Classifying spaces for commutativity in groups

IPPICTA

Omar Antolín Camarena
IMATE UNAM - CDMX

Spaces of commuting elements

Spaces of commuting elements

For a topological group G we can topologize the set of group homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ as a subspace of G^n by associating to each $\phi : \mathbb{Z}^n \to G$ the point $(\phi(e_1), \ldots, \phi(e_n))$.

$$\mathrm{Hom}(\mathbb{Z}^n, G) \cong \{(g_1, \ldots, g_n) \in G^n : g_i g_j = g_j g_i\}.$$

Example: commutativity in $SU(2)$

- $SU(2)$ is the group of unit quaternions.
- We write a quaternion as $a + u$ where $a \in \mathbb{R}$ is the real part, and $u = bi + cj + dk \in \mathbb{R}^3$ is the imaginary part.
- Multiplication is given by $uv = -u \cdot v + u \times v$.
- $a + u$ and $b + v$ commute if and only if u and v are parallel.

Example: commuting pairs in $SU(2)$, I

$$p : S^2 \times S^1 \times S^1 \to \mathrm{Hom}(\mathbb{Z}^2, SU(2))$$
$$(v, a_1 + a_2 i, b_1 + b_2 i) \mapsto (a_1 + a_2 v, b_1 + b_2 v)$$

- p is surjective and $p(v, a, b) = p(-v, \bar{a}, \bar{b})$.
- p descends to a map

$$\bar{p} : (S^2 \times S^1 \times S^1)/\sim \to \mathrm{Hom}(\mathbb{Z}^2, SU(2)).$$

Example: commuting pairs in $SU(2)$, II

- $p(v, a_1 + a_2 i, b_1 + b_2 v) = (a_1 + a_2 v, b_1 + b_2 v)$
- $\bar{p}([v, \pm 1, \pm 1]) = (\pm 1, \pm 1)$.
- $\bar{p}$ is an embedding when restricted to

$$S^2 \times (S^1 \times S^1 \setminus \{\pm 1\} \times \{\pm 1\}).$$

- So $\mathrm{Hom}(\mathbb{Z}^2, SU(2)$ is obtained from $(S^2 \times S^1 \times S^1)/\sim$ by collapsing each of four copies of $\mathbb{RP}^2$ to a point.

Homotopical behavior of $\mathrm{Hom}(\mathbb{Z}^n, G)$

If $f : H \to G$ is both a *group homomorphism* and a *homotopy equivalence*, then for many purposes H and G have the same homotopical behavior.

But $\mathrm{Hom}(\mathbb{Z}^n, G)$ and $\mathrm{Hom}(\mathbb{Z}^n, H)$ need not even have the same number of connected components! Not even if G is a Lie group and $H = K$ is its maximal compact subgroup.

Maximal compact subgroups

- A connected Lie group G always has a maximal compact subgroup K.
- All the maximal compact subgroups are conjugate to each other.
- G is homeomorphic to $K \times \mathbb{R}^d$ for some d, but not isomorphic as a group.
- Even if G is a complex Lie group, K is a real Lie group.
- Basic examples: $G = GL(n, \mathbb{R})$, $K = O(n)$; $G = GL(n, \mathbb{C})$, $K = U(n)$.

Reductive algebraic groups

Pettet and Souto (2013): If G is the group of (complex resp. real) points of a (complex resp. real) reductive algebraic group, and K is its maximal compact subgroup, then the inclusion of $\mathrm{Hom}(\mathbb{Z}^n, K)$ into $\mathrm{Hom}(\mathbb{Z}^n, G)$ is a homotopy equivalence.

Examples of reductive algebraic groups: $GL(n)$, $SL(n)$, $SU(n)$, $SO(n)$, $Sp(2n)$.

This result does not hold for non-algebraic groups!

The Heisenberg group

- $G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R}/\mathbb{Z} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}$. G is not algebraic.
- $\begin{pmatrix} 1 & a & [c] \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & x & [z] \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ commute if and only if $ay - bx \in \mathbb{Z}$.
- Thus $\mathrm{Hom}(\mathbb{Z}^n, G)$ has infinitely many connected components.
- The maximal compact subgroups is the $K = \mathbb{R}/\mathbb{Z}$ in the corner, so $\mathrm{Hom}(\mathbb{Z}^n, K) = K^n$ is connected.

A source of commuting elements

Let G be a compact, connected Lie group, let T be its maximal torus, and $W = N(T)/T$ be its Weyl group.

Consider the map

$$\varphi : (G/T \times T^n)/W \to \mathrm{Hom}(\mathbb{Z}^n, G)$$

given by

$$[gT, (t_1, \dots, t_n)] \mapsto (gt_1g^{-1}, \dots, gt_ng^{-1}).$$

It can be shown its image is the connected component of the trivial homomorphism, denoted by $\mathrm{Hom}(\mathbb{Z}^n, G)_1$.

Rational cohomology of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$

- Baird (2007) proved φ induces an isomorphism on rational cohomology, so

$$H^*(\mathrm{Hom}(\mathbb{Z}^n, G)_1) \cong (H^*(G/T) \otimes H^*(T)^{\otimes n})^W.$$

- Ramras and Stafa (2021) gave a formula for the Poincaré series of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$, that is, the generating function of the Betti numbers: if $H^*(G)$ is an exterior algebra on generators in degrees $2d_i - 1$, the Poincaré series is

$$\frac{1}{|W|} \prod (1 - t^{2d_i}) \prod_{w \in W} \frac{\det(1 + tw)^n}{\det(1 - t^2 w)}.$$

Torsion in the homology of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$

Kishimoto and Takeda (2022) showed that the integral homology of $\mathrm{Hom}(\mathbb{Z}^n, G)_1$ has p-torsion if and only if p divides $|W|$ for $G = SU(n), G_2, F_4, E_6$.

Homotopy groups of Hom($\mathbb{Z}^n, G$)

Let G be a compact, connected Lie group.

- Gómez, Pettet and Souto (2012) proved that $\pi_1(\mathrm{Hom}(\mathbb{Z}^n, G)_1) \cong \pi_1(G)^n$.
- Adem, Gómez, Gritschacher (2022) computed $\pi_2(\mathrm{Hom}(\mathbb{Z}^n, G))$ for $G = SU(m), Sp(m)$.
- Jaime García Villeda computed the rank of $\pi_3(\mathrm{Hom}(\mathbb{Z}^n, G)) \otimes \mathbb{Q}$ for $Q = SU(m), Sp(m)$.

Classifying spaces for commutativity

Brief reminder of simplicial spaces

A simplicial space $X_\bullet$ consists of a sequence of topological spaces X_n for $n = 0, 1, 2, \ldots$ and maps $d_i : X_n \to X_{n-1}$ for $0 \leq i \leq n$.

The points of X_n are called n-simplices, and you should think of each n-simplex as having a fixed numbering of its vertices with the numbers 0 through n. The map d_i gives you the face opposite the vertex numbered i.

The geometric realization is given by:

$$|X_\bullet| := \left(\coprod_{n \geq 0} X_n \times \Delta^n \right) / \sim .$$

The classifying space for commutativity

For a fixed G, as you vary n, the spaces $\mathrm{Hom}(\mathbb{Z}^n, G)$ assemble to form a simplicial space! The face maps multiply adjacent coordinates (except d_0 and d_n which simply drop the first or last coordinate).

$$B_{\mathrm{com}} G := |\mathrm{Hom}(\mathbb{Z}^\bullet, G)|$$

This is a simplicial subspace of a classic model for the classifying space of G, namely $BG := |G^\bullet|$.

Principle G-bundles

The space BG is called the classifying space of G because there is a bijection between isomorphism classes of principal G-bundles on a space X and homotopy classes of maps $X \to BG$.

A principle G-bundle on X is a space Y with a free G-action, together with a homeomorphism $X \cong Y/G$, and such that for any $x \in X$ there is some open $U \ni x$ where the quotient map $Y \to X$ has a section.

The universal principal G-bundle

The principal G-bundle corresponding to a map $X \to BG$ can be obtained by pulling back to X a fixed universal bundle $EG \to BG$.

There is a simplicial model for EG, namely $EG = |G^{\bullet+1}|$ where the face maps just drop the corresponding coordinate.

The quotient map $G^{n+1} \to G^n$ is given by

$$(g_0, \ldots, g_n) \mapsto (g_0^{-1} g_1, \ldots, g_{n-1}^{-1} g_n).$$

$E_{\text{com}}G$

Just like $B_{\text{com}}G$ came from a simplicial subspace of the model for BG, we can define a corrsponding $E_{\text{com}}G$ from a simplicial subspace of the model for EG.

$E_{\text{com}}G := |X_\bullet|$, where $X_n = \{(g_0, \dots, g_n) \in G^{n+1} : g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n \text{ commute pairwise}\}$.

Affinely commuting elements

The following are equivalent:

- $g_0^{-1}g_1, \ldots, g_{n-1}^{-1}g_n$ commute pairwise,
- all quotients $g_i^{-1}g_j$ commute pairwise,
- there is some abelian subgroup A of G such that $g_i \in g_0A$.

We say $g_0, \ldots, g_n$ are *affinely commutative*.

The commutator map

Consider the following map from the space of affinely commutative $(n+1)$-tuples in G to $[G, G]^n$:

$$\mathfrak{c}_n(g_0, g_1 \dots, g_n) = ([g_0, g_1], [g_1, g_2], \dots, [g_{n-1}, g_n])$$

This gives a simplicial map between the simplicial models for $E_{\text{com}}G$ and $B[G, G]$, whose geometric realization is called the *commutator map* $\mathfrak{c} : E_{\text{com}}G \to B[G, G]$.

The existence of $\mathfrak{c}$ is a bit of a miracle. The space $E_{\text{com}}G)$ is part of a family $E(q, G)$ defined in terms of nilpotent subgroups of class less than q. For $q > 2$ we don't know how to define something like $\mathfrak{c}$.

When is $E_{\mathrm{com}}G$ contractible?

If G is abelian, then $E_{\mathrm{com}}G = EG$ is contractible.
And for $G = SL(2,\mathbb{R})$, we have that
$E_{\mathrm{com}}G \simeq E_{\mathrm{com}}SO(2)$ is also contractible.

A., Gritschacher, Villarreal (2021): For a compact Lie group the following are equivalent:

- G is abelian.
- $E_{\mathrm{com}}G$ is contractible.
- $\mathfrak{c} : E_{\mathrm{com}}G \to B[G, G]$ is null-homotopic.
- $\pi_k(E_{\mathrm{com}}G) = 0$ for $k = 1, 2, 4$.

Homotopy-abelian groups

For compact connected Lie groups, the implication

"$\mathfrak{c}$ null-homotopic $\implies$ G is abelian",

can be deduced from a classic theorem of Araki, James and Thomas: If G is a compact connected Lie group, and the algebraic commutator map $G \times G \to G$, $(g, h) \mapsto [g, h]$ is null-homotopic, then G is a torus.

The proof relies on the classification of Lie groups.

Warning: This is false for disconnected groups!

Does $B_{\mathrm{com}}G$ classify some kind of bundle?

$B_{\mathrm{com}}G$ classifies principal G-bundles with a *transitionally commutative* structure.

To specify such a structure on a G-bundle $Y \to X$, pick an open cover of X on which there are local sections for which the corresponding transitions functions commute pairwise.

(Given two sections $s : U \to Y$ and $t : V \to Y$ the transition function between them is the unique function $\phi : U \cap V \to G$ such that $t(x) = \phi(x) \cdot s(x)$.)

Equivalence of TC-bundles

Giving such an open cover lets you factor the classifying map $X \to BG$ through $B_{\text{com}}G$ up to homotopy. We say two transitionally commutative bundles are equivalent if their classifying maps $X \to B_{\text{com}}G$ are homotopic.

Warnings

- A single principal G-bundle can have many different inequivalent transitionally commutative structures or none at all!
- Even the trivial bundle usually has many inequivalent transitionally commutative structures, which are in bijection with homotopy classes of maps $X \to E_{\text{com}}G$.

$B_{\text{com}} G_1$ and $E_{\text{com}} G_1$

Corresponding to the connected component $\text{Hom}(\mathbb{Z}^n, G)_1$ of $(1, 1, \ldots, 1)$ of the space of commuting n-tuples, we can define:

$$B_{\text{com}} G_1 := |\text{Hom}(\mathbb{Z}^\bullet, G)_1|$$

and $E_{\text{com}} G_1 := |Z_\bullet|$ where Z_n is the connected component of $(1, \ldots, 1)$ in the space of affinely commuting $(n+1)$-tuples.

Rational cohomology results

Let G be a compact, connected Lie group, let T be its maximal torus, and $W = N(T)/T$ be its Weyl group.

- Classical: $H^*(BG) \cong H^*(BT)^W$.
- Adem and Gómez (2015):

$$H^*(B_{\mathrm{com}}G_1) = (H^*(BT) \otimes H^*(G/T))^W$$
$$H^*(E_{\mathrm{com}}G_1) = (H^*(G/T) \otimes H^*(G/T))^W$$

Some specific calculations

A., Gritschaher, Villarreal (2019) computed for the low-dimensional Lie groups $SU(2), U(2), O(2), SO(3)$[1]:

- the integral cohomology ring of $B_{\text{com}}G$,
- the mod 2 cohomology ring of $B_{\text{com}}G$ and the action of the Steenrod algebra on it,
- the homotopy type of $E_{\text{com}}G$.

Jana (2023) computes the mod 2 and mod 3 cohomology groups of $E_{\text{com}}U(3)$.

[1] For $SO(3)$ the calculations are only for $B_{\text{com}}G_1$ and $EcomG_1$.

Homotopy type of $E_{\mathrm{com}}G$ for Lie groups

Gritschaher (2018): For the infinite unitary group we have $E_{\mathrm{com}}U \simeq BU\langle 4\rangle \times BU\langle 6\rangle \times BU\langle 8\rangle \times \cdots$ and $B_{\mathrm{com}}U \simeq BU \times E_{\mathrm{com}}U$, where $BU\langle 2n\rangle$ is the $(2n-1)$-connected cover of BU. Thus, $\pi_{2n}(B_{\mathrm{com}}U) = \mathbb{Z}^n$ y $\pi_{2n+1}(B_{\mathrm{com}}U) = 0$.

A., Gritschaher, Villarreal (2019): $E_{\mathrm{com}}O(2) \simeq S^3 \vee S^2 \vee S^2$ and $E_{\mathrm{com}}SU(2) \simeq S^4 \vee \Sigma^4\mathbb{RP}^2$. Thus, for example, $\pi_{10}(E_{\mathrm{com}}SU(2)) = \mathbb{Z}/4 \oplus (\mathbb{Z}/24)^2$ and $\pi_{10}(E_{\mathrm{com}}O(2)) = \mathbb{Z}^{308} \oplus (\mathbb{Z}/2)^{215} \oplus (\mathbb{Z}/3)^4 \oplus (\mathbb{Z}/15)^4 \oplus (\mathbb{Z}/24)^{34}$

Homotopy type of $E_{\text{com}}G$ for discrete G

Several people independently showed that when G is discrete $E_{\text{com}}G$ has the homotopy type of the order complex of the poset of cosets of abelian subgroups of G.

- Okay (2014): If G is an extraspecial group of order 32, then $\pi_1(E_{\text{com}}G) = \mathbb{Z}/2$ and the universal cover of $E_{\text{com}}G$ is homotopy equivalent to $\bigvee^{151} S^2$.
  Thus, for example, $\pi_2(E_{\text{com}}G) \cong \mathbb{Z}^{151}$, y $\pi_3(E_{\text{com}}G) \cong \mathbb{Z}^{11476}$. (!)

Geometric 3-manifolds

A *model geometry* is a simply connected manifold X with a transitive action of a Lie group with compact stabilizers; it is called *maximal* if G is maximal among groups acting transitively on X with compact stabilizers.

A *geometric manifold* is a manifold of the form X/Γ where (G, X) is some maximal model geometry and Γ is a discrete subgroup of G thats acts freely on X.

Thurston showed that there are eight 3-dimensional maximal model geometries for which some compact geometric manifold exists: S^3, $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{PSL}_2(\mathbb{R})$, Nil, and Sol.

$E_{\text{com}}G$ for geometric 3-manifold groups

A., García-Hernández, Sánchez-Saldaña (2023): Let G be the fundamental group of an orientable geometric 3-manifold. Then $E_{\text{com}}G$ is homotopically equivalent to $\bigvee_I S^1$, where I is a (possibly empty) countable index set.

- I is empty if and only if G is abelian.
- I is finite and non-empty if and only if G is non-abelian and is the fundamental group of a spherical 3-manifold.
- I is infinite if and only if G is infinite and nonabelian.