On the broken-circuit complex of graphs

Aurora Llamas; José Martínez-Bernal;
Departamento de Matemáticas, Cinvestav-IPN, México
APostal 14-740, México DF 07000
aurora@math.cinvestav.mx; jmb@math.cinvestav.mx

Criel Merino∗
Instituto de Matemáticas, UNAM, México
Circuito Exterior CU, Coyoacán 04510, México DF
merino@matem.unam.mx

Abstract

It is well-known that the Stanley-Reisner ring of a matroid complex is level; this is an algebraic
property between the Cohen-Macaulay and Gorenstein properties. A similar result for broken-
circuit complexes is no longer true, even for graphs. We show that the Stanley-Reisner ring of the
broken-circuit complex, of the cone of any simple graph, is level.

Keywords: Broken-circuit complex, chromatic polynomial, double Cohen-Macaulay, double pure
shellable, h-vector, level ring, Stanley-Reisner ring.

1. Introduction

Let ∆ be a simplicial complex with m vertices, and let k be a field. Following Stanley [Sta77],
the ring k[∆] is said to be a level ring if it is Cohen-Macaulay and its minimal free resolution
ends with a free module of the form k[x1, . . . , xm](−a)b, for some integers a and b. An important
algebraic property of the Stanley-Reisner ring of a matroid complex is that it is level [Sta96, Thm.
3.4]. Unfortunately, a similar result for broken-circuit complexes is no longer true, even for graphs;
see Example 11 for the complete bipartite graph K2,3.

Let G be a simple and connected graph with a total order on its edges. A broken-circuit of
G is a cycle with its least edge removed. The broken-circuit complex, denoted by ∆BC,G, is the
simplicial complex whose faces are those sets of edges containing no broken-circuits. Let G1 be the
cone of G, i.e., add a new vertex and all the edges joining this one to every vertex of G. Following
Bacławski [Bac82], one says that a pure shellable complex ∆ is doubly pure shellable, if for any
vertex w ∈ ∆, the complex ∆\w is pure shellable and has the same dimension as ∆. Let w0 be
the least vertex of ∆BC,G1 and let ∆BC,G1 := ∆BC,G1\w0 be the reduced broken-circuit complex
of G1. We describe a total order on the edges of G1 for which ∆BC,G1 is double pure shellable
(Theorem 9), and therefore, the Stanley-Reisner ring k[∆BC,G1], or, equivalently k[∆BC,G], is level.

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If the chromatic polynomial of $G$ is $t^v - f_1 t^{v-1} + \cdots + (-1)^{v-1} f_{v-1} t$, and the Hilbert series of the Stanley-Reisner ring $k[\Delta_{BC} G_1]$ is $F(t)$, then, as an immediate consequence of a classical result of Whitney [Whi32], it follows that $(1-t)^v F(t) = 1 + f_1 t + \cdots + f_{v-1} t^{v-1}$ (Corollary 4). Thus, algebraic and geometric properties of $k[\Delta_{BC} G_1]$ may be useful to study the chromatic polynomial. In this direction, we ask if there is a total order on the edges of $G_1$, and an artinian reduction of $k[\Delta_{BC} G_1]$, for which the Weak Lefschetz Property holds (see Section 5); this would imply, up to sign, the unimodality of the chromatic polynomial—a well known conjecture [Rea68].

For algebraic and combinatorial background we refer the reader to [Bj92], [BH93], [Sta77], [Sta96].

2. Broken-circuit complex

Let $G$ be a simple and connected graph with $v$ vertices. Fix a total order on its edges. A broken-circuit of $G$ is a cycle with its least edge removed. The broken-circuit complex of $G$, denoted by $\Delta_{BC} G$, is the simplicial complex whose faces are the subsets of edges containing no broken-circuits. Recall that the external activity of a spanning tree $T$ of $G$ is defined as the number of edges $x$ in $G\setminus T$ for which $x$ is the least edge in the unique cycle contained in $T \cup x$. Then, the facets of $\Delta_{BC} G$ are the spanning trees with zero external activity, and, hence, $\Delta_{BC} G$ has dimension $v - 2$; see [BO92, Section 6.6.A] or [Bj92, Section 7.3].

Let $\Delta$ be a simplicial complex of rank $r$; so $r$ is the largest cardinality of a face. Its $f$-polynomial is $f(\Delta, t) := t^r + f_1 t^{r-1} + \cdots + f_r$, where $f_i$ is the number of faces of cardinality $i$, and its $h$-polynomial is $h(\Delta, t) := f(\Delta, t-1)$. If $F(\Delta, t)$ denotes the Hilbert series of the Stanley-Reisner ring $k[\Delta]$, and $h(\Delta, t) = t^r + h_1 t^{r-1} + \cdots + h_s t^{r-s}$, with $h_s \neq 0$, then

\[(1-t)^r F(\Delta, t) = 1 + h_1 t + \cdots + h_s t^s.\]  (1)

Let $P(G, t)$ be the chromatic polynomial of $G$ and $f(\Delta_{BC} G, t) = t^r + f_1 t^{r-1} + \cdots + f_r$.

**Theorem 1** ([Whi32]) $P(G, t) = (-1)^v t f(\Delta_{BC} G, -t) = t^v - f_1 t^{v-1} + \cdots + (-1)^{v-1} f_{v-1} t$.

Denote by $G_n$ the join of $G$ and the complete graph $K_n$, i.e., the disjoint union of $G$ and $K_n$, with additional edges joining every vertex of $G$ to every vertex of $K_n$.

**Corollary 2** For $n \geq 0$,

(a) $f(\Delta_{BC} G_{n+1}, t) = (-1)^n(t + 1)(t + 2) \cdots (t + n) P(G, -t - n - 1)$.

(b) $h(\Delta_{BC} G_{n+1}, t) = (-1)^n t(t+1) \cdots (t+n-1) P(G, -t - n)$.

**Proof.** Use that $P(G_{n+1}, t) = t(t-1) \cdots (t-n) P(G, t-n - 1)$; see [Big93, Ch. 9].

**Remark 3** For $n = 0$, $h(\Delta_{BC} G_1, t) = (-1)^v P(G, -t)$. Then, by a result of Stanley in [Sta73], it follows that $h(\Delta_{BC} G_1, j)$ is the number of pairs $(\sigma, \mathcal{O})$, where $\sigma$ is any map from the vertices of $G$ to the set $\{1, \ldots, j\}$, and $\mathcal{O}$ is an orientation of $G$ such that it is acyclic and if $u \rightarrow v$ is in the orientation, then $\sigma(u) \geq \sigma(v)$. In particular, $h(\Delta_{BC} G_1, 1)$ can be interpreted as the number of acyclic orientations of $G$, or as the number of acyclic orientations of $G_1$ with a predetermined vertex as a source.

**Corollary 4** $(1-t)^v F(\Delta_{BC} G_1, t) = 1 + f_1 t + \cdots + f_{v-1} t^{v-1}$.
Proof. Since $h(\Delta_{BC}G_1, t) = (-1)^{w}P(G, -t)$, the result follows from Eq. (1). □

Thus, algebraic and geometric properties of $k[\Delta_{BC}G_1]$ may be useful to study the chromatic polynomial of $G$.

3. A combinatorial Lemma

For a simple and connected graph $G$, recall that $G_1$ denotes the cone of $G$. We ordered the vertices of $G_1$ as follows: The new vertex is 1. Fix an arbitrary spanning tree $B$ of $G$. Choose any vertex of $B$ as 2. Choose any vertex $B$-adjacent to 2 as 3. Choose any vertex $B$-adjacent to 2 or 3 as 4. Continue this process until all vertices of $G$ have been reached. Now, every edge has both ends labelled; we called the label of maximum value the largest entry of the edge. We define a total order on the edges of $G_1$ as follows: First we place those adjacent to the vertex 1, ordered according to its largest entry; we called this set $A$. Next come those of $B$, also ordered by its largest entry. Finally, place the remaining edges in an arbitrary order; we called this set $C$.

For a spanning tree $F$ of $G_1$, $x$ an edge and $i$ a vertex of $F$, denote by $F_i(x)$ the set of vertices which are in the same component as $i$ after $x$ has been removed from $F$; denote also by $b_F(x)$ the set of edges $y$ of $G_1$ such that $F(x, y) := (F\setminus x) \cup y$ is also a tree; i.e., $b_F(x)$ is the unique bond of $G_1$ induced by $F \setminus x$. By a 0-tree we mean a spanning tree of $G_1$ whose external activity is zero.

Some properties of this ordering are:

(1*) Any vertex $b$ on the $B$-path from 2 to $a$ satisfies that $2 \leq b \leq a$.

(2*) Let $F$ be a 0-tree. If $(1, b)$ is the edge on the $F$-path from 1 to $i$, then $b \leq i$.

Lemma 5 Let $F$ be a 0-tree and $x$ an edge of $F$, $x \neq (1, 2)$.

(a) If $y = \min b_F(x)$, then $F(x, y)$ is a 0-tree.

(b) If $b_F(x) \cap B \neq \emptyset$ and $y = \min b_F(x) \cap B$, then $F(x, y)$ is a 0-tree.

Proof. (a) Clear. (b) Say $y = (i, j)$, with $i < j$. Suppose there is a vertex $a$, with $2 \leq a < j$, such that $a \notin F_1(x)$. Since there is a $B$-path from 2 to $a$ (because $(1, 2)$ is in all 0-trees), hence there is an edge $y'$ on this path that also belongs to $b_F(x)$. By (1*), it holds that $y' < y$. This, together with the fact that $y' \in b_F(x) \cup B$, contradicts the minimality of $y$. Therefore, $1, \ldots, j - 1$ are in $F_1(x)$, and, in particular, $i \notin F_1(x)$ and $j \notin F_1(x)$. Let $(1, b)$ be the edge on the $F$-path from 1 to $i$. By (2*), it holds that $b \leq i$. Suppose there is an edge $z$ which is $e$-active (i.e., externally active) for $F(x, y)$. Since $z \in b_F(x, y) = b_F(x)$ and $z < y$, it follows that $z \in A \cup B$. Definition of $y$ implies that $z$ cannot be in $B$, so, $z = (1, c)$. As $z \in b_F(x)$, the above argument implies that $c \notin F_1(x)$, and then $j \leq c$. Let $[b, i]$ denote the $F$-path from $b$ to $i$. Because $(1, b) \cup [b, i] \cup y \cup [j, c] \cup z$ is the unique cycle contained in $F(x, y) \cup z$, it follows that $z$ is not $e$-active for $F(x, y)$; A contradiction. □

Corollary 6 Let $F$ be a 0-tree and $x$ an edge of $F$, $x \neq (1, 2)$.

(a) If $x \in A$, then there exists $y \in B$ such that $F(x, y)$ is a 0-tree.

(b) If $x \in B$, then there exists $y \in A$ such that $F(x, y)$ is a 0-tree.

(c) If $x \in C$, then there exist $y \in A$ and $y' \in B$ such that $F(x, y)$ and $F(x, y')$ are 0-trees.

Using our total order on the edges of $G_1$, we consider the 0-trees of $G_1$ lexicographically ordered.

Lemma 7 Let $w$ be a fixed edge of $G_1$, $w \neq (1, 2)$. Let $H < F$ be two 0-trees not containing $w$. Then, there is an edge $x$ in $F \setminus H$ and there is an edge $x < y$ not in $F$, with $y \neq w$, such that $F(x, y) < F$ is a 0-tree.
Proof. Put $x = \max F \setminus H$.

Case $x \in A$. This case is not possible: Let $y = \min H \setminus F$. $H < F$ implies $y < x$. Say $y = (1, a)$ and $x = (1, b)$, with $a < b$. That $F$ is a 0-tree implies that $a \in F_1(x)$. Let $(1, c)$ be the edge on the $F$-path from 1 to $a$. By $(2\ast)$, $c \leq a$. But, by definition of $x$ and $y$, this $F$-path is also a $H$-path. This one, jointly with $y$, form a cycle of $H$. This is not possible.

Case $x = (2, 3)$. Let $y = \min H \setminus F$. $H < F$ implies $y < x$. Say $y = (1, a)$. Suppose $a > 3$: If $a \in F_3(x)$, then there is an $F$-path from 3 to $a$. By definition of $x$, this is also a $H$-path. But then, $(1, 3)$ would be $e$-active for $H$. Hence $a \in F_1(x)$. Let $(1, b)$ be the edge on the $F$-path from 1 to $a$. By $(2\ast)$, $b \leq a$. Definitions of $x$ and $y$ imply that this $F$-path, together with $y$, form a cycle of $H$. This is not possible. So, $a = 3$; in particular $y \in bF(x)$. This $y$ satisfies the requirements.

Case $x \in B$, $x \neq (2, 3)$. In this case let $y_0 = \min bF(x) \in A$. If $y_0 \neq w$, then $F(x, y_0) < F$ and we are done. So, we can assume that $\min bF(x) \equiv w$; in particular $w \in A$. If $y_1 = \min bF(x) \cap B \neq x$, then $F(x, y_1) \neq F$, and we are also done. So, we can also assume that $\min bF(x) \cap B = x$.

Say $x = (i, j)$, with $i < j$. That $\min bF(x) \cap B = x$ and $(1\ast)$ imply that $1, \ldots, j-1 \in F_1(x)$; and then $w = (1, j)$. If there exists $x' \in (F \cap B) \setminus H$, with $x' \neq x$, then, by Lemma 5(a), $F(x', y') < F$; and we finish, because in that case $y' = \min bF(x') < w$. Therefore we can assume that $(F \cap B) \setminus H \subseteq H$. Let $y = \min H \setminus F$ again. $H < F$ implies $y < x$. If $(F \cap A) \subset H$, then $x < H \setminus y$, hence, $F(x, y) < F$ works. So we can assume $(F \cap A) \not\subseteq H$, and then $y \notin A$, say $y = (1, b)$. Let us see that this situation cannot happen. Suppose $b > j$: If $b \notin F_1(x)$, then the $F$-path from 1 to $b$, together with $y$, form a cycle in $H$. If $b \notin F_1(x)$, then, because there is a path in $F_j(x)$ from $j$ to $b$, $w$ would be $e$-active for $H$. Suppose $b < j$: Then $b \in F_1(x)$, and the $F$-path from 1 to $b$, jointly with $y$, form a cycle in $H$.

Case $x \in C$. In this case, at least one of $y$ or $y'$ in Corollary 6(c) is different of $w$, so, we conclude the proof. □

Remark 8 To simplify the exposition, all our results have been stated only for connected graphs. However, the definition of $\Delta_{BC}G$ does not depend on $G$ being connected. When dealing with a simple graph with $c$ components, the corresponding notion to spanning tree is that of spanning forest, and the appropriate rank $r$ of $\Delta_{BC}G$ is $v - c$. Theorem 1 is then: $P(G, t) = (-1)^{v-t} f(\Delta_{BC}G, -t)$. All the other notions have their corresponding definition for this case and it is straightforward to check that our results remain valid in this slightly more general setting.

4. Main result

A simplicial complex is pure if all its facets are equicardinal. It is shellable if its facets can be totally ordered $F_1, F_2, \ldots, F_t$, in such a way that: for each pair $F_i < F_j$, there is $F_k < F_j$ such that $F_i \cap F_j \subseteq F_k \cap F_j$ and $|F_k \cap F_j| = |F_j| - 1$. Following Baclawski [Bac82], one says that a Cohen-Macaulay complex $\Delta$ is doubly Cohen-Macaulay, if for any vertex $w \in \Delta$, the complex $\Delta \setminus w$ is Cohen-Macaulay and has the same rank as $\Delta$. Analogously, one says that a pure shellable complex $\Delta$ is doubly pure shellable, if for any vertex $w \in \Delta$, the complex $\Delta \setminus w$ is pure shellable and has the same rank as $\Delta$.

Let $0 \to \oplus_j R(-j)^{\beta_j} \to \cdots \to \oplus_j R(-j)^{\beta_j} \to R \to k[\Delta] \to 0$ be the minimal free resolution of $k[\Delta]$, where $R = k[x_1, \ldots, x_m]$, the $x_i$'s are indeterminates, and $m$ is the number of vertices in $\Delta$. 
\[ \Delta \] is said to be a level ring if it is Cohen-Macaulay and \( \min\{j : \beta_{pj} \neq 0\} = \max\{j : \beta_{pj} \neq 0\} \).

The Betti number \( \sum_j \beta_{pj} \) is known as the type of \( \Delta[\Delta] \). If \( \Delta \) has rank \( r \), it can be verified that

\[
\sum_{j=0}^p \sum_i \beta_{ij}(-1)^i t^j = (1-t)^{m-r}(1+h_1 t + \cdots + h_s t^s),
\]

where \( t^r + h_1 t^{r-1} + \cdots + h_s t^{r-s} \) is the \( h \)-polynomial of \( \Delta \).

Let \( w_0 \) be the least vertex of \( \Delta_{BC} G_1 \) and let \( \Delta_{BC} G_1 := \Delta_{BC} G_1 \backslash w_0 \) be the reduced broken-circuit complex of \( G_1 \).

**Theorem 9** Let \( G \) be a simple graph and \( n \geq 1 \). There is a total order on the edges of \( G_n \) such that

(a) \( \Delta_{BC} G_n \) is double pure shellable.

(b) \( \Delta_{BC} G_n \) is double Cohen-Macaulay.

(c) \( k[\Delta_{BC} G_n] \) is a level ring.

(d) \( k[\Delta_{BC} G_n] \) is a level ring.

**Proof.** (a) It is known that \( \Delta_{BC} G_n \) is pure shellable [Bjö92, Thm. 7.4.3]. From Corollary 6, it follows that \( \Delta_{BC} G_n \backslash w \) is pure and has the same rank as \( \Delta_{BC} G_n \). Lemma 7 implies the shellability of \( \Delta_{BC} G_n \backslash w \). (b) It is known that the property of being pure shellable implies being Cohen-Macaulay over any field [BH93, Thm. 5.1.13]. (c) It is known that the property of being doubly Cohen-Macaulay implies being level [BH93, Thm. 5.6.6]. (d) \( k[\Delta_{BC} G_n] \) and \( k[\Delta_{BC} G_n] \) have the same graded Betti numbers. \( \square \)

**Corollary 10**

\[
\text{type } k[\Delta_{BC} G_{n+1}] = \begin{cases} f_{v-1}, & \text{if } n = 0, \\ (-1)^n(n-1)! P(G, -n), & \text{if } n \geq 1. \end{cases}
\]

**Proof.** Set \( \Delta = \Delta_{BC} G_{n+1} \) and define \( M_i = \max\{j : \beta_{ij} \neq 0\}. \) Since \( \Delta \) is Cohen-Macaulay, i.e., \( p = m - r \), it holds that \( M_1 < \cdots < M_p \); see [BH95, Prop. 1.1]. Because \( k[\Delta] \) is level, applying Eq. (2), it follows that \( \text{type } k[\Delta] = \beta_{p M_p} = h_n \). But \( h_n \) is the absolute value of the linear coefficient in \( P(G, t) \). So, by Corollary 2, the result follows. \( \square \)

This corollary gives another interpretation for negative evaluations of the chromatic polynomial.

**Example 11** Let \( G \) be the complete bipartite graph \( K_{2,3} \). For any total ordering of the edges of \( G \), we obtain, using CoCoA [Coc98], that the minimal free resolution of \( \mathbb{Q}[\Delta_{BC} G] \) is

\[
0 \rightarrow R(-4) \oplus R(-5) \rightarrow R(-3)^3 \rightarrow R \rightarrow \mathbb{Q}[\Delta_{BC} G] \rightarrow 0,
\]

and \( (1-t)^3 F(\Delta_{BC} G, t) = 1 + 2t + 3t^2 + t^3 \). So, \( \mathbb{Q}[\Delta_{BC} G] \) is not a level ring; see also [Sta96, III, Prop. 3.3] for a different proof of this fact.

**Remark 12** We notice that in [Bro98, p. 61] it was incorrectly stated that if \( G \) is 2-connected, then \( \Delta_{BC} G \) is double Cohen-Macaulay. Example 11 is a contraexample.

**Example 13** Let \( G_1 \) be the cone of \( G \) as above. Using the ordering
The minimal free resolution for a tree $G$, we obtain the minimal free resolution

$$0 \to R(-10)^7 \to R(-8)^{11} \oplus R(-9)^{25} \to R(-6)^5 \oplus R(-7)^{41} \oplus R(-8)^{29} \to R(-4) \oplus R(-5)^{19} \oplus R(-6)^{50} \oplus R(-7)^{10} \to R(-3)^3 \oplus R(-4)^{26} \oplus R(-5)^{16} \to R(-2)^6 \oplus R(-3)^6 \to R \to \mathbb{Q}[\Delta_{BC}G_1] \to 0,$$

and $(1 - t)^4 F(\Delta_{BC}G_1, t) = 1 + 6t + 15t^2 + 17t^3 + 7t^4$. So, $\mathbb{Q}[\Delta_{BC}G_1]$ is a level ring.

**Example 14** Again, let $G_1$ be the cone of $G$ as above. But now, using the ordering

$$0 \to R(-10)^7 \to R(-8)^{10} \oplus R(-9)^{25} \to R(-6)^3 \oplus R(-7)^{40} \oplus R(-8)^{28} \to R(-5)^{17} \oplus R(-6)^{48} \oplus R(-7)^{9} \to R(-3)^2 \oplus R(-4)^{25} \oplus R(-5)^{14} \to R(-2)^6 \oplus R(-3)^5 \to R \to \mathbb{Q}[\Delta_{BC}G_1] \to 0,$$

and $(1 - t)^4 F(\Delta_{BC}G_1, t) = 1 + 6t + 15t^2 + 17t^3 + 7t^4$. So, $\mathbb{Q}[\Delta_{BC}G_1]$ is a level ring.

**Example 15** The minimal free resolution for a tree $T$ has the form $0 \to R \to k[\Delta_{BC}T] \to 0$; and for the cycle on $v + 1$ vertices, $C_{v+1}$, has the form $0 \to R(-v) \to R \to k[\Delta_{BC}C_{v+1}] \to 0$. So, in both cases, $k[\Delta_{BC}G]$ is a level ring.

As we have seen, examples of graphs for which $k[\Delta_{BC}G]$ is a level ring, include: trees, cycles, wheels, threshold and complete graphs, and of course, the cone of any simple graph. We propose the problem of characterize those graphs for which $k[\Delta_{BC}G]$ is a level ring.
5. Question

Following [HMNW03], an artinian graded $k$-algebra $A = \bigoplus_{i=0}^{s} A_i$, $A_s \neq 0$, has the Weak Lefschetz Property if there is an element $L \in A_1$ such that the linear transformations $L : A_i \to A_{i+1}$, $1 \leq i \leq s-1$, given by multiplication by $L$, are either injective or surjective. A main result is the following.

**Proposition 16** ([HMNW03, Prop. 3.5]) Let $h = (1, h_1, \ldots, h_s)$ be a finite sequence of positive integers. Then $h$ is the Hilbert function of a graded artinian $k$-algebra $k[x_1, \ldots, x_m]/J$ having the Weak Lefschetz Property if and only if $h$ is an unimodal $O$-sequence such that the positive part of the first difference is an $O$-sequence.

Two well-known open problems related with the chromatic polynomial of a simple graph $G$ are its unimodality [Rea68] and its characterization [Wil73]. Since algebraic properties of the ring $k[\Delta_{BC}G_1]$ may be useful to study the chromatic polynomial, a natural question is

**Question 17** Is there a total order on the edges of $G_1$, and an ideal $I$ in $k[\Delta_{BC}G_1]$, generated by a maximal regular sequence of linear forms, such that $k[\Delta_{BC}G_1]/I$ has the Weak Lefschetz Property?

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