The convex cone of weight matrices associated with a symmetric second order differential operator: some examples

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\(^1\)joint work with A. J. Durán
Outline

1 Introduction
   • Preliminaries
   • New phenomena

2 Convex cone of weight matrices
   • Definition
   • Method to find examples
   • Examples

3 New applications
Preliminaries

The theory of matrix valued orthogonal polynomials (MOP) on the real line was introduced by Krein in 1949.

Equivalence between \((P_n)_n\), orthonormal with respect to a (positive definite) weight matrix \(W\):

\[
\langle P_n, P_m \rangle_W = \int_a^b P_n(t)dW(t)P_m^*(t) = \delta_{nm}I, \quad n, m \geq 0
\]

and a three term recurrence relation:

\[
tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0
\]

\[
\det(A_{n+1}) \neq 0, \quad B_n = B_n^*
\]

- Systematic study: Asymptotics, zeros of MOP, quadrature formulae...
- Applications: scattering theory, times series and signal processing...
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Durán (1997): characterize orthonormal \((P_n)_n\) satisfying

\[ P''_n(t)F_2(t) + P'_n(t)F_1(t) + P_n(t)F_0(t) = \Lambda_n P_n(t), \quad n \geq 0 \]

\[ \text{grad } F_i \leq i, \quad \Lambda_n \quad \text{Hermitian} \]

Equivalent to the symmetry of

\[ D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^1 F_0(t), \quad \partial = \frac{d}{dt} \]

with \( P_n D = \Lambda_n P_n \)

\( D \) is symmetric with respect to \( W \) if \( \langle PD, Q \rangle_W = \langle P, QD \rangle_W \)

It has not been until very recently when the first examples appeared.
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How to get examples

- Matrix spherical functions associated to $P_n(\mathbb{C}) = SU(n+1)/U(n)$
  Grünbaum-Pacharoni-Tirao (2003)
- Durán-Grünbaum (2004):

Symmetry equations

$$F_2 W = WF_2^*$$

$$2(F_2 W)' = F_1 W + WF_1^*$$

$$(F_2 W)'' - (F_1 W)' + F_0 W = WF_0^*$$

$$\lim_{t \to x} F_2(t) W(t) = 0 = \lim_{t \to x} (F_1(t) W(t) - W(t) F_1^*(t)), \text{ for } x = a, b$$
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New phenomena

Algebra of differential operators: For a fixed family \((P_n)_n\) of MOP we consider the algebra over \(\mathbb{C}\)

\[
\mathcal{D}(W) = \left\{ D = \sum_{i=0}^{k} \partial^i F_i(t) : P_n D = \Lambda_n(D) P_n, \ n = 0, 1, 2, \ldots \right\}
\]

Scalar case: If \(F\) is the second order differential operator (Hermite, Laguerre or Jacobi), then any operator \(U\) such that \(U p_n = \lambda_n p_n\)

\[
U = \sum_{i=0}^{k} c_i F^i, \quad c_i \in \mathbb{C}
\]

\(\Rightarrow \mathcal{D}(\omega) \simeq \mathbb{C}[t]\)
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**Algebra of differential operators:** For a fixed family $(P_n)_n$ of MOP we consider the algebra over $\mathbb{C}$

$$\mathcal{D}(W) = \left\{ D = \sum_{i=0}^{k} \partial^i F_i(t) : P_nD = \Lambda_n(D)P_n, \ n = 0, 1, 2, \ldots \right\}$$

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$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{F}^i, \quad c_i \in \mathbb{C}$$

$$\Rightarrow \mathcal{D}(\omega) \simeq \mathbb{C}[t]$$
Matrix case: This algebra can be noncommutative and generated by several elements

- Existence of several linearly independent second order differential operators having a fixed family of MOP as eigenfunctions
- Existence of families of MOP satisfying odd order differential equations

Algebras: conjectures (Castro, Durán, Grünbaum, MdI) except one (Tirao) due to Castro–Grünbaum (2006)
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3. New applications
**Convex cone of weight matrices**

**Dual situation** to $\mathcal{D}(W)$: given a **fixed** differential operator $D$ we study:

$$\mathcal{Y}(D) = \{ W : \langle PD, Q \rangle_w = \langle P, QD \rangle_w, \text{ for all } P, Q \}$$

- If $\mathcal{Y}(D) \neq \emptyset$, it is a **convex cone**:
  $$W_1, W_2 \in \mathcal{Y}(D) \implies \gamma W_1 + \zeta W_2 \in \mathcal{Y}(D), \gamma, \zeta \geq 0 \text{ (one of them } \neq 0)$$

The weight matrices $W$ going along with a symmetric second order differential operator $D$ give examples where $\mathcal{Y}(D) \neq \emptyset$ (one dimensional)

We show the first examples of symmetric second order differential operators $D$ for which $\mathcal{Y}(D)$ is a **two dimensional** convex cone.

$\implies$ New phenomenon: (Monic) MOP $P_{n, \zeta/\gamma}$ with respect to $\gamma W_1 + \zeta W_2$

$$P_{n, \zeta/\gamma} D = \Gamma_n P_{n, \zeta/\gamma}$$
Convex cone of weight matrices

**Dual situation** to $\mathcal{D}(W)$: given a **fixed** differential operator $D$ we study:

$$\Upsilon(D) = \{ W : \langle PD, Q \rangle_W = \langle P, QD \rangle_W, \text{ for all } P, Q \}$$

- If $\Upsilon(D) \neq \emptyset$, it is a **convex cone**:
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$$P_{n, \zeta/\gamma} D = \Gamma_n P_{n, \zeta/\gamma}$$
Adding a Dirac delta distribution

All examples we consider are of the form

\[ \gamma W + \zeta M(t_0)\delta_{t_0}, \quad \gamma > 0, \zeta \geq 0, \quad t_0 \in \mathbb{R}, \]

where \( W \) is a weight matrix having several linearly independent symmetric second order differential operators and \( M(t_0) \) certain positive semidefinite matrix.

Scalar case \((\omega + m\delta_{t_0})\)

- Second order: there are NOT symmetric second order differential operators.
- Fourth order: \( t_0 \) at the endpoints of the support, which is NOT symmetric with respect to the original weight (Krall, 1941):
  - Laguerre type \( e^{-t} + M\delta_0 \)
  - Legendre type \( 1 + M(\delta_{-1} + \delta_1) \)
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Method to find examples

**Theorem (Durán–MdI, 2008)**

Let $W$ be a weight matrix and $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$. Assume that associated with the real point $t_0 \in \mathbb{R}$ there exists a Hermitian positive semidefinite matrix $M(t_0)$ satisfying

\[
F_2(t_0)M(t_0) = 0,
\]
\[
F_1(t_0)M(t_0) = 0,
\]
\[
F_0M(t_0) = M(t_0)F_0^*
\]

Then

$D$ is symmetric with respect to $W$ $\iff$ $D$ is symmetric with respect to $\gamma W + \zeta M(t_0)\delta t_0$
Example where \( t_0 \in \mathbb{R} \)

\[
W(t) = e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}
\]

Durán–Grünbaum (2004): weight matrix

**Symmetry equations** \( \Rightarrow \) Expression for the 5-dimensional (real) linear space of symmetric differential operators of order at most two

**Constraints:**

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F_2(t_0)M(t_0) = 0,
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$$t_0 = 0$$

\[ D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t), \]

\[ F_2(t) = \begin{pmatrix} 1 - at & -1 + a^2 t^2 \\ -1 & 1 + at \end{pmatrix} \]

\[ F_1(t) = \begin{pmatrix} -2a - 2t & 2a + 2(2 + a^2)t \\ 0 & -2t \end{pmatrix} \]

\[ F_0(t) = \begin{pmatrix} -1 & \frac{2 + a^2}{a^2} \\ \frac{4}{a^2} & 1 \end{pmatrix} \]

\[ M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]

⇒ \( D \) is symmetric with respect to the family of weight matrices

\[ \gamma(D) = \left\{ \gamma e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix} + \zeta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_0(t), \quad \gamma > 0, \zeta \geq 0 \right\} \]
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\[ D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t), \]

\[ F_2(t) = \begin{pmatrix}
-\xi_{a,t_0} + at_0 - at & -1 - (a^2t_0)t + a^2t^2 \\
-1 & -\xi_{a,t_0} + at
\end{pmatrix} \]

\[ F_1(t) = \begin{pmatrix}
-2a + 2\xi_{a,t_0}t & -2t_0 - 2a\xi_{a,t_0} + 2(2 + a^2)t \\
2t_0 & 2(\xi_{a,t_0} - at_0)t
\end{pmatrix} \]

\[ F_0(t) = \begin{pmatrix}
\xi_{a,t_0} + 2t_0/a & 2\frac{2 + a^2}{a^2} \\
\frac{4}{a^2} & -\xi_{a,t_0} - 2\frac{t_0}{a}
\end{pmatrix} \]

\[ M(t_0) = \begin{pmatrix}
(\xi_{t_0,a}^\pm)^2 & \xi_{t_0,a}^\pm \\
\xi_{t_0,a}^\pm & 1
\end{pmatrix}, \quad \xi_{a,t_0}^\pm = \frac{at_0 \pm \sqrt{4 + a^2t_0^2}}{2} \]

⇒ \[ \gamma(D) = \{ \gamma W + \zeta M(t_0) \delta_{t_0}, \quad \gamma > 0, \zeta \geq 0 \} \]
\[ D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t), \]

\[ F_2(t) = \begin{pmatrix} -\xi_{a,t_0}^\mp + at_0 - at & -1 - (a^2 t_0) t + a^2 t^2 \\ -1 & -\xi_{a,t_0}^\mp + at \end{pmatrix} \]

\[ F_1(t) = \begin{pmatrix} -2a + 2\xi_{a,t_0}^\mp t & -2t_0 - 2a\xi_{a,t_0}^\mp + 2(2 + a^2)t \\ 2t_0 & 2(\xi_{a,t_0}^\mp - at_0) t \end{pmatrix} \]

\[ F_0(t) = \begin{pmatrix} \xi_{a,t_0}^\mp + 2\frac{t_0}{a} & \frac{2 + a^2}{a^2} \\ \frac{4}{a^2} & -\xi_{a,t_0}^\mp - 2\frac{t_0}{a} \end{pmatrix} \]

\[ M(t_0) = \begin{pmatrix} (\xi_{t_0,a}^\pm)^2 & \xi_{t_0,a}^\pm \\ \xi_{t_0,a}^\pm & 1 \end{pmatrix}, \quad \xi_{a,t_0}^\pm = \frac{at_0 \pm \sqrt{4 + a^2 t_0^2}}{2} \]

\[ \Rightarrow \gamma(D) = \{ \gamma W + \zeta M(t_0) \delta t_0, \quad \gamma > 0, \zeta \geq 0 \} \]
Another example where $t_0 \in \mathbb{R}$

$$W(t) = t^\alpha e^{-t} \begin{pmatrix} t^2 + a^2(t - 1)^2 & a(t - 1) \\ a(t - 1) & 1 \end{pmatrix}, \quad t > 0, \quad \alpha > -1$$


$t_0 = -1, \alpha = 0, a = 1$

$$D = \partial^2 \begin{pmatrix} -\sqrt{2}(\sqrt{2}+2t) \quad -1 + 2t^2 \\ 2 \quad \sqrt{2}(\sqrt{2}-2t) \end{pmatrix} +$$

$$\partial^1 \begin{pmatrix} (1 - \sqrt{2})(5 + 2\sqrt{2} - t) \quad -2\sqrt{2} + 6t \\ -2 \quad (1 + \sqrt{2})(t - 1) \end{pmatrix} + \partial^0 \begin{pmatrix} -1 + \frac{\sqrt{2}}{2} \quad \frac{3}{2} \\ \frac{1}{2} \quad 1 - \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} 3 + 2\sqrt{2} \quad -1 - \sqrt{2} \\ -1 - \sqrt{2} \quad 1 \end{pmatrix}$$
Another example where $t_0 \in \mathbb{R}$

$$W(t) = t^\alpha e^{-t} \begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}, \quad t > 0, \quad \alpha > -1$$


$t_0 = -1, \alpha = 0, a = 1$

$$D = \partial^2 \left( -\frac{\sqrt{2}(\sqrt{2}+2t)}{2} \begin{pmatrix} -1 + 2t^2 \\ 1 \end{pmatrix} + \sqrt{2}(\sqrt{2}-2t) \right) +$$

$$\partial^1 \left( \begin{pmatrix} (1 - \sqrt{2})(5 + 2\sqrt{2} - t) \\ -2 \end{pmatrix} -2\sqrt{2} + 6t \begin{pmatrix} 1 + \sqrt{2}(t-1) \end{pmatrix} \right) + \partial^0 \left( -1 + \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \begin{pmatrix} -\frac{1}{2} \end{pmatrix} & -\frac{1}{2} \end{pmatrix} \right)$$

$$M = \begin{pmatrix} 3 + 2\sqrt{2} & -1 - \sqrt{2} \\ -1 - \sqrt{2} & 1 \end{pmatrix}$$
Example where $\delta_0$ of size $N \times N$

\[
W_{\alpha, \nu_1, \ldots, \nu_{N-1}}(t) = t^\alpha e^{-t} e^{At} t^{1/2} J t^{1/2} J^* e^{A^* t}, \quad \alpha > -1, \quad t > 0
\]

\[
A = \begin{pmatrix}
0 & \nu_1 & 0 & \cdots & 0 \\
0 & 0 & \nu_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{N-1} \\
0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad \nu_i \in \mathbb{R}\setminus\{0\}, \quad J = \begin{pmatrix}
N-1 & 0 & \cdots & 0 & 0 \\
0 & N-2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

Durán–MdI (2008)

Second order differential operators

\[
D_1 = \partial^2 t l + \partial^1 [(\alpha + 1) l + J + t(A - I)] + \partial^0 [(J + \alpha l)A - J]
\]

\[
D_2 = \partial^2 t (J - At) + \partial^1 ((1 + \alpha) l + J) J + Y - t (J + (\alpha + 2) A + Y^* - ad_A Y)
\]

\[
+ \partial^0 \frac{N-1}{\nu_{N-1}^2} [J - (\alpha l + J) A]
\]

Durán–MdI (2008)
Example where $\delta_0$ of size $N \times N$

$$W_{\alpha, \nu_1, \ldots, \nu_{N-1}}(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}} J t^{\frac{1}{2}} J^* e^{A^* t}, \quad \alpha > -1, \ t > 0$$

$$A = \begin{pmatrix}
0 & \nu_1 & 0 & \cdots & 0 \\
0 & 0 & \nu_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_{N-1} \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \nu_i \in \mathbb{R}\{0\}, \quad J = \begin{pmatrix}
N-1 & 0 & \cdots & 0 & 0 \\
0 & N-2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

Durán–MdI (2008)

Second order differential operators

$$D_1 = \partial^2 tl + \partial^1 [((\alpha + 1)I + J + t(A - I)] + \partial^0 [(J + \alpha l)A - J]$$

$$D_2 = \partial^2 t(J - At) + \partial^1 (1 + (1 + \alpha)I + J)J + Y - t(J + (\alpha + 2)A + Y^* - \text{ad}_A Y) + \partial^0 \frac{N-1}{\sqrt{N-1}} [J - (\alpha l + J)A]$$
Example where $\delta_0$ of size $N \times N$

**Symmetry equations** $\Rightarrow$ Expression for the 3-dimensional (real) linear space of symmetric differential operators of order at most two

\[ t_0 = 0 \]

\[ D = -(N - 1)D_1 + D_2 \]

\[ M = v^*v \]

\[ v = \sum_{j=1}^{N-1} \left( \prod_{k=1}^{N-j} \frac{\nu_{N-k}(\alpha + k)}{k} \right) e_j + e_N \]

$\Rightarrow D$ is symmetric with respect to the family of weight matrices

\[ \gamma(D) = \{ \gamma W_{\alpha, \nu_1, \ldots, \nu_{N-1}}(t) + \zeta M\delta_0(t), \quad \gamma > 0, \ \zeta \geq 0 \} \]
Example where $\delta_0$ of size $N \times N$

Symmetry equations $\Rightarrow$ Expression for the 3-dimensional (real) linear space of symmetric differential operators of order at most two

$t_0 = 0$

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D = -(N - 1)D_1 + D_2
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\]

$\Rightarrow$ $D$ is symmetric with respect to the family of weight matrices

\[
\gamma(D) = \{\gamma W_{\alpha, \nu_1, \ldots, \nu_{N-1}}(t) + \zeta M\delta_0(t), \quad \gamma > 0, \zeta \geq 0\}
\]
Example where \( \delta_0 \) of size \( N \times N \)

**Symmetry equations** \( \Rightarrow \) Expression for the 3-dimensional (real) linear space of symmetric differential operators of order at most two

\[
\begin{align*}
    D &= -(N - 1)D_1 + D_2 \\
    M &= v^*v \\
    v &= \sum_{j=1}^{N-1} \left( \prod_{k=1}^{N-j} \frac{\nu_{N-k}(\alpha + k)}{k} \right) e_j + e_N
\end{align*}
\]

\( \Rightarrow \) \( D \) is symmetric with respect to the family of weight matrices

\[
\gamma(D) = \{ \gamma W_{\alpha, \nu_1, \ldots, \nu_{N-1}}(t) + \zeta M \delta_0(t), \quad \gamma > 0, \ \zeta \geq 0 \}
\]
Outline

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New applications

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Time-and-band limiting


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Quantum mechanics


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New applications

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