Higher generation of compact Lie groups by abelian subgroups

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What is this talk about?

- This talk is about a space $E(2, G)$ that one can assign to any topological group G.
- The space $E(2, G)$ knows something about which pairs of elements of G commute and which do not. In particular, it will follow from its definition that:
  If G is abelian then $E(2, G)$ is contractible.
- Simon Gritschacher, Bernardo Villarreal and I proved a strong converse for compact Lie groups:
  If G is a compact Lie group and $E(2, G)$ has $\pi_1 = \pi_2 = \pi_4 = 0$, then G is abelian.
- Background:
  - When G is discrete there is a better result due to Cihan Okay:
    If G is a discrete group and $\pi_1(E(2, G)) = 0$, then G is abelian.
  - A theorem of Araki, James and Thomas:
    If G is a compact, connected Lie group, and the commutator map $G \times G \to G$, $(g, h) \mapsto g^{-1}h^{-1}gh$ is null-homotopic, then G is abelian.

The Plan

- First I will tell you about the case of discrete groups.
  - A brief reminder about simplicial *complexes*.
  - Abels' and Holz's idea of "higher generation".
  - Cihan Okay's theorem: $\pi_1(E(2, G)) = 0$ implies G is abelian.
- Then I'll tell you about the Lie group case.
  - A brief reminder about simplicial *spaces*.
  - The definition of $E(2, G)$.
  - The main tool in our proof: the commutator map $E(2, G) \to B[G, G]$.

Simplicial complexes

Definition

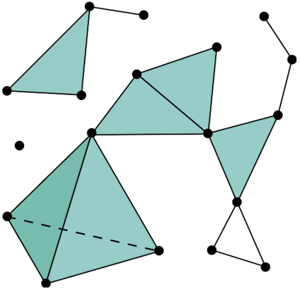
A *simplicial complex* K is a family of non-empty finite sets such that $\emptyset \neq \sigma \subset \tau \in K \implies \sigma \in K$.
The *vertices* are the elements of the singletons:

$$V(K) := \{v : \{v\} \in K\} = \bigcup_{\sigma \in K} \sigma.$$

Geometric realization $|K|$

We think of the elements of K as *simplices.*

- $\{u\} \in K$ is a vertex
- $\{u, v\} \in K$ is an edge
- $\{u, v, w\} \in K$ is a triangle
- etc.



Simplicial complexes associated to families of subsets

Let $\mathcal{U} = \{U_j : j \in J\}$ be a family of subsets of a set X.
Consider the following simplicial complexes:

- $N_{\mathcal{U}}$, the nerve of $\mathcal{U}$:
  $\{j_0, \ldots, j_n\}$ is a simplex of $N_{\mathcal{U}}$ is $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_n} \neq \emptyset$.
- $S_{\mathcal{U}}$, the subset complex of $\mathcal{U}$:
  $\{x_0, \ldots, x_n\}$ is a simplex of $S_{\mathcal{U}}$ if there is some $j \in J$ such that $\{x_0, \ldots, x_n\} \subseteq U_j$.
- $P_{\mathcal{U}}$, the order complex of the poset $(\mathcal{U}, \subseteq)$, whose simplices are chains $\{U_{j_0} \subseteq U_{j_1} \subseteq \cdots \subseteq U_{j_n}\}$.

Theorem (Abels and Holz)

- *$N_{\mathcal{U}}$ and $S_{\mathcal{U}}$ are homotopy equivalent.*
- *If $U_i \cap U_j \neq \emptyset \implies U_i \cap U_j \in \mathcal{U}$, then $P_{\mathcal{U}}$ is also homotopy equivalent to $N_{\mathcal{U}}$ and $S_{\mathcal{U}}$.*

Cosets

Let $\mathcal{F}$ be a family of subgroups of some discrete group G, closed under intersection.
The collection $G\mathcal{F} = \{gH : g \in G, H \in \mathcal{F}\}$ of all cosets of elements of $\mathcal{F}$ is a family of subsets of G to which we can apply the previous constructions.
We obtain three homotopy equivalent simplicial complexes whose simplices are:

- $N_{G\mathcal{F}}$ sets of cosets with non-empty intersection
- $S_{G\mathcal{F}}$ subsets of G contained in a single coset
- $P_{G\mathcal{F}}$ chains of cosets ordered by inclusion

If $G \in \mathcal{F}$, these are contractible.

Higher generation

Let $\mathcal{F}$ be a family of subgroups of some discrete group G, closed under intersection.

The group $H = \operatorname{colim}_{F \in \mathcal{F}} F$ has the following presentation:

generators x_g for $g \in \bigcup \mathcal{F}$,

relations $x_{gh} = x_g x_h$ whenever $g, h \in F$ for some $F \in \mathcal{F}$.

There is a canonical homomorphism $\kappa : H \to G$ given by $\kappa(x_g) = g$.

Theorem (Abels and Holz)

- $\pi_0(N_{G\mathcal{F}}) = G / \langle \bigcup \mathcal{F} \rangle$
- $\pi_1(N_{G\mathcal{F}}) = \ker \kappa$

Definition (Abels and Holz)

The family $\mathcal{F}$ is n-generating $\pi_k(N_{G\mathcal{F}}) = 0$ for all $k < n$.

The family of abelian subgroups

For the family $\mathcal{A}$ of abelian subgroups of G we can say more. First, recall that if G itself is abelian, $N_{G\mathcal{A}}$, $S_{G\mathcal{A}}$ and $P_{G\mathcal{A}}$ are contractible.

Theorem (Okay)

If $\pi_1(P_{G\mathcal{A}}) = 1$, then G is abelian.

Proof.

By Abels' & Holz's theorem, the group

$$H = \langle x_g : g \in G \mid x_{gh} = x_g x_h \text{ if } [g, h] = 1 \rangle$$

is isomorphic to G via $\kappa(x_g) = g$.
Since $(gh)^{-1} = g^{-1}h^{-1}$ whenever $[g, h] = 1$, the formula $x_g \mapsto x_g^{-1}$ defines an endomorphism of H. Therefore $g \mapsto g^{-1}$ defines an endomorphism of G and so G is abelian. □

Affine commutativity

Let G be a group and let $g_0, \ldots, g_n \in G$. Let's describe the simplices of $S_{G\mathcal{A}}$, where $\mathcal{A}$ = abelian subgroups of G.
The following are equivalent:

- $\{g_0, \ldots, g_n\}$ is contained in some coset of an abelian subgroup.
- $\{g_0^{-1}g_1, \ldots, g_0^{-1}g_n\}$ commute pairwise.
- $\{g_i^{-1}g_j : 0 \leq i, j \leq n\}$ commute pairwise.

If these conditions hold we say that $\{g_0, \ldots, g_n\}$ is an *affinely commutative* set. We'll call $S_{G\mathcal{A}}$ the *affine commutativity complex* of G.

Observations

- Any set of 1 or 2 elements is affinely commutative.
- A set of more than 3 elements is affinely commutative if and only if all of its 3 element subsets are affinely commutative.

The fundamental group of a simplicial complex

Let X be a connected simplicial complex and let T be a spanning tree for its 1-skeleton.
The fundamental group of the geometric realization of X has the following presentation:

generators x_{uv} for each $\{u, v\} \in X$.

relations
- $x_{vv} = 1$
- $x_{uv} = x_{vu}^{-1}$
- $x_{uv} = 1$ if $\{u, v\} \in T$
- $x_{uv}x_{vw} = x_{uw}$ if $\{u, v, w\} \in X$

(In particular, the fundamental group only depends on the vertices, edges and triangles of X, not on simplices of higher dimension.)

The fundamental group of the complex of affine commutativity

Since $S_{G\mathcal{A}}$ has all possible edges, we can pick T as the star centered at $1 \in G$ and obtain the following presentation $\pi_1(S_{G\mathcal{A}})$:

generators x_{gh} with $g, h \in G$

relations
- $x_{g1} = x_{1g} = 1$
- $x_{gh}x_{hk} = x_{gk}$ if $\{g, h, k\}$ is affinely commutative.

The commutator homomorphism

Lemma

$\{g, h, k\}$ is affinely commutative $\implies [g,h][h,k] = [g,k]$.

Proof.

$$\begin{aligned}(g^{-1}h^{-1}gh)(h^{-1}k^{-1}hk) &= g^{-1}(h^{-1}g)(k^{-1}h)k \\ &= g^{-1}(k^{-1}h)(h^{-1}g)k = [g,k]\end{aligned}$$

□

Therefore, there is a homomorphism $c : \pi_1(S_{G\mathcal{A}}) \to [G,G]$ defined on the generators by $c(x_{gh}) := [g,h]$.
Obviously c is surjective: its image includes all generators of $[G,G]$. Therefore, if $[G,G] \neq 1$, then $\pi_1(S_{G\mathcal{A}}) \neq 1$.

Theorem

If $S_{G\mathcal{A}}$ is simply connected, then G is abelian.

Simplicial spaces

A simplicial space $X_\bullet$ comprises:

- for each $n \geq 0$ a space X_n whose points we call n-simplices —we'll imagine each n-simplex as equipped with a fixed numbering of its vertices from 0 to n—, and
- continuous functions $d_j : X_n \to X_{n-1}$ for $0 \leq j \leq n$, that we interpret as follows: given an n-simplex $x \in X_n$, the simplex $d_j(x)$ is the face opposite vertex number j in x.

Geometric realization

Let $\iota_j : \Delta^{n-1} \to \Delta^n$ be the inclusion of the face opposite vertex j, and set $|X_\bullet| = \left(\coprod_{n\geq 0} X_n \times \Delta^n\right) / \sim$, where $(x, \iota_j(p)) \sim (d_j(x), p)$ for $x \in X_n$, $p \in \Delta^{n-1}$.

$E(2, G)$

Let G be a topological group.
We define a simplicial space $E_\bullet(2, G)$ with

$$E_n(2, G) = \{(g_0, \dots, g_n) : \{g_0, \dots, g_n\} \text{ is affinely commutative}\}.$$

$E_n(2, G) \subseteq G^{n+1}$ and we give it the subspace topology.
Vertex number j of $(g_0, \dots, g_n)$ is g_j and
$d_j(g_0, \dots, g_n) = (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_n)$.
The space $E(2, G)$ is the geometric realization of $E_\bullet(2, G)$.
(The original definition by Adem, F. Cohen and Torres Giese is different but isomorphic to this one).

What is known about these spaces?

- Not much!
- If G is abelian, $E(2,G) = EG$ is contractible.
- If G is discrete, $E(2,G)$ is homotopy equivalent to the complex of affine commutativity of G.
- We don't know much about the case of general topological groups, research has been focused on compact Lie groups.
- There is a variant $E(2,G)_1$ obtained by taking from $E_n(2,G)$ just the connected component $E_n(2,G)_1$ of $(1,\ldots,1)$. This space is more tractable, at least for rational cohomology calculations:

Theorem (Adem and Gómez)

If G is a compact connected Lie group, $E(2,G)_1$ has the homotopy type of a CW complex with finitely many cells and $H^*(E(2,G)_1,\mathbb{Q}) \cong \left(H^*(G/T;\mathbb{Q}) \otimes H^*(G/T;\mathbb{Q})\right)^W$, where T is a maximal torus in G and $W = N_G(T)/T$ is the Weyl group.

Concrete calculations of the homotopy type of $E(2, G)$?

Very few!

Theorem (Okay)

If G is an extraspecial group of order 32, then $\pi_1(E(2, G)) = \mathbb{Z}/2$ and the universal cover of $E(2, G)$ is homotopy equivalent to $\bigvee^{151} S^2$. Thus, for example, $\pi_2(E(2, G)) \cong \mathbb{Z}^{151}$, y $\pi_3(E(2, G)) \cong \mathbb{Z}^{11476}$. (!)

Theorem (Gritschacher)

$E(2, U) \simeq BU \times BU\langle 6 \rangle \times BU\langle 8 \rangle \times \cdots$, where $BU\langle 2n \rangle$ is the $(2n-1)$-connected cover of BU. Thus, $\pi_{2n}(E(2, U)) = \mathbb{Z}^{n-1}$ y $\pi_{2n+1}(E(2, U)) = 0$.

Theorem (A., Gritschacher, Villarreal)

$E(2, O(2)) \simeq S^3 \vee S^2 \vee S^2$ and $E(2, SU(2)) \simeq S^4 \vee \Sigma^4 \mathbb{RP}^2$. Thus, for example, $\pi_{10}(E(2, SU(2))) = \mathbb{Z}/4 \oplus (\mathbb{Z}/24)^2$ y $\pi_{10}(E(2, O(2))) = \mathbb{Z}^{308} \oplus (\mathbb{Z}/2)^{215} \oplus (\mathbb{Z}/3)^4 \oplus (\mathbb{Z}/15)^4 \oplus (\mathbb{Z}/24)^{34}$

Commutator map

A big advantage of the definition I gave you of $E(2, G)$ over the original isomorphic but slightly different definition, is that it suggests defining the following simplicial map:

$$\mathfrak{c}_\bullet : E_\bullet(2, G) \to B_\bullet[G, G]$$
$$\mathfrak{c}_n(g_0, g_1 \ldots, g_n) = ([g_0, g_1], [g_1, g_2], \ldots, [g_{n-1}, g_n])$$

Its geometric realization is the *commutator map*
$\mathfrak{c} : E(2, G) \to B[G, G]$.
When G is discrete, $\mathfrak{c}$ induces the homomorphism
$c : \pi_1(E(2, G)) \to [G, G]$ given by $c(x_{gh}) = [g, h]$ that we used before.
The existence of $\mathfrak{c}$ is a bit of a miracle. The space $E(2, G)$ is called that because it is a part of a family $E(q, G)$ defined in terms of nilpotent subgroups of class less than q. For $q > 2$ we don't know how to define something like $\mathfrak{c}$.

Our main theorem

Theorem (A., Gritschacher, Villarreal)

For a compact Lie group the following are equivalent:

- G *is abelian.*
- $E(2, G)$ *is contractible.*
- $\mathfrak{c} : E(2, G) \to B[G, G]$ *is null-homotopic.*
- $\pi_k(E(2, G)) = 0$ *for* $k = 1, 2, 4$.

Homotopy-abelian groups

For compact connected Lie groups, the implication
"$\mathfrak{c}$ null-homotopic $\implies$ G is abelian",
can be deduced from a classic theorem of Araki, James and Thomas:

Theorem (Araki, James, Thomas)

If G is a compact connected Lie group, and the algebraic commutator map $G \times G \to G$, $(g, h) \mapsto [g, h]$ is null-homotopic, then G is a torus.

The proof relies on the classification of Lie groups.
Warning: This is false for disconnected groups! The group $G = (S^1 \times Q_8)/D$ where $D = \langle(-1, -1)\rangle \leq S^1 \times Q_8$ is homotopy-abelian, but not abelian. Indeed, $[G, G] = (1, \pm 1) \pmod{D} = (\mp 1, 1) \pmod{D}$. So $[G, G]$ is contained in the image of $S^1 \times \{1\}$ in G, which is $\cong S^1$ and thus path-connected.

$\mathfrak{c}$ null-homotopic $\implies$ G is abelian

If $X_\bullet$ is a simplicial space, we can take a truncated geometric realization, $F_N|X_\bullet|$, given by the image of $\coprod_{n=0}^{N} X_n \times \Delta^n$ in $|X_\bullet|$. There is a commutative square:

$$\begin{array}{ccccccc} \Sigma G \wedge G & = & F_1 E(2,G) & \xrightarrow{F_1\mathfrak{c}} & F_1 B[G,G] & = & \Sigma[G,G] \\ & & \downarrow & & \downarrow & & \\ & & E(2,G) & \xrightarrow{\mathfrak{c}} & B[G,G] & & \end{array}$$

The top horizontal composition is the suspension of the map $G \wedge G \to G$, $g \wedge h \mapsto [g,h]$.
If $\mathfrak{c}$ is null-homotopic, then so is the composite $\Sigma G \wedge G \to B[G,G]$, and so is its adjunct $G \wedge G \to [G,G] \simeq \Omega B[G,G]$.
Since the commutator $G \times G \to G$ factors through $G \wedge G$, the commutator is then also null-homotopic, and by the theorem of Araki, James and Thomas we deduce that G is a torus.

Sketch of the proof of the main theorem

- $\Omega\mathfrak{c} : \Omega E(2, SU(2)) \to SU(2)$ has a section.
- If $\pi_4(E(2, G)) = 0$, then the identity component G_0 of G is a torus.
  This step uses that or any simply-connected simple Lie group K there is homomorphism $SU(2) \hookrightarrow K$ inducing an isomorphism on π_3.
- If G_0 is a torus and $E(2, G)$ is 2-connected, then $\mathfrak{c} : E(2, G) \to B[G, G]$ is null-homotopic.
  $[G, G] \subset G_0$ is also a torus, say of rank r, and maps like that correspond to classes in $H^2(E(2, G); \mathbb{Z}^r) = 0$.

The End

Thank you!

The nerve theorem

The nerve of a cover

If $\mathcal{U} = \{U_j : j \in J\}$ is a family of subsets of X, the *nerve* of $\mathcal{U}$ is the simplicial complex $N_{\mathcal{U}}$ with:

- **vertices** the elements of J
- **simplices** the $\{j_0, \ldots, j_n\}$ such that $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_n} \neq \emptyset$.

The Nerve Theorem

If $\mathcal{U}$ satisfies that for any $\{j_0, \ldots, j_n\} \subseteq J$ the intersection $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_n} \neq \emptyset$ is either *empty* or *contractible*, then X is homotopy equivalent to $N_{\mathcal{U}}$.

The fine print

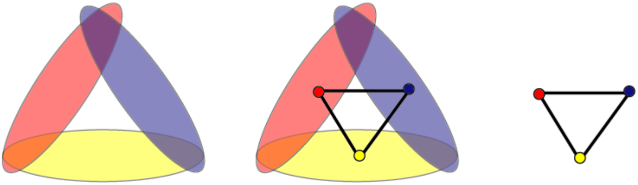
The theorem holds if:

- $\mathcal{U}$ is an *open* cover of a space X, or
- $\mathcal{U}$ is cover of a simplicial complex X by *subcomplexes*.

Example of the Nerve Theorem

Say $X = U_1 \cup U_2 \cup U_3$ with:

- U_i contractible,
- $U_i \cap U_j$ contractible, and
- $U_1 \cap U_2 \cap U_3$ empty.



Then $N_{\{U_1, U_2, U_3\}}$ is the boundary of a triangle.

The nerve and the subset complex

- For $x \in X$, let $J_x = \{j \in J : x \in U_j\}$.
- All non-empty finite subsets of J_x are simplices of $N_{\mathcal{U}}$.
- Let N_x be the subcomplex of $N_{\mathcal{U}}$ formed by these simplices. $\mathcal{N} := \{N_x : x \in X\}$ is a cover of $N_{\mathcal{U}}$.
- $N_{x_0} \cap \cdots \cap N_{x_n}$ consists of all non-empty finite subsets of $\{j \in J : \{x_0, \ldots, x_n\} \subseteq U_j\}$. Therefore, this intersection is empty when that set is and contractible otherwise.
- By the nerve theorem, $N_{\mathcal{U}} \simeq N_{\mathcal{N}} \cong S_{\mathcal{U}}$.

The order complex of the cover

- For $x \in X$, let $\mathcal{U}_x = \{U_j \in J : x \in U_j\}$.
- Let P_x be the order complex of $(\mathcal{U}_x, \subseteq)$.
  $\mathcal{P} := \{P_x : x \in X\}$ is a cover of $P_{\mathcal{U}}$.
- $P_{x_0} \cap \cdots \cap P_{x_n}$ is the order complex of $\mathcal{I} := \{U_j : \{x_0, \ldots, x_n\} \subseteq U_j\}$. Therefore the intersection is empty if $\mathcal{I} = \emptyset$.
- If $\mathcal{I} \neq \emptyset$ and $\mathcal{U}$ is closed under non-empty intersection, then $\mathcal{I}$ is directed and thus its order complex is contractible.
- By the nerve theorem, $P_{\mathcal{U}} \simeq N_{\mathcal{P}} \cong P_{\mathcal{U}}$.