A family of quasi-birth-and-death processes coming from the theory of matrix valued orthogonal polynomials*,†

F. A. Grünbaum and **M. D. de la Iglesia**

Department of Mathematics. University of California, Berkeley. Berkeley, CA 94720 U.S.A., **Dpto. de Análisis Matemático, Universidad de Sevilla. Apdo (P. O. BOX) 1160. 41080 Sevilla. Spain**

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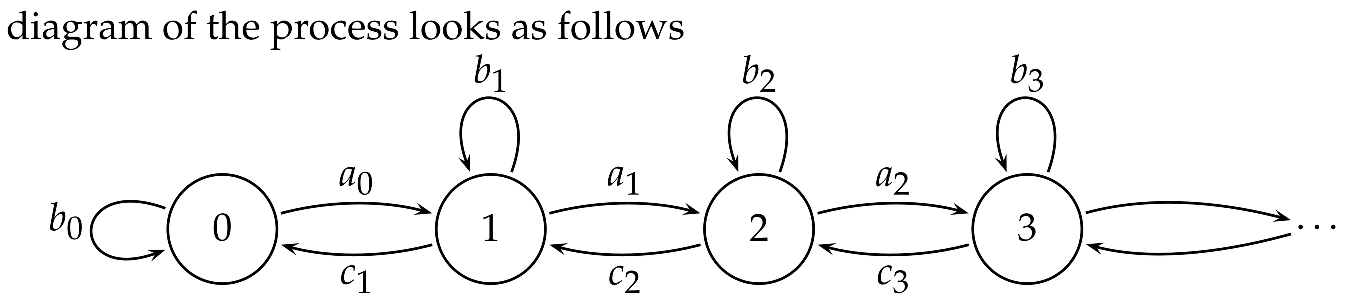
Birth and death processes

The birth-and-death processes are a very special kind of Markov chain on the space of non-negative integers. At each discrete unit of time a transition is allowed from state i to state j with probability P_{ij} and we put $P_{ij}=0$ if $|i-j|>1$. The one-step transition probability matrix is given by

$$P=\begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The diagram of the process looks as follows



The problem here is to obtain an expression of the so called n-step transition probability matrix, giving the probability of going between any two states in n steps. S. Karlin and J. McGregor, [3], obtained a neat representation formula for this quantity of interest. Introducing the sequence of polynomials $(q_n(x))_n$ by the conditions $q_{-1}(x)=0$, $q_0(x)=1$, using the notation $\phi=(q_0(x),q_1(x),\cdots)^T$ and insisting on the recursion relation $P\phi=x\phi$, it is possible to prove the existence of a unique measure $d\psi(x)$ supported in $[-1,1]$ that makes the polynomials $(q_n(x))_n$ orthogonal.
Hence, one gets the Karlin-McGregor representation formula

$$P_{ij}^n=\int_{-1}^1 x^n q_i(x)q_j(x)d\psi(x)\Big/\int_{-1}^1 q_j(x)^2 d\psi(x)$$

Another probabilistic object of interest is the so called the invariant (stationary) distribution, i.e. the (unique up to scalars) row vector $\boldsymbol{\pi}=(\pi_0,\pi_1,\pi_2,\dots)$ such that $\boldsymbol{\pi}P=\boldsymbol{\pi}$. In this case, the coefficients of $\boldsymbol{\pi}$ can be computed either from the matrix P itself or from the norms of the orthogonal polynomials:

$$\pi_i=\frac{a_0a_1\cdots a_{i-1}}{c_1c_2\cdots c_i}=\frac{1}{\int_{-1}^1 q_i^2(x)d\psi(x)}=\frac{1}{\|q_i\|^2}.$$

Quasi-birth-and-death processes

The quasi-birth-and-death processes are a natural extension of the birth-and-death processes where the one-step transition probability matrix is a block tridiagonal matrix (each block of size $d\times d$):

$$P=\begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1}$$

We have now a two dimensional Markov chain with discrete time. The state space consists of the pair of integers (i,j), $i\in\{0,1,2,\dots\}$, $j\in\{1,\dots,d\}$. The first component is usually called the *level* and the second one the *phase*. This indicates that in one unit of time a transition can change the phase without changing the level, or can change the level (and possibly the phase) to either of the adjacent levels. The probability of going in one step from state (i,j) to state (i',j') is given by the (j,j') element of the block $P_{i,i'}$.
As before, introducing the matrix polynomials $(Q_n(x))_n$ by the conditions $Q_{-1}(x)=0$, $Q_0(x)=I$, using the notation $\Phi=(Q_0^T(x),Q_1^T(x),\cdots)^T$ and insisting on the recursion relation $P\Phi=x\Phi$, it is possible to prove (under certain technical conditions over the coefficients A_n,B_n,C_n) the existence of a unique weight matrix $dW(x)$ supported in $[-1,1]$ that makes the polynomials $(Q_n(x))_n$ orthogonal (with respect to the matrix valued inner product $(R,S)_W=\int_{\mathbb{R}}R(x)dW(x)S^T(x)$).
Again, one gets the Karlin-McGregor representation formula (see [1, 2])

$$P_{ij}^n=\left(\int_{-1}^1 x^nQ_i(x)dW(x)Q_j^*(x)\right)\left(\int_{-1}^1 Q_j(x)dW(x)Q_j^*(x)\right)^{-1} \tag{2}$$

Now, the problem of computing an invariant distribution row vector with non-negative entries π_j^i

$$\boldsymbol{\pi}=(\boldsymbol{\pi}^0;\boldsymbol{\pi}^1;\cdots)\equiv(\pi_1^0,\pi_2^0,\dots,\pi_d^0;\pi_1^1,\pi_2^1,\dots,\pi_d^1;\cdots)$$

such that $\boldsymbol{\pi}P=\boldsymbol{\pi}$ leads to a complicated system of equations. A formula that relates the entries π_j^i with the (matrix valued) norms of the orthogonal matrix polynomials has not been found yet.

The example

In what follows we shall use E_{ij} to denote the matrix with entry (i,j) equal 1 and 0 elsewhere, where the indices i,j run from 0 on.
Let $d\in\{1,2,3,\dots\}$, $\alpha,\beta>-1$ and $0<k<\beta+1$. Let us consider the $d\times d$ weight matrix $\widetilde{W}=T^*WT$, where the weight matrix

$$W(x)=x^\alpha(1-x)^\beta\sum_{i,j=0}^{d-1}\left(\sum_{r=0}^{d-1}\binom{r}{i}\binom{r}{j}\binom{d+k-r-2}{d-r-1}\binom{\beta-k+r}{r}(1-x)^{i+j}x^{d-r-1}\right)E_{ij},\quad x\in[0,1],$$

comes from group representation theory (see [4]) and T is the nonsingular upper triangular matrix

$$T=\sum_{i\le j}(-1)^i\frac{(-j)_i}{(1-d)_i}\frac{(\alpha+\beta-k+j+1)_i}{(\beta-k+1)_i}E_{ij},$$

where $(a)_n$ denotes the usual Pochhammer symbol.
This conjugation allows us to produce a particular family of matrix valued orthogonal polynomials $(Q_n(x))_n$ with leading coefficient

$$\sum_{i\le j}(-1)^{i+j}\binom{j}{i}(\alpha+\beta-k+2i+1)\frac{(n)_{j-i}(k+n)_{d-j-1}(\alpha+\beta-k+n+j+1)_i(\alpha+\beta+n+d+j)_n}{(k)_{d-j-1}(\beta+d)_n(\alpha+\beta-k+i+1)_{j+1}}E_{ji},$$

such that we have the following theorem:

Theorem 1. *The family of orthogonal polynomials introduced above satisfies the three term recursion relation*

$$xQ_n(x)=A_nQ_{n+1}(x)+B_nQ_n(x)+C_nQ_{n-1}(x),\quad n\ge 0,$$

where $Q_{-1}(x)=0$, $Q_0(x)=I$,

$$A_n=\sum_{i=0}^{d-1}a_1^2(n,i)b_3^2(n+1,i)E_{ii}+\sum_{i=1}^{d-1}a_1^2(n,i)b_2^2(n+1,i)E_{i,i-1},\quad n\ge 0,$$

$$C_n=\sum_{i=0}^{d-1}a_3^2(n,i)b_1^2(n,i)E_{ii}+\sum_{i=0}^{d-2}a_2^2(n,i)b_1^2(n,i+1)E_{i,i+1},\quad n\ge 1,\quad \textit{and}$$

$$B_n=\sum_{i=0}^{d-1}(a_1^2(n,i)b_1^2(n+1,i)+a_2^2(n,i)b_2^2(n,i+1)+a_3^2(n,i)b_3^2(n,i))E_{ii}+\sum_{i=0}^{d-2}a_2^2(n,i)b_3^2(n,i+1)E_{i,i+1}+\sum_{i=1}^{d-1}a_3^2(n,i)b_2^2(n,i)E_{i,i-1},\quad n\ge 0,$$

where

$$a_1^2(n,i)=\frac{(k+n)(\beta+n+d)}{(k+n+d-i-1)(\alpha+\beta+2n+d+i)},\quad a_2^2(n,i)=\frac{(d-i-1)(\beta-k+i+1)}{(k+n+d-i-1)(\alpha+\beta-k+n+2i+1)},$$

$$a_3^2(n,i)=\frac{(\alpha+\beta-k+n+d+i)(\alpha+n+i)}{(\alpha+\beta-k+n+2i+1)(\alpha+\beta+2n+d+i)},\ b_1^2(n,i)=\frac{n(k+n+d-1)}{(k+n+d-i-1)(\alpha+\beta+2n+d+i-1)},$$

$$b_2^2(n,i)=\frac{i(k+d-i-1)}{(k+n+d-i-1)(\alpha+\beta-k+n+2i)},\quad b_3^2(n,i)=\frac{(\alpha+\beta-k+n+i)(\alpha+\beta+n+d+i-1)}{(\alpha+\beta-k+n+2i)(\alpha+\beta+2n+d+i-1)}$$

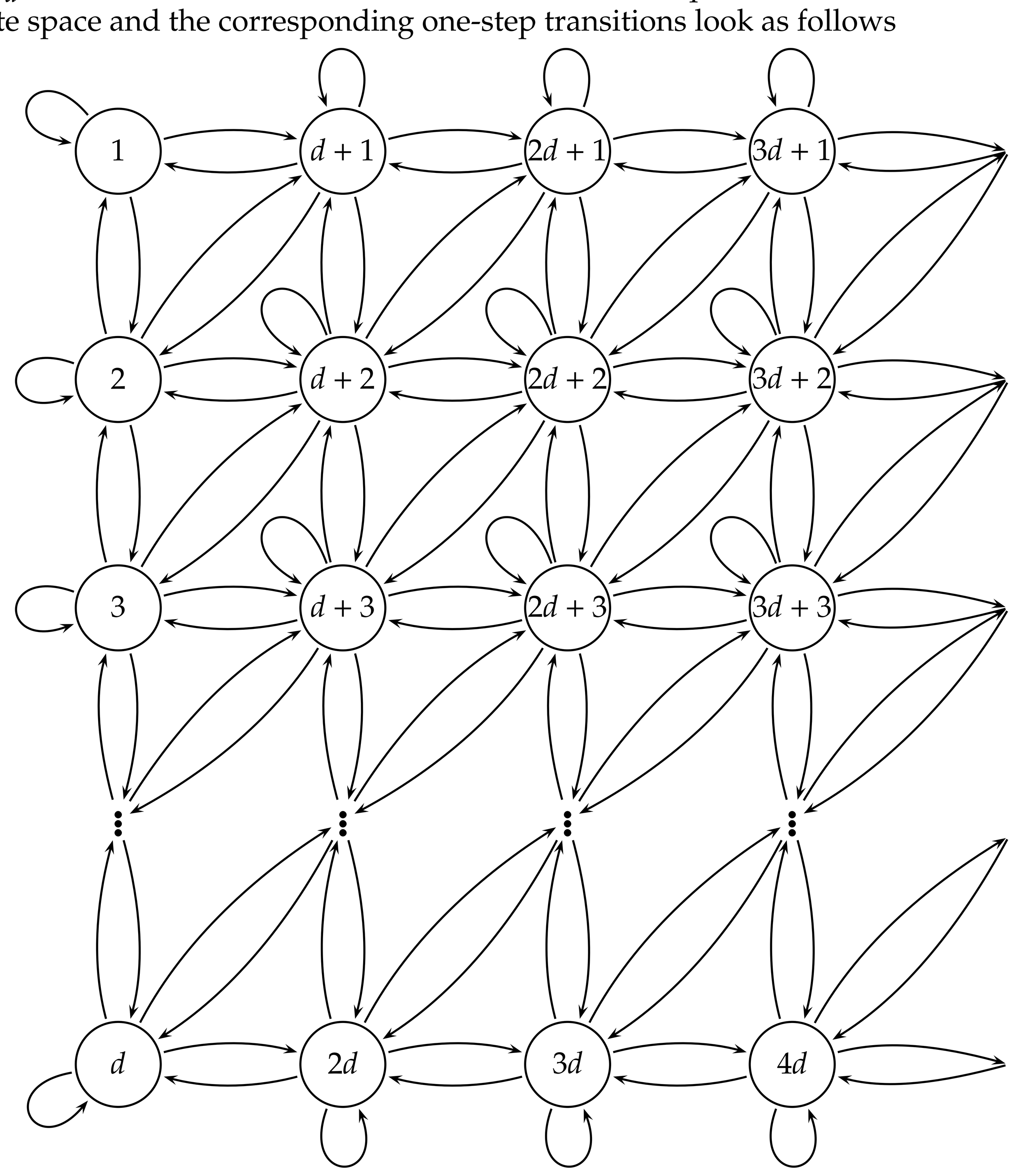
Moreover, the block tridiagonal Jacobi matrix (like in (1)) is stochastic and the matrix valued norms of $(Q_n(x))_n$ *are given by the diagonal matrices*

$$\|Q_n\|_{\widetilde{W}}^2=\sum_{i=0}^{d-1}(-1)^i\frac{\Gamma(n+1)\Gamma(\beta+d)\Gamma(\alpha+n+i+1)(1-k-d-n)_i}{\binom{d-1}{i}\Gamma(d)\Gamma(\alpha+\beta+d+i+2n+1)}\times\frac{(k+d-i-1)_n(\alpha+\beta+d+i+n)_n(\alpha+\beta-k+i+n+1)_d}{(\alpha+\beta-k+2i+n+1)(k)_n(\beta-k+1)_i(\beta+d)_n}E_{ii},$$

Hence, we obtain an example of a several parameter family of quasi-birth-and-death processes with an arbitrary (finite) number of phases, where we can make use of the Karlin-McGregor formula (2), since we have an explicit expression of the weight matrix $\widetilde{W}$.
Also we get the explicit computation of an invariant distribution. Since our family $(Q_n(x))_n$ has the remarkable property that their norms are diagonal matrices, this provides us, for each n, with d scalars which could be used (inspired by the case of birth-and-death processes) as a way of getting an invariant row vector $\boldsymbol{\pi}=(\boldsymbol{\pi}^0;\boldsymbol{\pi}^1;\cdots)$ with the property that $\boldsymbol{\pi}P=\boldsymbol{\pi}$ by the recipe

$$\boldsymbol{\pi}^n=\mathbf{e}_d(\|Q_n\|_{\widetilde{W}}^2)^{-1},\quad n\ge 0,$$

where $\mathbf{e}_d$ is the d-dimensional row vector with all entries equal to 1.
The state space and the corresponding one-step transitions look as follows



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Manuel D. de la Iglesia E-mail: mdi29@us.es WWW: http://euler.us.es/~mdi29/

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