The convex cone of weight matrices associated with a symmetric second order differential operator: some examples[1]

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[1]joint work with A. J. Durán

Outline

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1. Introduction
   - Preliminaries
   - New phenomena

2. Convex cone of weight matrices
   - Definition
   - Method to find examples
   - Examples

3. New applications

Preliminaries

The theory of matrix valued orthogonal polynomials (MOP) on the real line was introduced by Krein in 1949

Equivalence between $(P_n)_n$, orthonormal with respect to a (positive definite) weight matrix W

$$\langle P_n, P_m \rangle_W = \int_a^b P_n(t) dW(t) P_m^*(t) = \delta_{nm} I, \quad n, m \geqslant 0$$

and a three term recurrence relation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geqslant 0$$
$$\det(A_{n+1}) \neq 0, \quad B_n = B_n^*$$

- Systematic study: Asymptotics, zeros of MOP, quadrature formulae... Applications: scattering theory, times series and signal processing...

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Durán (1997): characterize orthonormal $(P_n)_n$ satisfying

$$P_n''(t)F_2(t) + P_n'(t)F_1(t) + P_n(t)F_0(t) = \Lambda_n P_n(t), \quad n \geqslant 0$$

$$\text{grad } F_i \leqslant i, \quad \Lambda_n \quad \text{Hermitian}$$

Equivalent to the symmetry of

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^1 F_0(t), \quad \partial = \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\text{with} \quad P_n D = \Lambda_n P_n$$

D is symmetric with respect to W if $\langle PD, Q\rangle_W = \langle P, QD\rangle_W$

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How to get examples

- Matrix spherical functions associated to $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$ Grünbaum-Pacharoni-Tirao (2003)
- Durán-Grünbaum (2004):

Symmetry equations

$$F_2 W = W F_2^*$$

$$2(F_2 W)' = F_1 W + W F_1^*$$

$$(F_2 W)'' - (F_1 W)' + F_0 W = W F_0^*$$

$$\lim_{t \longrightarrow x} F_2(t) W(t) = 0 = \lim_{t \longrightarrow x} (F_1(t) W(t) - W(t) F_1^*(t)), \quad \text{for} \quad x = a, b$$

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New phenomena

Algebra of differential operators: For a fixed family $(P_n)_n$ of MOP we consider the algebra over $\mathbb{C}$

$$\mathcal{D}(W) = \left\{ D = \sum_{i=0}^{k} \partial^i F_i(t) : P_n D = \Lambda_n(D) P_n, \; n = 0, 1, 2, \ldots \right\}$$

Scalar case: If $\mathcal{F}$ is the second order differential operator (Hermite, Laguerre or Jacobi), then any operator $\mathcal{U}$ such that $\mathcal{U} p_n = \lambda_n p_n$

$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{F}^i, \quad c_i \in \mathbb{C}$$

$$\Rightarrow \mathcal{D}(\omega) \simeq \mathbb{C}[t]$$

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Matrix case: This algebra can be noncommutative and generated by several elements

- Existence of several linearly independent second order differential operators having a fixed family of MOP as eigenfunctions
- Existence of families of MOP satisfying odd order differential equations

Algebras: conjectures (Castro, Durán, Grünbaum, MdI)
except one (Tirao) due to Castro–Grünbaum (2006)

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Convex cone of weight matrices

Dual situation to $\mathcal{D}(W)$: given a fixed differential operator D we study:

$$\Upsilon(D) = \{W : \langle PD, Q\rangle_W = \langle P, QD\rangle_W, \quad \text{for all} \quad P, Q\}$$

- If $\Upsilon(D) \neq \emptyset$, it is a convex cone:
  $W_1, W_2 \in \Upsilon(D) \Rightarrow \gamma W_1 + \zeta W_2 \in \Upsilon(D)$, $\gamma, \zeta \geqslant 0$ (one of them $\neq 0$)

The weight matrices W going along with a symmetric second order differential operator D give examples where $\Upsilon(D) \neq \emptyset$ (one dimensional)
We show the first examples of symmetric second order differential operators D for which $\Upsilon(D)$ is a two dimensional convex cone.
$\Rightarrow$ New phenomenon: (Monic) MOP $P_{n,\zeta/\gamma}$ with respect to $\gamma W_1 + \zeta W_2$

$$P_{n,\zeta/\gamma} D = \Gamma_n P_{n,\zeta/\gamma}$$

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Adding a Dirac delta distribution

All examples we consider are of the form

$$\gamma W + \zeta M(t_0)\delta_{t_0}, \quad \gamma > 0, \zeta \geqslant 0, \quad t_0 \in \mathbb{R},$$

where W is a weight matrix having several linearly independent symmetric second order differential operators and $M(t_0)$ certain positive semidefinite matrix.

Scalar case $(\omega + m\delta_{t_0})$

- Second order: there are NOT symmetric second order differential operators.
- Fourth order: t_0 at the endpoints of the support, which is NOT symmetric with respect to the original weight (Krall, 1941):

$$\text{Laguerre type } e^{-t} + M\delta_0$$
$$\text{Legendre type } 1 + M(\delta_{-1} + \delta_1)$$
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Method to find examples

Theorem (Durán–MdI, 2008)

Let W be a weight matrix and $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$. Assume that associated with the real point $t_0 \in \mathbb{R}$ there exists a Hermitian positive semidefinite matrix $M(t_0)$ satisfying

$$F_2(t_0)M(t_0) = 0,$$
$$F_1(t_0)M(t_0) = 0,$$
$$F_0 M(t_0) = M(t_0)F_0^*$$

Then

D is symmetric with respect to W

$$\Leftrightarrow$$

D is symmetric with respect to $\gamma W + \zeta M(t_0)\delta_{t_0}$

Example where $t_0 \in \mathbb{R}$

$$W(t) = e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}$$

Durán–Grünbaum (2004): weight matrix

Castro–Grünbaum (2006): Algebra of differential operators

Symmetry equations $\Rightarrow$ Expression for the 5-dimensional (real) linear space of symmetric differential operators of order at most two

Constraints:

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$t_0 = 0$

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t),$$

$$F_2(t) = \begin{pmatrix} 1 - at & -1 + a^2t^2 \\ -1 & 1 + at \end{pmatrix}$$

$$F_1(t) = \begin{pmatrix} -2a - 2t & 2a + 2(2 + a^2)t \\ 0 & -2t \end{pmatrix}$$

$$F_0(t) = \begin{pmatrix} -1 & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ D is symmetric with respect to the family of weight matrices

$$\Upsilon(D) = \left\{ \gamma e^{-t^2} \begin{pmatrix} 1 + a^2t^2 & at \\ at & 1 \end{pmatrix} + \zeta \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_0(t), \quad \gamma > 0, \zeta \geqslant 0 \right\}$$

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$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t),$$

$$F_2(t) = \begin{pmatrix} -\xi^{\mp}_{a,t_0} + at_0 - at & -1-(a^2t_0)t + a^2t^2 \\ -1 & -\xi^{\mp}_{a,t_0} + at \end{pmatrix}$$

$$F_1(t) = \begin{pmatrix} -2a + 2\xi^{\mp}_{a,t_0} t & -2t_0 - 2a\xi^{\mp}_{a,t_0} + 2(2+a^2)t \\ 2t_0 & 2(\xi^{\mp}_{a,t_0} - at_0)t \end{pmatrix}$$

$$F_0(t) = \begin{pmatrix} \xi^{\mp}_{a,t_0} + 2\frac{t_0}{a} & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & -\xi^{\mp}_{a,t_0} - 2\frac{t_0}{a} \end{pmatrix}$$

$$M(t_0) = \begin{pmatrix} (\xi^{\pm}_{t_0,a})^2 & \xi^{\pm}_{t_0,a} \\ \xi^{\pm}_{t_0,a} & 1 \end{pmatrix}, \quad \xi^{\pm}_{a,t_0} = \frac{at_0 \pm \sqrt{4+a^2t_0^2}}{2}$$

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$$\Rightarrow \Upsilon(D) = \{\gamma W + \zeta M(t_0)\delta_{t_0}, \quad \gamma > 0, \zeta \geqslant 0\}$$

Another example where $t_0 \in \mathbb{R}$

$$W(t) = t^{\alpha} e^{-t} \begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}, \quad t > 0, \quad \alpha > -1$$

Durán–Grünbaum (2004)

$t_0 = -1, \alpha = 0, a = 1$

$$D = \partial^2 \begin{pmatrix} -\frac{\sqrt{2}(\sqrt{2}+2t)}{2} & -1 + 2t^2 \\ 1 & \frac{\sqrt{2}(\sqrt{2}-2t)}{2} \end{pmatrix} +$$

$$\partial^1 \begin{pmatrix} (1-\sqrt{2})(5+2\sqrt{2}-t) & -2\sqrt{2}+6t \\ -2 & (1+\sqrt{2})(t-1) \end{pmatrix} + \partial^0 \begin{pmatrix} -1 + \frac{\sqrt{2}}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 - \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} 3 + 2\sqrt{2} & -1 - \sqrt{2} \\ -1 - \sqrt{2} & 1 \end{pmatrix}$$

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$t_0 = -1, \alpha = 0, a = 1$

$$D = \partial^2 \begin{pmatrix} -\frac{\sqrt{2}(\sqrt{2}+2t)}{2} & -1 + 2t^2 \\ 1 & \frac{\sqrt{2}(\sqrt{2}-2t)}{2} \end{pmatrix} +$$
$$\partial^1 \begin{pmatrix} (1-\sqrt{2})(5+2\sqrt{2}-t) & -2\sqrt{2}+6t \\ -2 & (1+\sqrt{2})(t-1) \end{pmatrix} + \partial^0 \begin{pmatrix} -1+\frac{\sqrt{2}}{2} & \frac{3}{2} \\ \frac{1}{2} & 1-\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} 3+2\sqrt{2} & -1-\sqrt{2} \\ -1-\sqrt{2} & 1 \end{pmatrix}$$

Example where δ_0 of size $N \times N$

$$W_{\alpha,\nu_1,\dots,\nu_{N-1}}(t) = t^{\alpha} e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^* t}, \ \alpha > -1, \ t > 0$$

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \nu_i \in \mathbb{R}\backslash\{0\}, \ J = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Durán–MdI (2008)

Second order differential operators

$$D_1 = \partial^2 tI + \partial^1[(\alpha+1)I + J + t(A - I)] + \partial^0[(J + \alpha I)A - J]$$

$$D_2 = \partial^2 t(J - At) + \partial^1((1+\alpha)I + J)J + Y - t(J + (\alpha+2)A + Y^* - \mathrm{ad}_A Y)$$

$$+ \partial^0 \frac{N-1}{\nu_{N-1}^2}[J - (\alpha I + J)A]$$

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Example where δ_0 of size $N \times N$

Symmetry equations $\Rightarrow$ Expression for the 3-dimensional (real) linear space of symmetric differential operators of order at most two

$t_0 = 0$

$$D = -(N-1)D_1 + D_2$$

$$M = v^* v$$

$$v = \sum_{j=1}^{N-1} \left(\prod_{k=1}^{N-j} \frac{\nu_{N-k}(\alpha + k)}{k} \right) e_j + e_N$$

$\Rightarrow$ D is symmetric with respect to the family of weight matrices

$$\Upsilon(D) = \{\gamma W_{\alpha,\nu_1,\dots,\nu_{N-1}}(t) + \zeta M \delta_0(t), \quad \gamma > 0, \zeta \geqslant 0\}$$

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