QUANTUM GEOMETRY AND NEW CONCEPT OF SPACE

MICHO ĐURĐEVICH

1. Introduction

Quantum geometry is a new branch of mathematics. It introduces a completely new concept of space, by unifying methods of classical geometry with non-commutative C*-algebras and functional analysis, and incoprorating into geometry various ideas from quantum physics.

Every geometry deals with some kind of *spaces*. Quantum geometry deals with *quantum spaces*, including the classical concept of space as a very special case. In classical geometry spaces are always understandable as *collections of points* equipped with the appropriate additional structure (as for example a topological structure given by the collection of open sets, or a smooth structure given by the atlas). In contrast to classical geometry, quantum spaces are not interpretable in this way. In general, quantum spaces have no points at all! They exhibit non-trivial 'quantum fluctuations' of geometry at all scales.

A very interesting potential application of quantum geometry in physics is to provide a mathematically coherent description of the physical space-time, at all scales—in particular at the level of ultra-small distances, characterized by the *Planck lenght.* This lenght is a universal physical constant, defined as a unique combination of gravitational constant γ, Planck's constant $\hbar$ and the velocity of light c. Explicitly,

$$l = \sqrt{\frac{\gamma\hbar}{c^3}}.$$

As we can see, it is an exorbitantly small number! There are many reasons to believe that Planck's lenght marks a boundary for the applicability of classical concepts of space and time in physics.

Indeed, the assumption that the underlying space-time is a smooth manifold is contained in the roots of various mathematical *inconsistency problems* appearing in quantum field theory. The same assumption is in the roots of the failure of many attempts to unify gravity and quantum theory. The difficulties with such classical concepts about space and time appear at the very small distances, precisely of the order of magnitude of the Planck lenght.

Quantum geometry introduces much more flexibility in the game, allowing us to express the idea that the space-time exhibits certain quantum fluctuations of the structure which are neglectable at the macroscopic level, but which become essential at the level of the Planck scale. In particular, the very concept of a space-time point is loosing the sense at the quantum level. The same applies to the space-time coordinates.

The formalism of quantum geometry is a symbiosis of global methods from classical diffferential geometry, with non-commutative algebras and functional analysis.

Quantum spaces are described by certain non-commutative complex *-algebras. The elements of these algebras are intuitively interpreted as the appropriate functions (continuous or smooth for example) over the associated quantum spaces. The mentioned *-algebras are always associated, in the appropriate sense, to certain C*-algebras representing the quantum spaces at the topological level.

When the algebras are commutative, we are back in the classical geometry. In other words, classical geometry is understandable as the *commutative sector* of quantum geometry. Quantum geometry is also called *non-commutative geometry.*

Non-commutative geometry has a great conceptual value for the study of classical spaces. In many situations, the proofs of the theorems of classical geometry become more elegant and transparent if performed at the quantum level. The language of local coordinates, open sets and points, characteristic for classical geometry, sometimes hides the true geometrical structure. On the other hand, in non-commutative geometry we are a priory forced to work with the global entities inherently connected with the existing geometrical structure.

In generalizing classical geometry to the non-commutative level, there are two important conceptual steps: Translation of geometry into commutative algebra language, and non-commutative generalizations.

The first step consists in re-expressing a geometrical structure existing on a classical space X in terms of the algebraic structure of the associated (commutative) *-algebra of the appropriate complex-valued functions on X. The definition of this algebra depends on the geometrical level of our considerations. For example:

$$\text{Measure Theory} \leftrightarrow \text{Measurable Functions}$$
$$\text{Topology} \leftrightarrow \text{Continuous}$$
$$\text{Algebraic Geometry} \leftrightarrow \text{Polynomial}$$
$$\text{Differential Geometry} \leftrightarrow \text{Smooth Functions}$$

It turns out that the geometrical structure on X is always *completely expressible* at the language of the associated *-algebra. The second step consists in the appropriate noncommutative generalization of such algebraically-reformulated geometry. The idea is to replace the function algebras by more general non-commutative *-algebras, and enlarge in such a way the concept of space–by introducing quantum spaces.

In what follows we shall explain both conceptual steps in more details. Then we shall discuss some concrete examples of quantum spaces.

2. Reformulating Basic Geometrical Concepts

We shall now explain how some of the most important geometrical concepts are translated into the language of algebra.

2.1. **Points**

Let us assume that X is a compact topological space, and let A be the *-algebra of continuous complex-valued functions on X. The algebraic operations in A are the standard multiplication and addition of functions. The *-operation is the standard complex conjugation.

Every element $x \in X$ naturally gives rise to a linear functional $\kappa = \kappa_x : A \to \mathbb{C}$ defined by $\kappa(f) = f(x)$. This map is *multiplicative* in the sense that $\kappa(fg) = \kappa(f)\kappa(g)$ for each $f, g \in A$, and hermitian in the sense that $\kappa(f^*) = \kappa(f)^*$. It is also non-trivial ($\kappa \neq 0$). In other words κ is a *character* on A.

Conversely, let us consider an arbitrary character $\kappa : A \to \mathbb{C}$. Then it can be shown that there exists a unique point $x \in X$ such that $\kappa = \kappa_x$. In other words, we have a natural bijection between points of X and characters of A. It is worth noticing that this characterization of points remains valid at the smooth level, too. In this case X is a compact smooth manifold and the associated *-algebra consists of smooth functions on X.

2.2. **Gelfand-Naimark Theorem**

The algebra $A = \mathrm{C}(X)$ of complex-valued continuos functions on a compact topological space X, equipped with the *maximum norm*

$$|f| = \max_{x \in X} |f(x)|$$

is a commutative C*-algebra. The classical theorem of Gelfand and Naimark characterizes the algebras of the form $A = \mathrm{C}(X)$, as commutative unital C*-algebras. In other words, for every commutative unital C*-algebra A there exists (up to the homeomorphisms) a unique compact topological space X such that $A \cong \mathrm{C}(X)$.

As we have just mentioned, the points of the space X are recovered as characters of the associated algebra A. In terms of this identification, the topology on X coincides with the *-weak topology, induced from the dual space A^*, consisting of continuous linear functionals on A. It turns out that homomorphisms between C*-algebras are automatically continuous, in particular characters are continuous linear functionals.

The theory of compact topological spaces is the same as the theory of commutative unital C*-algebras. If we relax the unitality assumption (dealing with arbitrary commutative C*-algebras), then the category of corresponding spaces is enlarged to the level of locally-compact spaces. If X is non-compact then A is consisting of continuous functions on X that vanish at infinity.

If X is a measurable space (without any extra structure) then the relevant *-algebra is consisting of all essentially bounded measurable functions on X. It becomes a (commutative) von Neumann algebra, if equipped with the essential supremum norm. It can be shown that every commutative von Neumann algebra is of this form. The entire measure theory is essentially the same as the theory of commutative von Neumann algebras.

2.3. **Continuous Maps and Direct Products**

Now let us consider two compact topological spaces X and Y, and let us denote by A and B the *-algebras of continuous functions over X and Y respectively. Let $F : X \to Y$ be an arbitrary continuous map between X and Y. To this map, we can associate another map $\Phi = \Phi_F : B \to A$, defined via the composition $\Phi(g) = Fg$. It is easy to see that the map Φ is a unital *-homomorphism between B and A.

Conversely, let us consider an arbitrary unital *-homomorphism $\Phi : B \to A$. Then it can be shown that Φ is always of the form $\Phi = \Phi_F$, for a uniquely determined continuous map $F : X \to Y$. In other words we have a natural bijection between continuous maps from X to Y, and unital *-homomorphisms from B to A. The same algebraic characterization holds at the smooth level, too (smooth maps between compact smooth manifolds are in a one-to-one correspondence with unital *-homomorphisms between the associated algebras of smooth functions).

Properties of the map F are reflected as properties of Φ_F and vice versa. For example, F is surjective if and only if Φ_F is injective, and F will be injective if and only of Φ_F is surjective. If C is the C*-algebra of continuous functions on the direct product $X \times Y$, then the following natural identification holds:

$$C \leftrightarrow A \otimes B,$$

where the product $\otimes$ here is a C*-algebraic tensor product.

2.4. **Symmetry**

Symmetry transformations of the space X can be understood as certain homeomorphisms of X. In accordance with the previous paragraph, homeomorphisms of X are in one-to-one correspondence with automorphisms of A. The same holds at the smooth level: If X is a smooth manifold then diffeomorphisms of X are in a natural bijection with automorphisms of the corresponding algebra of smooth functions.

2.5. **Probability Measures**

Let us assume that X is a metrizable compact topological space (metrizability of X is equivalent to the separability of $A = \mathrm{C}(X)$ in its uniform norm). Let us consider a probability measure μ defined on the σ-field $\mathrm{B}(X)$ of Borel subsets of X. Let $\rho = \rho_\mu : A \to \mathbb{C}$ be a linear functional defined as the Lebesgue intergral

$$\rho(f) = \int_X f(x) d\mu(x).$$

The map ρ is linear, normalized ($\rho : 1 \mapsto 1$) and positive (if $f \geq 0$ then $\rho(f) \geq 0$) functional on A.

Conversely, if $\rho : A \to \mathbb{C}$ is an arbitrary positive and normalized linear functional on A, then there exists a unique probability measure $\mu : \mathrm{B}(X) \to [0, 1]$ such that $\rho = \rho_\mu$. This is the classical Riesz representation theorem, establishing a natural correspondence between probability measures on X and positive normalized functionals on A.

The theorem remains true in the non-metrizable case, however Borel sets should be replaced by Baire sets.

2.6. **Compact Topological Groups**

Let X be a compact topological group. This means that X is a compact topological space equiped with a group structure, such that the product map $\circ : X \times X \to X$ is continuous (it can be shown that in the compact case continuity of the product implies continuity of the inverse map). At the dual level, the product map is represented by a *-homomorphism $\phi : A \to A \otimes A$. The associativity property of the product is equivalent to the coassociativity property

$$(\phi \otimes \mathrm{id})\phi = (\mathrm{id} \otimes \phi)\phi.$$

It can be shown that the remaining two group axioms (the existence of the neutral element and the existence of the inverse elements) are equivalent to a single assumption that the elements of the form $a\phi(b)$ as well as of the form $\phi(b)a$, where $a, b \in A$, span two everywhere dense linear subspaces of $A \otimes A$.

2.7. **Vector Bundles**

Let us now assume that X is a compact smooth manifold, and let $\mathcal{E}$ be a smooth complex vector bundle over X. Let $\Gamma = \Gamma(\mathcal{E})$ be the space of smooth sections of $\mathcal{E}$. This space is a finite and projective module over the *-algebra $\mathcal{A} = \mathrm{C}^\infty(X)$ of smooth functions on X.

Conversely, if Γ is an arbitrary finite and projective module over $\mathcal{A}$, then there exists a unique (up to the isomorphisms) smooth vector bundle $\mathcal{E}$ over X such that $\Gamma = \Gamma(\mathcal{E})$.

2.8. **Vector Fields**

Let ξ be a smooth vector field on X and let $D = D_\xi : \mathcal{A} \to \mathcal{A}$ be the corresponding Lie derivative (the derivation along ξ). The map D satisfies the Leibniz rule

$$D(fg) = D(f)g + fD(g) \qquad \forall f, g \in \mathcal{A},$$

in other words it is a derivation on $\mathcal{A}$. Moreover, D is hermitian in the sense that $D* = *D$, as long as ξ is a real vector field.

Conversely, if $D : \mathcal{A} \to \mathcal{A}$ is an arbitrary hermitian derivation on $\mathcal{A}$ then there exists a unique (real) vector field ξ on X such that $D = D_\xi$. In other words, there exists a natural bijection between vector fields on X and hermitian derivations on $\mathcal{A}$. If we relax the hermicity assumption, then we obtain a correspondence between all derivations on $\mathcal{A}$ and complex vector fields on X.

2.9. **Differential Forms**

The algebra $\Omega(X)$ of smooth differential forms over a compact smooth manifold X can be constructed as follows.

Let us first consider the Lie algebra $\Xi = \mathrm{Der}(\mathcal{A})$ of all derivations of $\mathcal{A}$ (vector fields, in accordance with the previous paragraph). This algebra is naturally acting in $\mathcal{A}$, so we can construct the Chevalley complex $C(\Xi, \mathcal{A})$. This space possesses a natural graded-differential *-algebra structure. By definition, the elements of $C(\Xi, \mathcal{A})^k$ are all possible k-linear antisymmetric maps

$$\omega : \overbrace{\Xi \times \ldots \times \Xi}^{k} \to \mathcal{A}.$$

The product, differential and the *-structure in $C(\Xi,\mathcal{A})$ are given by

$$\begin{aligned} d\omega(\xi_1,\dots,\xi_{k+1}) &= \sum_{i=1}^{k}(-1)^{i-1}\xi_i\omega(\xi_1,\dots,\widehat{\xi_i},\dots,\xi_{k+1}) \\ &\quad+\sum_{i<j}(-1)^{i+j}\omega([\xi_i,\xi_j],\dots,\widehat{\xi_i},\dots,\widehat{\xi_j},\dots,\xi_{k+1}) \\ \omega^*(\xi_1\dots,\xi_k) &= \omega(\xi_1^*,\dots,\xi_k^*)^* \\ (\omega\eta)(\xi_1,\dots,\xi_{k+l}) &= \sum_{\pi\in S_{kl}}(-1)^{\partial\pi}\omega(\xi_{\pi(1)},\dots,\xi_{\pi(k)})\eta(\xi_{\pi(k+1)},\dots,\xi_{\pi(k+l)}) \end{aligned}$$

where $\eta\in C(\Xi,\mathcal{A})^l$ and S_{kl} is a subset cosisting of all $(k+l)$-permutations preserving the orders of the first k and the last l elements.

By definition, we put $C(\Xi,\mathcal{A})^0=\mathcal{A}$. The algebra of differential forms $\Omega(X)$ can be viewed as a differential subalgebra of $C(\Xi,\mathcal{A})$ generated by $\mathcal{A}$. In other words, the spaces $\Omega(X)^k$ are linearly spanned by the elements of the form

$$w=a_0d(a_1)\dots d(a_k),$$

where $a_i\in\mathcal{A}$.

We can summarize our discussion so far in the following table, which is a kind of a mini *dictionary* between geometry and algebra:

Compact topological spaces X	Unital commutative C*-algebras A
X=compact topological space	A = C(X) = {complex continuous functions on X}
Points $x\in X$	Characters $\kappa=\kappa_x$ of A
Continuous maps between compacts X and $Y\leftrightarrow B$	Unital *-homomorphisms from B to A
The direct product $X\times Y$	The C*-tensor product $A\otimes B$
Symmetries of X	Automorphisms of A
Group structure on X	Coproduct map $\phi: A\to A\otimes A$
Probabilty measures on a metrizable compact X	Positive normalized linear functionals on A
Locally-compact noncompact topological spaces X	Non-unital commutative C*-algebras A
Measure theory	Commutative von Neumann algebras
X=compact smooth manifold	$\mathcal{A}$ = C$^\infty(X)$ = {complex smooth functions on X}
Vector bundles over X	Finite projective modules over $\mathcal{A}$
Vector fields on X	Hermitian derivations on $\mathcal{A}$
Differential forms on X	Graded-differential algebra $\Omega(X)\subset C(\Xi,\mathcal{A})$ generated by $\mathcal{A}$

Elementary Geometry-Algebra Dictionary

3. Noncommutative Generalizations

The second main conceptual step consists in generalizing the re-formulated classical geometry, by relaxing the assumption of commutativity of the algebras $\mathcal{A}=$ C$^\infty(X)$ and $A=$ C(X), and allowing them to be the appropriately chosen *noncommutative* *-algebras. In such a way we arive to *quantum spaces,* the main objects of study in quantum geometry. The elements of these non-commutative *-algebras are intuitively interpreted as smooth functions (or continuous, measurable–depending on the context) over quantum spaces. However, in contrast to classical geometry, the 'existence' of quantum spaces is implicit, as they generally appear in the formalism exclusively via the corresponding *-algebras.

For example, in accordance with this philosophy and the classical Gelfand-Naimark theory, we may say that C*-algebras generalize classical topology of compact (and locally-compact) topological spaces. Such a new class of quantum topological spaces was introduced by Woronowicz in [25] and developed in [27]–[30] mainly in the context of quantum groups. In the case of non-compact structures, we meet an essentially new situation consisting in the existence of various different types of 'continuous functions'—the analogs of functions with a compact support, functions vanishing at infinity, bounded functions, unbounded functions. The counterparts of these functions play a very important role in the theory of non-compact quantum spaces and groups [20, 26], and accordingly we have to play with various types of algebras. In particular, besides the 'vanishing-at-infinity functions' $A \sim \mathrm{C}_0(X)$, it is necessary to introduce the multiplier C*-algebra $\mathrm{M}(A)$ representing the bounded functions.

In accordance with the above mentioned Riesz representation theorem, we can say that probability measures on the quantum space X are given by positive normalized linear functionals $\rho : A \to \mathbb{C}$. Such functionals are called *states*. In the general C*-algebraic context, positive elements are defined as those satisfying $a = b^*b$ for some $b \in A$. The set A_+ of all positive elements is a closed strict conus in A. A linear functional is positive if it takes positive values on positive elements. Positive functionals are automatically continuous. The set $S(A)$ of all states of A is convex, and compact in the *-weak topology of the dual space A^*. According to the Krein-Millman theorem, $S(A)$ coincides with the *-weak closure of the convex hull over its extremal elements. The extremal elements of $S(A)$ are called pure states.

There exists a deep connection between C*-algebra representations, and states. Let us consider an arbitrary unital C*-algebra A and let $D : A \to \mathrm{B}(H)$ be a representation of A in a Hilbert space H. Every unit vector $\psi \in H$ gives rise to a state $\rho : A \to \mathbb{C}$, via the formula

$$\rho(a) = \langle \psi, D(a)\psi \rangle.$$

If the vector ψ is cyclic for the representation D, then the representation D is *completely determined* by the associated state ρ. More precisely, the above formula establishes a natural bijection between the states in A and the (isomorphism classes of) triples (H, D, ψ) consisting of a Hilbert space H, representation D of A in H with a cyclic unit vector $\psi \in H$. This is the idea of the GNS-construction, which is one of the most important tools in the study of C*-algebras. In terms of the GNS-construction, irreducible representations of A are characterized as those associated to the pure states.

Going back to the classical (=commutative) context—we have $A = \mathrm{C}(X)$ and the GNS-representation D associated to a state ρ is acting in the Hilbert space $H = L^2(X, \mu_\rho)$. The cyclic vector ψ is represented by the unit function, and the operators $D(a)$ are given by the left multiplication. Irreducible representations are 1-dimensional and given by characters of A, in other words the points of the associated space X. The pure states are also given by characters of A. The associated probability measures are simply δ-measures concentrated in points of X.

The theory of von Neumann algebras can be viewed as a quantum generalization of classical measure theory. Commutative von Neumann algebras describe classical measurable spaces, and non-commutative von Neumann algebras represent *quantum measure spaces.* The roots of quantum geometry are present in the foundational papers by Murray and von Neumann [16, 17, 18].

Various fundamental topics of non-commutative differential geometry, including cyclic cohomogy as topological invariants of quantum spaces, their incorporation into algebraic K-theory, quantum elliptic operators and non-commutative index theory, are developed in the works by Connes [3, 4] [13, 14].

Before passing to some concrete examples of quantum spaces, let us observe that one and the same concept of classical geometry may have several very different generalizations in quantum geometry. This is because the procedure of generalizing objects of commutative algebra into the objects of non-commutative algebras is not at all unique and straightforward.

Perhaps one of the best examples of this phenomena is the quantum differential calculus. In accordance to what we have mentioned above, one natural way of introducing the concept of differential forms in quantum geometry would be to start from an apropriate non-commutative *-algebra $\mathcal{A}$ (representing smooth functions) and define differential forms as the elements of the $\mathcal{A}$-generated graded-differential *-subalgebra $\Omega(X)$ of the Chevalley complex $C[\Xi = \mathrm{Der}(\mathcal{A}), \mathcal{A}]$ associated to the Lie algebra of derivations and its natural representation in $\mathcal{A}$. The product, differential and the *-structure are given by the same formulas as in the classical case (with the difference that in the classical case $C(\Xi, \mathcal{A})$ is a graded-commutative algebra, in general case $C(\Xi, \mathcal{A})$ will be highly non-commutative).

This idea was sistematically followed in [6]–[8]. On the other hand, the mentioned construction is not appropriate in considerations involving quantum groups and differential structures over them, when we must take care of the quantum group symmetry of the calculus. In the quantum group context, a very different construction looks more natural [29]. This construction incorporates from the very beginning the idea of a quantum group covariance. Both constructions include the classical situation (classical differential calculus over classical Lie groups) as a very special case.

The same remark applies to the very concept of symmetry—one possible way to introduce symmetries in quantum geometry is to consider automorphisms of the corresponding noncommutative algebras. An essentially different way is to play with quantum groups, and define the action of quantum groups on quantum spaces, generalizing the classical concept of a group action.

4. Examples of Quantum Spaces

4.1. Spaces with Indistinguishable Points

Non-commutative geometry provides a set of tools for the study of certain 'strange-behaving' spaces that naturally appear in classical geometry.

Example 1—Foliated Manifolds

These are the spaces of leaves of foliations of smooth manifolds. As a rule, such spaces behave very irregularly, from the classical point of view. A general phenomena is that it is not possible to introduce a smooth manifold structure or a reasonable topological structure on them.

As an extreme case, let us mention *ergodic foliations*, and in particular the spaces of orbits of ergodic dynamical systems.

In the ergodic case, it is not possible to introduce a reasonable concept of measurability into the leaf space. The reason for this is in the *effective indistinguishability* of points of the spaces of leaves of ergodic foliations. And if there are no non-trivial measurable sets, there is no geometry in the standard sense and all the tools of standard analysis loose their validity.

However, it turns out that all the above mentioned spaces can be naturally treated associating to them certain non-commutative *-algebras. One can then speak about differential and integral calculus, cohomological invariants and geometric structures over such spaces. Actually, all basic constructions of classical geometry can be generalized at the quantum level.

Example 2—Spectrums of Discrete Groups

As a similar type of 'quantum' spaces, we can mention the space of equivalence classes of irreducible unitary representations of certain discrete groups Γ.

If the von Neumann algebra A generated by the left-regular representation $\lambda\colon \Gamma \to B[l^2(\Gamma)]$ of Γ is a non-type-I factor, then the spectrum of Γ exhibits a similar undistingushibility-of-points property. It is worth mentioning that A is hyperfinite iff Γ is amenable. A group Γ is called amenable if there exists a left-invariant state on the C*-algebra $l^\infty(\Gamma)$ of all bounded continuous functions on Γ. Every finitely generated discrete group Γ is viewable as the fundamental group of a compact smooth 4-dimensional manifold.

Example 3—Penrose Tilings

The space of equivalence classes of certain tilings of the Euclidean plane, such as the Penrose tilings [4].

This space is defined as follows. Let us consider two triangles T_1 and T_2 of the Euclidean plane, defined by the lengths of edges—$(1,\tau,\tau)$ for the first triangle, and $(\tau,1,1)$ for the second. Here

$$\tau = \frac{1+\sqrt{5}}{2}$$

is the golden ratio number. Both triangles are naturally coming from a regular pentagon.

Let us also assume that edges of both triangles are oriented, in the sense $(+,+,-)$ and $(-,-,+)$ respectively (say, relative to counterclockwise orientation). Let $\mathcal{X}$ be the set of all tilings of the Euclidean plane that can be obtained using the above two triangles, and the rules:

(i) It is allowed to perform reflections of triangles;

(ii) The oriented edges are paired so that their orientation is the same.

Such tilings exist. The 3-parameter group $E(2)$ of isometric motions of the Euclidean plane is naturally acting on the space $\mathcal{X}$. Let X be the corresponding orbit space. It can be shown that X possesses (uncountably) infinitely many points. However the points of X are effectively indistinguishable, because of the following remarkable property:

Let $T_1, T_2 \in \mathcal{X}$ be arbitrary two non-equivalent tilings. Then for every finite portion (consisting of finitely many triangles) of T_1 there exists the same (modulo $E(2)$ portion of T_2.

It is important to mention that there exist two different interpretations of the relations between quantum spaces and the associated *-algebras. The first one is already explained—it assumes that spaces determine algebras and algebras determine spaces. The second interpretation (originally proposed by Connes) assumes that geometry is determined by the class of Morita-equivalent C*-algebras. In other words non-isomorphic but Morita equivalent C*-algebras describe the same quantum space. By definition, two C*-algebras A and B are Morita-equivalent if

$$A \otimes \mathcal{K}_\infty \cong B \otimes \mathcal{K}_\infty,$$

where $\mathcal{K}_\infty$ is the ideal of compact operators of a separable Hilbert space. Morita-equivalent algebras have the same cyclic cohomology and K-groups.

The second approach is more suitable for constructions involving the factor-spaces, as for example the structures mentioned in this subsection. However, it is not appropriate for considerations involving quantum groups and quantum bundles, where passing to a Morita-equivalent algebras destroys the entire geometrical structure.

4.2. **Quantum Groups and Quantum Bundles**

A very important class of examples of quantum spaces is given by *quantum groups*. These are, by definition, quantum spaces equipped with a 'group structure'. Here we shall outline how the concept of a compact group can be incorporated at the quantum level.

In accordance with our basic dictionary, it is natural to assume that the group structure on a quantum space G is described by a *-homomorhism $\phi : A \to A \otimes A$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A \otimes A \\ \phi\big\downarrow & & \big\downarrow \mathrm{id} \otimes \phi \\ A \otimes A & \xrightarrow[\phi \otimes \mathrm{id}]{} & A \otimes A \otimes A \end{array}$$

is commutative, and such that

$$A \otimes A = \overline{\Bigl\{ \sum a\phi(b) \Bigm| a, b \in A \Bigr\}}$$
$$A \otimes A = \overline{\Bigl\{ \sum \phi(b)a \Bigm| a, b \in A \Bigr\}}.$$

In such a way we arrive to *compact quantum groups* .

As a very important special class of compact quantum groups, let us mention *matrix* groups. These structures are specified by a C*-algebra A, together with a *-homomorphism $\phi\colon A \to A \otimes A$ and a matrix $u \in M_n(A)$ such that the *-algebra $\mathcal{A}$ generated by the entries u_{ij} is everywhere dense in A, and such that

$$\phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

It is also assumed that both u and its conjugate $\bar{u}$ are *invertible* matrices. It follows that

$$\phi(\mathcal{A}) \subseteq \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$$

and that the above mentioned coassociativity and density properties are satisfied automatically. Matrix groups generalize the idea of a compact Lie group. The algebra $\mathcal{A}$ plays the role of polynomial functions over G. The matrix u correspond to the fundamental representation of the group G. The theory of compact quantum groups was systematically developed in [28, 30].

As a basic example of a compact matrix quantum group, let us mention a quantum version of the $SU(2)$ group [27]. By definition the corresponding C*-algebra A is generated by elements α and γ, and relations

$$\alpha\alpha^* + \mu^2\gamma\gamma^* = 1 \qquad \alpha^*\alpha + \gamma^*\gamma = 1$$
$$\gamma\gamma^* = \gamma^*\gamma$$
$$\alpha\gamma = \mu\gamma\alpha \qquad \alpha\gamma^* = \mu\gamma^*\alpha \qquad \alpha^*\gamma = \frac{1}{\mu}\gamma\alpha^* \qquad \alpha^*\gamma^* = \frac{1}{\mu}\gamma^*\alpha^*$$

where $\mu \in [-1, 1] \setminus \{0\}$. The coproduct is specified by the above mentioned matrix rule

$$\phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj},$$

where the elements u_{ij} are the entries of a 2×2 A-matrix

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

The defining relations for A are actually equivalent to the *unitarity* of the above matrix.

Quantum groups provide a conceptual framework for generalizing the classical *concept of symmetry*. Indeed, in classical geometry symmetries of the space X are interpretable as automotphisms of the associated algebra A. This is straghtforwardly generalizable to the quantum level—we can define symmetries of a quantum space as (appropriate) automorphisms of the associated non-commutative algebra A. So symmetries always form a subgroup of the automorphism group $\mathrm{Aut}(A)$. Another way of incorporating the idea of symmetry is to generalize the concept of the *group action* rather than the one of the individual symmetries. In such a way we arrive to the concept of *an action of a quantum group on a quantum space.*

A fundamental class of quantum spaces possessing a built-in quantum group symmetry is given by *quantum principal bundles.* These objects are quantum counterparts of classical principal bundles. Quantum groups play the role of structure groups and general quantum spaces play the role of the base manifolds. The main geometrical idea is the same as in the classical theory—that of a fibered space on which the structure group acts freely on the right, so that the fibers are the orbits of this action.

If the quantum principal bundle and the structure group are represented by C*-algebras B and A respectively, then the right action of G on P is represented by a *-homomorphism $F: B \to B \otimes A$, such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & B \otimes A \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ B \otimes A & \xrightarrow[F\otimes\mathrm{id}]{} & B \otimes A \otimes A \end{array}$$

is commutative. The classical condition that the structure group is acting freely on the bundle is expressed as a density condition

$$B \otimes A = \overline{\left\{ \sum bF(q) \;\middle|\; b, q \in B \right\}}.$$

The above diagram corresponds to the requirement that the structure group is really 'acting' on the bundle.

The base manifold M is described by the F-fixed point subalgebra of B. Geometrically, this means that the functions on the base are just the functions on the bundle, constant along the action orbits.

As a quantum object, the structure group G is not understandable as a collection of elements. Therefore the action F is not reducible to a collection of single symmetry transformations. In other words the action of the whole quantum group is considered as a quantum symmetry (in the commutative case, F contains the information about all possible transformations of P induced by the elements of G).

A general theory of quantum principal bundles has been developed in [9, 10]. All basic topics of the classical theory (including a differential calculus, the formalism of connections, the theory of characteristic classes, classifying spaces and frame bundles geometry) can be naturally generalized and incorporated in the non-commutative context. A natural and potentially interesting application of quantum principal bundles in theoretical physics is to develop Yang-Mills theories over a quantum space-time, with a quantum local symmetry group.

4.3. **Quantum Tori and Finite Quantum Spaces**

A large array of purely quantum phenomenas can be illustrated on two very interesting 'completely pointless' quantum spaces—quantum 2-tori, and 'finite' quantum spaces based on matrix algebras.

By definition, a quantum 2-torus is based on the C*-algebra generated by the elements U and V and relations

$$U^{-1} = U^* \qquad V^* = V^{-1} \qquad VU = zUV,$$

where z is a fixed unit complex number. The algebra A_z will be commutative if and only if $z = 1$, and in this case it describes the classical 2-torus. If $z \neq 1$ then A_z describes a purely quantum object. This space is called *a quantum torus*. It has no points at all. Quantum torus has the same $U(1) \times U(1)$ symmetry of the classical torus, because the group $U(1) \times U(1)$ acts by automorphisms on A_z, as in the classical case–by phase shifts of generators U and V. It is worth noticing that if z is not a root of unity, then (and only then) the algebra A_z is simple. If z is a root od unity, then A_z is Morita-equivalent to the classical 2-torus algebra.

The quantum torus can be naturally viewed as a *quantum principal* $U(1)$*-bundle* over the base $M = U(1)$. This is a nice example of a quantum principal bundle, with the classical base and the classical structure group. The existence of such bundles have a deep impact to the classification problematics of bundles in non-commutative geometry. For example, the classifying space for $U(1)$ is still a quantum object.

Let us now consider algebras of the form $A = M_n(\mathbb{C})$. By definition, it means that A is consisting of complex $n \times n$-matrices. The algebraic operations are the standard addition and multiplication of matrices, and the *-structure is given by the adjoint operation. All matrix algebras are Morita-equivalent to complex numbers.

Quantum spaces based on such algebras are finite, because the algebras are finite-dimensional (and a classical space is finite iff its function algebra is finite-dimensional, and in this case the dimension of the algebra is the same as the number of points of the space). The space X has no points at all (because matrix algebras are always simple, and so do not admit characters). A purely quantum phenomena is that finite quantum spaces possess nontrivial *continuous symmetries*. In the case of X, all automorphisms of A are inner—given by the similarity transformations by unitary matrices. So the group of symmetries of X is $\mathrm{Aut}(A) = U(n)/U(1)$. Another purely quantum phenomena consists in *a topological non-triviality* of such spaces, and the nontriviality of differential structures over them. For example, the calculus based on derivations as explained above. The matrix spaces can be viewed as quantum counterparts of a 2-sphere, where the classical sphere is replaced by a quantum object consisting of n elementary quantum 'cells' in such a way that the classical $SO(3)$ symmetry is preserved. A detailed analysis of such quantum spaces can be found in [7, 15].

A very interesting possible application of these examples in physics is in formulating a Kaluza-Klein type theory [8, 15], where the internal space-time manifold is one of the quantum spaces based on matrix algebras. One of new purely quantum phenomenas appearing in such Kaluza-Klein theories is a possibility to interpret Higgs fields as parts of Yang-Mills multiplets.

A similar philosophy is applied in the Connes geometric model [4] of electroweak interactions, where the internal space-time manifold is described by a finite quantum space Λ of a more elaborated geometrical nature. The pure electrodynamics on the total space-time is reduced to the standard Weinberg-Salam model of electrodynamics and weak interactions, when viewed in terms of the classical 4-dimensional space-time.

4.4. **Supergeometry**

Supergeometry generalizes classical geometry by introducing the appropriate graded-commutative extensions of the algebras of smooth functions:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathrm{S}(M) \longrightarrow \mathrm{C}^\infty(M) \longrightarrow 0$$

where M is a smooth manifold and $\mathcal{K}$ is an ideal posessing the nilpotency property

$$\mathcal{K}^k = \{0\}.$$

for some $k \geq 2$. Supergeometry deals with supermanifolds, which are formally defined as dual objects to the extensions $\mathcal{A} = \mathrm{S}(M)$.

From the sheaf-theoretic point of view, a supermanifold is defined as a ringed space $(M, \mathcal{F})$ consisting of a smooth manifold M and a sheaf of $\mathbb{Z}/2$-graded algebras $\mathcal{F}$, which is locally isomorphic to the coordinate sheaf $\mathcal{F}_{m,M} = C^\infty_M \otimes [\mathbb{C}^m]^\wedge$, where C^∞_M is the standard sheaf of smooth functions and $m + 1 = k$. The algebra $\mathcal{A} = S(M)$ consists of global sections of the sheaf.

In terms of 'local coordinates' supermanifolds are described by local coordinate systems containing, besides standard local coordinates $\{x_1, \ldots, x_n\}$ for M also some mutually anticommuting coordinates $\{\theta_1, \ldots, \theta_m\}$. It is assumed that coordinates x_i and θ_j commute.

From the theoretical physics prospective, supergeometry plays a central role in the foundations of *supersymmetry*. A principal goal of supersymmetry was to provide a unifying view of bosonic and fermionic fields, and to establish a framework for a mathematically consistent formulation of quantum theory of gravity. The space-time is viewed as a supermanifold, and the symmetry is described by *supergroups*, which are the supergeometric counterparts of Lie groups. Elementary particles are grouped into supermultiplets, that generally contain both bosons and fermions.

For more informations about supergeometry, we refer to [21, 22, 23].

Many constructions of supergeometry are naturally generalizable to the level of *braided structures*. In this context, the *-algebra $\mathcal{A}$ is equipped with an additional structure, given by the appropriate operators $\sigma: \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ satisfying the braid equation

$$(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})=(\mathrm{id}\otimes\sigma)(\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\sigma).$$

The operator σ replaces the standard transposition, and is generally not involutive. In the context $\mathbb{Z}/2$-graded supercommutative algebras $\mathcal{A}$ (used in supergeometry) the braiding reduces to the involution

$$\sigma: a\otimes b\mapsto(-1)^{\partial a\partial b}b\otimes a.$$

In classical geometry, all the braidings are the standard transpositions. In quantum geometry, an interesting phenomena appears. It turns out that quantum objects with a sufficiently 'rich' geometrical structure (as quantum groups and quantum bundles) are always intrinsically braided, in the sense that the geometrical structure allows us to construct very interesting braid operators.

4.5. **Deformation Quantization of Symplectic Manifolds**

A very interesting class of examples of noncommutative *-algebras can be obtained by deforming the algebras of smooth functions over a symplectic manifold M, so that a *quantum correspondence principle* holds. This requirement is actually a central problem in the deformation quantization of symplectic manifolds. More precisely, let $\mathcal{A}$ be the (commutative) algebra of smooth functions on a symplectic manifold M. Let $\mathcal{A}[\nu]$ be the associated algebra of formal power series over a formal parameter ν. We say that a new associative product $\star$ introduced in the space $\mathcal{A}[\nu]$ satisfies the correspondence principle iff

$$f\star g=fg+\frac{\nu}{2i}\{f,g\}+\nu^2r(f,g),$$

where $\{,\}$ are Poisson brackets associated to M. The motivating idea behind this definition is that M plays the role of the phase space of a classical mechanical system. We assume that this classical system has a quantum couterpart, described by a noncommutative algebra, which is in fact $\mathcal{A}[\nu]$ equipped with the new product $\star$. The parameter ν plays the role of the Planck constant. The correspondence principle tells us that quantum commutator $i/\nu[*,*]$ coincides with the classical Poisson bracket, modulo terms of the order of ν.

There exists an intrinsically geometrical construction [12] of a noncommutative product $\star$ of the described type, for every symplectic manifold M. The following is a very brief sketch of this construction. We start from the formal Weyl algebra bundle $W[M]$ associated to (M,ν,ω), where ω is the initial symplectic form on M.

In other words, the fibers of $W[M]$ are the Weyl algebras associated to the tangent spaces $T_x(M), x \in M$, equipped with the symplectic scalar product $\nu\omega_x$. Let $\mathcal{W}$ be the algebra of formal power series with coefficients in the smooth sections of $W[M]$. It turns out that every symplectic torsion-free connection ∇ on M induces an injective map $j_\nabla : \mathcal{A}[\nu] \to \mathcal{W}$ with the following properties:

(i) The image of j_∇ is a subalgebra of $\mathcal{W}$. In fact this image coincides with the kernel of a naturally associated differential D, acting in the algebra $\Omega(M, W[M])$.

(ii) A new product $\star$ in $\mathcal{A}[\nu]$ defined by

$$f \star g = j_\nabla^{-1}[j_\nabla(f) j_\nabla(g)]$$

satisfies the above mentioned correspondence principle.

The final and crucial (from the physical viewpoint) step in the quantization of the considered system is to incorporate the construction in the conceptual framework of C*-algebraic physics [1]. This is done by constructing a C*-algebra $\widehat{A}$, by completing the appropriate *-subalgebra of $\mathcal{A}[\nu]$, and considering irreducible representations and superselection sectors of the completed algebra $\widehat{A}$.

4.6. C*-algebraic extensions and BDF-Theory

Let us consider a metrizable compact topological space Ω, and let us consider the classification problem for all possible short exact sequences of C*-algebras of the form

$$0 \longrightarrow \mathcal{K}_\infty \longrightarrow A \longrightarrow \mathrm{C}(\Omega) \longrightarrow 0$$

where $\mathcal{K}_\infty$ is the ideal of compact operators in a separable Hilbert space. It coincides with the commutant of A, in other words Ω is the space of characters of A. It can be shown [2] that homotopy classes of such extensions are in one-to-one correspondence with the elements of the first K-homology group of Ω. In other words, described non-commutative extensions reflect the topology of Ω. It is also possible to consider more general situations, where $\mathcal{K}_\infty$ is replaced by a different C*-algebra. For a general introduction to C*-algebraic extensions we refer to [24].

As a paradigmic example of a non-trivial extension of the described type, let us mention the extension generated by the shift operator T acting in $H = l^2(\mathbb{N})$. By definition, this operator is defined by

$$Te_n = e_{n+1} \qquad \forall n \in \mathbb{N}$$

where the vectors e_n form the canonical orthonormal basis in H. The algebra A is defined as the C*-algebra generated by T. We have $\Omega = U(1)$, as characters of A are completely determined by their values on T, and the values cover the whole unit circle $U(1)$. This extension plays a central role in a very elegant proof [24] of the Bott periodicity for general C*-algebras.

It is worth mentioning that extensions of the described type naturally appear in a C*-algebraic foundation of a causal *subquantum mechanics* [11] where A plays the role of the algebra of subquantum variables and Ω is the subquantum space of a given physical system.

References

[1] Brateli O, Robinson D: *Operator Algebras and Quantum Statistical Mechanics,* Volumes 1/2, Springer-Verlag (1979)

[2] Brown L G, Douglas R G, Filmore P G: *Extensions of C*-algebras and K-Homology,* Ann Math, 105 265–324 (1977)

[3] Connes A: *Non-commutative differential geometry,* Extrait des Publications Mathematiques–IHES 62 (1986)

[4] Connes A: *Noncommutative Geometry,* Academic Press (1994)

[5] Dixmier J: *Les algebres d'operateurs dans l'espace Hilbertien,* Gauthier-Villars (1957)

[6] Dubois-Violette: *Dérivations et calcul différentiel non-commutatif,* CR Acad Sci Paris 307 (1988)

[7] Dubois-Violette, R Kerner, J Madore: *Noncommutative Differential Geometry of Matrix Algebras,* J Math Phys 31 316–322 (1990)

[8] Dubois-Violette, R Kerner, J Madore: *Noncommutative Differential Geometry and New Models of Gauge Theory,* J Math Phys 31 323–335 (1990)

[9] Đurđevich M: *Geometry of Quantum Principal Bundles I,* Commun Math Phys 175 (3) 457–521 (1996)

[10] Đurđevich M: *Geometry of Quantum Principal Bundles II,* Rev Math Phys 9 (5) 531–607 (1997)

[11] Đurđevich M: *Algebro-Geometric Constructions of Subquantum Theories,* Doctoral Thesis, Faculty of Physics, University of Belgrade, Serbia (*in Serbian* :)) (1993)

[12] Fedosov B: *Deformation Quantization and Index Theory,* Akademie Verlag GmbH, Berlin (1996)

[13] Kastler D: *Cyclic Cohomology Within Differential Envelope,* Hermann, Paris (1988)

[14] Kastler D: *Introduction to Alain Connes' Non-Commutative Differential Geometry,* Karpacz Lectures (1986)

[15] Madore J: *An Introduction to Noncommutative Differential Geometry and Its Physical Applications,* Book–Second Edition (1998)

[16] Murray F J, von Neumann J: *On Rings of Operators,* Ann Math **37** 116–229 (1936)

[17] Murray F J, von Neumann J: *On Rings of Operators 2,* Trans Amer Math Soc **41** 208–248 (1937)

[18] Murray F J, von Neumann J: *On Rings of Operators 4,* Ann Math **44** 716–808 (1943)

[19] Podleś P: *Quantum Spheres,* Lett Math Phys 14 193–202 (1987)

[20] Podleś P, Woronowicz S L: *Quantum Deformation of Lorentz Group,* Commun Math Phys 130 381–431 (1990)

[21] Sanchez-Valenzuela O A: *On Supergeometric Structures,* Ph. D. Thesis, Harvard University, Cambridge (1986)

[22] Sanchez-Valenzuela O A: *Differential-Geometric Approach to Supervector Bundles,* Preprint, CIMAT, Guanajuato, México (1996)

[23] Rothstein M: *The Axioms of Supermanifolds and a New Structure Arising From Them,* Trans Amer Math Soc **297** 159–180 (1986)

[24] Wegge-Olsen N E: *K-Theory and C*-Algebras,* Oxford Science Publications (1993)

[25] Woronowicz S L: *Pseudospaces, Pseudogroups and Pontryagin duality,* Lecture Notes in Physics **116**, Proceedings of the International Conference on Mathematical Physics, Lausanne (1979)

[26] Woronowicz S L: *C^*-algebras Generated by Unbounded Elements,* Rev Math Phys, 7 (3) 481–521 (1995)

[27] Woronowicz S L: *Twisted $SU(2)$ group: An example of a noncommutative differential calculus,* RIMS, Kyoto University 23, 117–181 (1987)

[28] Woronowicz S L: *Compact Matrix Pseudogroups,* Commun Math Phys 111, 613–665 (1987)

[29] Woronowicz S L: *Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups),* Commun Math Phys 122, 125–170 (1989)

[30] Woronowicz S L: *Tannaka-Krein Duality for Compact Matrix Pseudogroups; Twisted SU(n) groups,* Invent Math 93, 35–76 (1988)

Instituto de Matematicas, UNAM, Area de la Investigacion Cientifica, Circuito Exterior, Ciudad Universitaria, México DF, CP 04510, MEXICO

E-mail address: micho@matem.unam.mx

http://www.matem.unam.mx/~micho