The Riemann-Hilbert problem for matrix-valued orthogonal polynomials[1]

Manuel Domínguez de la Iglesia

Department of Mathematics, K. U. Leuven

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[1]joint work with Andrei Martínez Finkelshtein

Outline

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Let W be a $N \times N$ a weight matrix such that $dW(x) = W(x)dx$.

We can construct a family of OMP such that

$$\int_{\mathbb{R}} P_n(x)W(x)P_m^*(x)dx = \delta_{n,m}I, \quad n, m \geqslant 0$$

$$P_n(x) = \gamma_n(x^n + a_{n,n-1}x^{n-1} + \cdots) = \gamma_n\widehat{P}_n(x)$$

The matrix-valued polynomials of the second kind, defined by

$$Q_n(x) = \int_{\mathbb{R}} \frac{P_n(t)W(t)}{t-x}dt, \quad n \geqslant 0$$

$(P_n)_n$ and $(Q_n)_n$ satisfy a three term recurrence relation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geqslant 0$$

$$\det(A_{n+1}) \neq 0, \quad B_n = B_n^*$$

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The coefficients of the TTRR satisfy

$$A_n = \gamma_{n-1}\gamma_n^{-1}, \quad B_n = \gamma_n(a_{n,n-1} - a_{n+1,n})\gamma_n^{-1}$$

The TTRR for monic OMP

$$x\widehat{P}_n(x) = \widehat{P}_{n+1}(x) + \alpha_n \widehat{P}_n(x) + \beta_n \widehat{P}_{n-1}(x), \quad n \geqslant 0$$
$$\alpha_n = a_{n,n-1} - a_{n+1,n}, \quad \beta_n = (\gamma_n^* \gamma_n)^{-1}(\gamma_{n-1}^* \gamma_{n-1})$$

Second-order differential equations of hypergeometric type

$$P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0(x) = \Lambda_n P_n(x), \quad n \geqslant 0$$
$$\deg F_i \leqslant i, \quad \Lambda_n \quad \text{Hermitian}$$

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Outline

Solution of the RH for OMP

$Y^n : \mathbb{C} \to \mathbb{C}^{2N\times 2N}$ such that

1. **Analyticity.** Y^n is analytic in $\mathbb{C} \setminus \mathbb{R}$
2. **Jump Condition.** $Y^n_+(x) = Y^n_-(x) \begin{pmatrix} I & W(x) \\ 0 & I \end{pmatrix}$ when $x \in \mathbb{R}$
3. **Normalization.** $Y^n(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n I & 0 \\ 0 & z^{-n} I \end{pmatrix}$ as $z \to \infty$

For $n \geqslant 1$ the unique solution of the RH problem above is given by

$$Y^n(z) = \begin{pmatrix} \widehat{P}_n(z) & \dfrac{1}{2\pi i} \displaystyle\int_{\mathbb{R}} \frac{\widehat{P}_n(t) W(t)}{t - z} dt \\ -2\pi i \gamma^*_{n-1} \gamma_{n-1} \widehat{P}_{n-1}(z) & -\gamma^*_{n-1} \gamma_{n-1} \displaystyle\int_{\mathbb{R}} \frac{\widehat{P}_{n-1}(t) W(t)}{t - z} dt \end{pmatrix}$$

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Also we find a solution of the inverse

$$(Y^n)^{-1} = \begin{pmatrix} -\left(\int_{\mathbb{R}} \frac{W(t)\widehat{P}^*_{n-1}(t)}{t-z}dt\right)\gamma^*_{n-1}\gamma_{n-1} & -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{W(t)\widehat{P}^*_n(t)}{t-z}dt \\ 2\pi i\widehat{P}^*_{n-1}(z)\gamma^*_{n-1}\gamma_{n-1} & \widehat{P}^*_n(z) \end{pmatrix}$$

- The Liouville-Ostrogradski formula

$$Q_n(z)P^*_{n-1}(z) - P_n(z)Q^*_{n-1}(z) = A_n^{-1}$$

- The Hermitian property

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The three-term recurrence relation

If we call $R = Y^{n+1}(Y^n)^{-1}$ and denoting

$$Y^n(z) = \left(I + \frac{1}{z}Y_1^n + \mathcal{O}_n(1/z^2)\right)\begin{pmatrix} z^n I & 0 \\ 0 & z^{-n} I \end{pmatrix}, \quad z \to \infty$$

$$Y^{n+1}(z) = \begin{pmatrix} zI + \left(Y_1^{n+1}\right)_{11} - \left(Y_1^n\right)_{11} & -\left(Y_1^n\right)_{12} \\ \left(Y_1^{n+1}\right)_{21} & 0 \end{pmatrix} Y^n(z)$$

- $\gamma_{n-1}^* \gamma_{n-1} = -\frac{1}{2\pi i}\left(Y_1^n\right)_{21} = -\frac{1}{2\pi i}\left(Y_1^{n-1}\right)_{12}^{-1}$
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The kernel

Let $(P_n)_n$ an orthonormal family. Then we have

$$K_n(x,y) = \sum_{j=0}^{n-1} P_j^*(y)P_j(x) = \frac{P_{n-1}^*(y)A_nP_n(x) - P_n^*(y)A_n^*P_{n-1}(x)}{x-y}$$

This kernel has the following properties

1. $K_n(x,y) = K_n^*(y,x)$
2. $K_n(x,y) = \int_{\mathbb{R}} K_n(s,y)W(s)K_n(x,s)ds$

We also have that

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & I \end{pmatrix} (Y^n)_+^{-1}(y)(Y^n)_+(x) \begin{pmatrix} I \\ 0 \end{pmatrix}$$

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A differential equation

We consider weight matrices of the form

$$W(x) = \rho(x) T(x) T^*(x),$$

where T satisfies $T'(x) = G(x) T(x)$.

Consider

$$X^n(z) = Y^n(z) J(z) = Y^n(z) \begin{pmatrix} \rho(z)^{1/2} T(z) & 0 \\ 0 & \rho(z)^{-1/2} (T(z))^{-*} \end{pmatrix}$$

Then $\frac{d}{dz} X^n(z) X^n(z)^{-1}$ is entire and near infinity it behaves like

$$\left(I + \frac{Y_1}{z} + \mathcal{O}(z^{-2}) \right) \begin{pmatrix} \frac{1}{2} \frac{\rho'(z)}{\rho(z)} + G(z) & 0 \\ 0 & -\frac{1}{2} \frac{\rho'(z)}{\rho(z)} - G(z)^* \end{pmatrix} \left(I - \frac{Y_1}{z} + \mathcal{O}(z^{-2}) \right)$$

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Outline

We will study in detail the RH problem for the weight matrix

$$W(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad x \in \mathbb{R},$$

where

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \nu_i \in \mathbb{C} \setminus \{0\}$$

Therefore, $\rho(z) = e^{-z^2}$ and $T(z) = e^{Az}$ with $T'(z) = AT(z)$.

It was shown by Durán-Grünbaum that

$$\widehat{P}_n''(z) + \widehat{P}_n'(z)(2A - 2zI) + \widehat{P}_n(z)(A^2 - 2J) = (-2nI + A^2 - 2J)\widehat{P}_n(z),$$

where J is the diagonal matrix $J = \sum_{i=1}^{N}(N-i)E_{i,i}$.

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$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \nu_i \in \mathbb{C} \setminus \{0\}$$

Therefore, $\rho(z) = e^{-z^2}$ and $T(z) = e^{Az}$ with $T'(z) = AT(z)$.
It was shown by Durán-Grünbaum that

$$\widehat{P}_n''(z) + \widehat{P}_n'(z)(2A - 2zI) + \widehat{P}_n(z)(A^2 - 2J) = (-2nI + A^2 - 2J)\widehat{P}_n(z),$$

where J is the diagonal matrix $J = \sum_{i=1}^{N} (N - i) E_{i,i}$.

The Lax pair

Let Y^n be the solution of the RH for W and consider

$$X^n(z) = Y^n(z) \begin{pmatrix} e^{-z^2/2} e^{Az} & 0 \\ 0 & e^{z^2/2} e^{-A^* z} \end{pmatrix}$$

$$X^{n+1}(z) = \begin{pmatrix} zI + \left(Y_1^{n+1}\right)_{11} - \left(Y_1^n\right)_{11} & -\left(Y_1^n\right)_{12} \\ \left(Y_1^{n+1}\right)_{21} & 0 \end{pmatrix} X^n(z)$$

$$\frac{d}{dz} X^n(z) = \begin{pmatrix} -zI + A & 2\left(Y_1^n\right)_{12} \\ -2\left(Y_1^n\right)_{21} & zI - A^* \end{pmatrix} X^n(z)$$

Compatibility conditions

$$2(\beta_{n+1} - \beta_n) = A\alpha_n - \alpha_n A + I$$

$$\alpha_n = \frac{1}{2}\left(A + (\gamma_n^* \gamma_n)^{-1} A^* (\gamma_n^* \gamma_n)\right)$$

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Ladder operators

Lowering operator

$$\widehat{P}_n'(z) = A\widehat{P}_n(z) - \widehat{P}_n(z)A + 2\beta_n \widehat{P}_{n-1}(z)$$

Therefore

$$\beta_n = \frac{1}{2}(nI + a_{n,n-1}A - Aa_{n,n-1})$$

Raising operator

$$\widehat{P}_n'(z) = -2\widehat{P}_{n+1}(z) + 2z\widehat{P}_n(z) + A\widehat{P}_n(z) - \widehat{P}_n(z)A - 2\alpha_n \widehat{P}_n(z)$$

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Second-order differential equations

Introduce the following differential/difference operators

$$R_1 = \partial^1 + \partial^0 A, \quad L_1 = 2\beta_n E^{-1} + AE^0$$
$$R_2 = \partial^1 + \partial^0 (A - 2zI), \quad L_2 = -2E^1 + (A - 2\alpha_n)E^0$$
$$\partial^k = \frac{d^k}{dz^k}, \quad E^k f(n) = f(n+k)$$

The ladder operators are equivalent to

$$\widehat{P}_n(z)R_1 = L_1\widehat{P}_n(z), \quad \text{and} \quad \widehat{P}_n(z)R_2 = L_2\widehat{P}_n(z)$$

Therefore, the OMP $\widehat{P}_n$ satisfy two second-order differential equations

$$\widehat{P}_n(z)R_1R_2 = L_1\widehat{P}_n(z)R_2 = L_1L_2\widehat{P}_n(z),$$
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We will proof the following:

- Both equations are equivalent
- They are also equivalent to the second-order differential equation of hypergeometric type

$$\widehat{P}_n''(z)+\widehat{P}_n'(z)(2A-2zI)+\widehat{P}_n(z)(A^2-2J)=(-2nI+A^2-2J)\widehat{P}_n(z)$$

The first one is

$$\widehat{P}_n''(z)+2\widehat{P}_n'(z)(A-zI)+\widehat{P}_n(z)A(A-2zI)=$$
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Subtracting two last equations we get an important relation

$$\beta_n(A - 2\alpha_{n-1}) = (A - 2\alpha_n)\beta_n$$

Using ladder operators we get that both equations are

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To proof that the equations above are of hypergeometric type we use that

$$(A-\alpha_n)\widehat{P}_n'(z)+(A-\alpha_n+zI)(\widehat{P}_n(z)A-A\widehat{P}_n(z))-2\beta_n\widehat{P}_n(z)=\widehat{P}_n(z)J-J\widehat{P}_n(z)-n\widehat{P}_n(z)$$

Remarks

- The OMP $\widehat{P}_n$ satisfy a first-order differential equation (not of hypergeometric type), something that is not possible in the scalar case.
- These results are consistent if we compare them with the scalar situation (Hermite polynomials with $\alpha_n=0$ and $\beta_n=\frac{1}{2}n$).

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