ such that $\mathbf{MT} = \mathbf{TM}$.

Main goal: Explore this example in depth or find other examples.

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5 years ahead II

2. **Noncommutative harmonic oscillators**

It is very well known that the quantum harmonic oscillator can be studied using Hermite or wave functions $\psi_n(x)$. These functions are eigenfunctions of the Schrödinger equation

$$\psi_n''(x) - x^2\psi_n(x) = -(2n+1)\psi_n(x)$$

In the matrix case Parmeggiani-Wakayama, 2002 consider 2×2 differential operators of the form

$$\mathcal{A} = \mathbf{A}\left(-\frac{d^2}{dx^2} + \frac{x^2}{2}\right) + \mathbf{B}\left(x\frac{d}{dx} + \frac{1}{2}\right), \quad x \in \mathbb{R}$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$, $\mathbf{A} > 0, \mathbf{B} = -\mathbf{B}^T$ and $\mathbf{A} + i\mathbf{B} > 0$.

Main goal: relate these differential operators with MOP or switching diffusion models on $\mathbb{R}$, or extend to $N \times N$ examples.

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