Classifying crystalline interacting topological phases through equivariant cohomology

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A disclaimer

- I don't really know what I'm talking about!
- I'm a mathematician.
- My coauthor Daniel Sheinbaum is the one who understands the physics.
- He's kindly agreed to help me answer questions.

The homotopy model of adiabatic evolution

We will work only with gapped Hamiltonians —that's already a strong(ish) restriction!

We will define phases for a class of gapped Hamiltonians as follows:

Two systems are said to be in the same phase if there is a continuous one-parameter family of gapped Hamiltonians in the given class $H(s)$, $0 \leq s \leq 1$ connecting the Hamiltonians $H(0)$ and $H(1)$ of the two systems.

Note that this says two things:

1. During adiabatic evolution Hamiltonians vary continuously.
2. *Any* continuous variation of Hamiltonians can be realized physically as adiabatic evolution.

Condition 1 is widely accepted, condition 2 less so.

This is already a bit of topology!

What that the definition says is that the set of phases for a class of Hamiltonians is given by what topologists would call $\pi_0(\mathcal{H})$, where $\mathcal{H}$ is the *space* of Hamiltonians in the given class.

$\pi_0(\mathcal{H})$ denotes the set of *path components* of the space $\mathcal{H}$. A path component is a subset of $\mathcal{H}$ consisting of all points that can be connected to a fixed one via a continuous path.

We can think of a Hamiltonian for a single-particle d-dimensional system as an operator on the complex Hilbert space $L^2(\mathbb{R}^d; \mathbb{C})$. So spaces of Hamiltonians are spaces of operators on a Hilbert space and much is known about them.

A single particle on a crystal

Now let's restrict to infinite crystals.

This means the Hamiltonian commutes with translations from a lattice: $[H, T_r] = 0$ for all $r \in \Lambda$, where $T_r(f)(x) = f(x - r)$.

For example, a Hamiltonian for a single particle might look like $H = -\nabla^2 + V(x)$ where the potential satisfies $V(x + r) = V(x)$ whenever $r \in \Lambda$.

Bloch's theorem

There is a basis of wave functions that are energy eigenstates of the form $\psi(x) = e^{ik\cdot x}u(x)$ for some function $u(x)$ with periods given by Λ, that is, such that $u(x + r) = u(x)$ for $r \in \Lambda$.

A functional analysis view of Bloch's theorem

Using ideas from functional analysis, we can interpret Bloch's theorem in the following way:

- For each k the functions of the form $e^{ik\cdot x}u(x)$ for some Λ-periodic u form a Hilbert subspace $\mathfrak{h}_k$ of the Hilbert $\mathfrak{h}$ space on which the Hamiltonian acts.
- These "small" Hilbert spaces form a *direct integral decomposition* of $\mathfrak{h}$ over the space of all possible k, the Brillouin zone $\widehat{\Lambda}$, which is a torus, $\mathbb{T}^d$!

$$\mathfrak{h} = \int_{\widehat{\Lambda}}^{\oplus} \mathfrak{h}_k \, dk.$$

- Bloch's theorem then says that a translation-invariant Hamiltonian H has a direct integral decomposition too, in terms of the *Bloch Hamiltonians* H_k acting on the $\mathfrak{h}_k$, which are also (almost all) gapped:

$$H \simeq \int_{\widehat{\Lambda}}^{\oplus} H_k \, dk.$$

Symmetry begets topology

Bloch's theorem brought with it the gift of a torus, but that torus has nothing to do with the Hamiltonian, it came just from the translation symmetries. What to do with the Hamiltonian?

Kuiper's theorem implies any bundle of Hilbert spaces is trivial. So we can identify all the "small" Hilbert spaces $\mathfrak{h}_k$ with a single one $\mathfrak{h}'$.

Now, for every point k in the Brillouin zone, we can take the ground state $\mathfrak{G}_k$ of the Bloch Hamiltonian H_k. This is an m-dimensional subspace of $\mathfrak{h}'$, where m is the ground state degeneracy.

The space of all m-dimensional subspaces of $\mathfrak{h}'$ is called a *Grassmanian* and is denoted by $\mathrm{Gr}_m(\mathfrak{h}')$.

Thus to any translation-invariant Hamiltonian H we have associated a "Bloch ground state map" $g_H : \mathbb{T}^d \to \mathrm{Gr}_m(\mathfrak{h}')$, given by $g_H(k) = \mathfrak{G}_k$.

Phases are homotopy classes of maps

We're now working with ground state maps $g_H : \mathbb{T}^d \to \mathrm{Gr}_m(\mathfrak{h}')$ in place of the original Hamiltonian. What does adiabatic evolution mean for them?

Same as before! They live in a space $\mathrm{Map}(\mathbb{T}^d, \mathrm{Gr}_m(\mathfrak{h}'))$ and two are in the same phase if they can be connected by a continuous path in that space.

But paths in spaces of maps can be confusing and there's a more down-to-earth way to think of them, as *homotopies*.

A homotopy between two maps $H_0, H_1 : \mathbb{T}^d \to \mathrm{Gr}_m(\mathfrak{h}')$ is a single map $\bar{H} : \mathbb{T}^d \times [0,1] \to \mathrm{Gr}_m(\mathfrak{h}')$ such that $\bar{H}(x,0) = H_0(x)$ and $\bar{H}(x,1) = H_1(x)$.

The set of homotopy classes of maps is denoted by $[\mathbb{T}^d, \mathrm{Gr}_m(\mathfrak{h}')] = \pi_0(\mathrm{Map}(\mathbb{T}^d, \mathrm{Gr}_m(\mathfrak{h}')))$.

Homotopies are a central concept in algebraic topology

Exaggerating only a little, you could say that algebraic topology is the study of continuous functions up to homotopy.

Homotopy groups

- The homotopy groups of a space X are $\pi_n(X) := [S^n, X]_\bullet$.
- There are groups, that is, they come with an addition operation that is associative and, for $n > 1$, commutative, and has a corresponding operation of subtraction.
- They are very hard to compute, even for spheres!

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$

Systems with interaction

Traditionally the Brillouin zone is only used for single-particle systems, but *there's no mathematical reason to stop there*!

A Hamiltonian for a many-particle system with interaction, it will still operate on a Hilbert space: a Fock space, or a bosonic or fermionic variant. If $\mathfrak{h}_1$ is the Hilbert space for the single-particle Hamiltonian, the Hilbert space for the many-particle system will be $\overline{\bigoplus_{n\geq 0} \mathfrak{h}_1^{\otimes n}}$, $\overline{\bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathfrak{h}_1)}$ or $\overline{\bigoplus_{n\geq 0} \Lambda^n(\mathfrak{h}_1)}$.

All separable Hilbert spaces are isomorphic and so the direct integral decomposition in Bloch's theorem can be adapted$\hat{}\ast$ to Fock space.

This means our homotopy classes of maps, $[\mathbb{T}^d, \mathrm{Gr}_m(\mathfrak{h}')]$, are also invariants for crystalline many-particle systems with interaction!

A subtle point about translation for interacting systems

Consider a typical Hamiltonian for a system with N particles wit a periodic potential V:

$$H = \sum_{1 \le i \le N} \left(-\nabla_i^2 + V(x_i) \right) + \sum_{1 \le i < j \le N} \frac{e^2}{|x_i - x_j|^2}.$$

Translating a single x_i is not a symmetry of that Hamiltonian: the Laplacian and potential remain unchanged, but the Coulomb interaction terms change.

Translating all x_i simultaneously by the same vector *is* a symmetry.

When we say a Hamiltonian on Fock space is translation invariant we mean with respect to this action of simultaneous translation. This is required for the direct integral decomposition of the Hamiltonian in our extension of the Bloch decomposition to Fock space.

Crystallographic groups

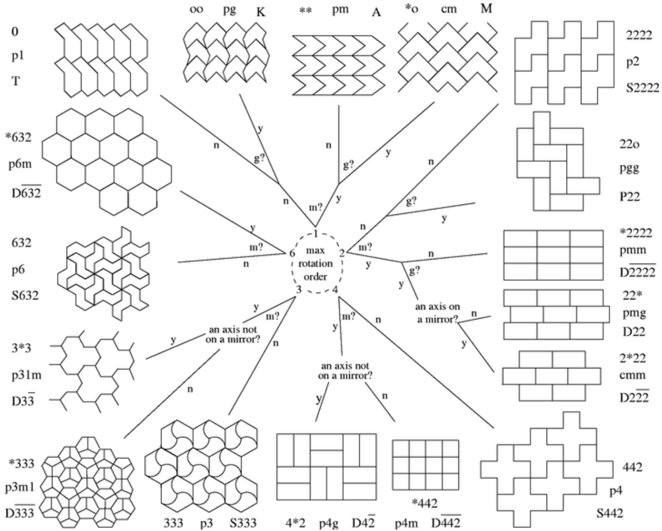
We can require our systems and adiabatic evolution to preserve a larger group of symmetry than just the translations.

The possible symmetry groups of crystals have been widely studied. It's a hard problem. There are *very* many groups even in low dimensions.

dim	Groups	Year	Authors
2	17	1891	Fedorov
3	230	1891	Fedorov, Schönflies
4	4783	1973	Brown, Bülow, Neubüser, Wondratschek, Zassenhaus
5	222018	2000	Plesken, Schulz
6	28927922	2000	Plesken, Schulz

I believe the number is not known for any higher dimension.

Wallpaper groups



Dror Bar-Natan's variant of Brian Sanderson's algorithm for identifying 2-dimensional crystallographic groups.

More symmetry begets equivariant topology

Let G be a *symmorphic* crystallographic symmetry group. This means the point group $P = G/\Lambda$ is a subgroup of G.

If the Hamiltonian for a system is G-invariant, then we have:

- an action of P on the Brillouin zone,
- an action of P on the Hilbert space $\mathfrak{h}'$ in the Bloch decomposition,
- a corresponding action of P on the Grassmannian $\mathrm{Gr}_m(\mathfrak{h}')$, and
- the ground state map $g_H : \mathbb{T}^d \to \mathrm{Gr}_m(\mathfrak{h}')$ is *P-equivariant*, meaning that for a point symmetry $\gamma \in P$, we have $g_H(\gamma \cdot k) = \gamma \cdot g_H(k)$.

Main theoretical result

Daniel Sheinbaum, Omar Antolín Camarena. *Crystallographic Interacting Topological Phases and Equivariant Cohomology: To assume or not to assume*. JHEP (2021) https://doi.org/10.1007/JHEP07(2021)139

Result

Given a symmorphic crystallographic G with point group P, the phases of gapped interacting systems with symmetry group G are given by P-equivariant homotopy classes of maps, from the Brillouin zone to the Grassmannian of m-dimensional subspaces of a Hilbert space, $[\mathbb{T}^d, \mathrm{Gr}_m(\mathfrak{h}')]_P$, where m is the degeneracy of the ground state.

Calculations?

Computing homotopy classes of maps is difficult in general, and equivariant homotopy classes maps are typically harder still. But in the non-degenerate case we can express our answer in terms of group cohomology, which is much easier to compute.

de Rham Cohomology

The invariant that we'll use later on in this talk is *cohomology*. It may be most familiar through de Rham cohomology.

Whether an integral over a closed submanifold is 0 or not depends on topology!

$$\int_{S^1} \frac{dz}{z} = 2\pi i$$

Define $H^d_{dR}(X) = \{\omega \in \Omega^d(X) : d\omega = 0\}/\{d\alpha : \alpha \in \Omega^{d-1}(X)\}$.

This knows about the hole in $\mathbb{C} \setminus \{0\}$:

$$H^1_{dR}(\mathbb{C} \setminus \{0\}) = \mathbb{R}, \text{ while } H^1_{dR}(\mathbb{C}) = 0.$$

Cohomology as homotopy classes of maps

There are some special, quite ungeometrical spaces, called Eilenberg–MacLane spaces.

$$\pi_k(K(A,n)) = \begin{cases} A & \text{if } k = n \\ 0 & \text{if not} \end{cases}$$

They can be used to define cohomology: $H^d(X;A) := [X, K(A,n)]$.

In these terms, $H^d_{dR}(X) \cong H^d(X;\mathbb{R})$.

$H^d(X;\mathbb{R})$ is a real vector space, but algebraic topologists tend to be more interested in $H^d(H;\mathbb{Z})$ which is just an abelian group.

The space $K(\mathbb{Z},2)$ which gives $H^2(X;\mathbb{Z})$ happens to be the Grassmannian of 1-dimensional subspaces of a complex Hilbert space!

The non-degenerate case, $m = 1$

Recall that phases of gapped interacting crystalline d-dimensional systems with only translation symmetry were given by $[\mathbb{T}^d, \mathrm{G}_m(\mathfrak{h}')]$.

When $m = 1$, the Grassmannian $\mathrm{G}_m(\mathfrak{h}')$ is a $K(\mathbb{Z}, 2)$, so those homotopy classes of maps are in bijection with $H^2(\mathbb{T}^d; \mathbb{Z}) \cong \mathbb{Z}^{\binom{d}{2}}$.

Similarly, for systems with symmetry given by a symmorphic crystallographic group G with point group P, phases are given by $H^2_P(\mathbb{T}^d; \mathbb{Z})$, which is a Borel equivariant cohomology group.

This turns out to always agree with $H^2(G^*; \mathbb{Z})$, the second group cohomology of the *reciprocal group* G^*: the group generated by the point group P together with translations in the reciprocal lattice Λ^*.

Obtaining cohomology groups is great news, because algebraic topologists have many more tools for computing those than general sets of homotopy classes of maps.

A comparison for the non-degenerate bosonic case

$H^2_P(\mathbb{T}^2;\mathbb{Z})$ $m = 1$, $d = 2$ phases.

$H^4(G;\mathbb{Z})$ $d = 2$ SPT phases (Thorngren–Else).

G	P	$H^2_P(\mathbb{T}^2;\mathbb{Z})$		$H^4(G;\mathbb{Z})$
p2	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^3$	$>$	$\mathbb{Z}_2^4$
p3	$\mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_3^2$	$>$	$\mathbb{Z}_3^3$
p4	$\mathbb{Z}_4$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	$>$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^2$
p6	$\mathbb{Z}_6$	$\mathbb{Z} \oplus \mathbb{Z}_6$	$>$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3^2$
pm	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$=$	$\mathbb{Z}_2^2$
cm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$=$	$\mathbb{Z}_2$
pmm	D_2	$\mathbb{Z}_2^4$	$<$	$\mathbb{Z}_2^8$
cmm	D_2	$\mathbb{Z}_2^3$	$<$	$\mathbb{Z}_2^5$
p31m	D_3	$\mathbb{Z}_2$	$<$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$
p3m1	D_3	$\mathbb{Z}_2 \oplus \mathbb{Z}_3$	$>$	$\mathbb{Z}_2$
p4m	D_4	$\mathbb{Z}_2^3$	$<$	$\mathbb{Z}_2^6$
p6m	D_6	$\mathbb{Z}_2^2$	$<$	$\mathbb{Z}_2^4$

Why are the results different?

R. Thorngren and D. V. Else, *Gauging Spatial Symmetries and the Classification of Topological Crystal line Phases*, Phys. Rev. X 8 (2018). https://doi.org/10.1103/PhysRevX.8.011040

There are several possible reasons, and we don't know which of them actually matter!

Unlike our classification, the classification of SPT phases:

- is for systems with short-range entanglement,
- declares that stacking with a trivial phase does not change the phase, and
- counts phases of relativistic topological fields theories.

But why sometimes more phases and sometimes fewer?

Short-range entanglement

Two states in *distinct* SPT phases cannot be connected via adiabatic evolution through states with short-range entanglement, but they might still be connected through unconstrained adiabatic evolution. Those two SPT phases would merge into a single phase in our classification.

Stacking

Our classification may include *fragile* phases, which become trivial after stacking with systems in the trivial phase, but which cannot adiabatically evolve to the trivial phase (without stacking). These are considered to be trivial in the SPT classification.

Relativistic TFTs

If there is a genuine physical difference between emergent relativistic topological field theories and gapped topological phases, then *all bets are off!*.

That's all systems, what about *my* system?

With our approach it's hard to answer questions about specific systems:

- Given a Hamiltonian, which phase does it belong to?
- Or even: given two Hamiltonians, do they belong to the same phase?

And we can't easily give examples of the different phases.

Daniel likens the situation to the Fundamental Theorem of Algebra:

- The theorem tells you how many roots a polynomial has: the number of roots equals the degree of the polynomial.
- But it is of no help if you want to *find* the roots!

The end

Thank you!