MATH 340 Coffee stain problem Richard Anstee

Consider the following 'puzzle' which uses our duality theory. Determine the maximum value of z (the objective function) in the LP:

$$
\begin{array}{rrrrl}
\max & c_1x_1 & +c_2x_2 & -2x_3 & \\
 & x_1 & -x_2 & & \leq 2 \\
 & -x_1 & +x_2 & -x_3 & \leq -3 \\
 & 2x_1 & -2x_2 & -2x_3 & \leq 2.13 \\
 & x_1 & -2x_2 & -2x_3 & \leq 3
\end{array}
\qquad x_1, x_2, x_3 \geq 0
$$

We do not know c_1, c_2 (which you can imagine were lost in a coffee stain) but we are told an optimal dual solution $(y_1^*, y_2^*, y_3^*, y_4^*)$ exists with $y_1^* = 1$. Do not try to solve using the simplex method. Instead use our duality theorems and some clever inequalities.

Solution: The dual to the given LP is:

$$
\begin{array}{rrrrrl}
\min & 2y_1 & -3y_2 & +2.13y_3 & +3y_4 & \\
 & y_1 & -y_2 & +2y_3 & +y_4 & \geq c_1 \\
 & -y_1 & +y_2 & -2y_3 & -2y_4 & \geq c_2 \\
 & & -y_2 & -2y_3 & -2y_4 & \geq -2
\end{array}
\qquad y_1, y_2, y_3, y_4 \geq 0
$$

We do not know c_1, c_2. We are told an optimal dual solution $(y_1^*, y_2^*, y_3^*, y_4^*)$ exists with $y_1^* = 1$. By the Theorem of Complementary Slackness, we deduce that $x_1^* - x_2^* = 2$. Substituting this into the second inequality of the primal, results in the inequality $x_3^* \geq 1$. Now by Complementary Slackness, we deduce that $-y_2^* - 2y_3^* - 2y_4^* = -2$. We substitute $x_1^* - x_2^* = 2$ in the third inequality of the primal, and find that the inequality becomes $4 - 2x_3^* \leq 2.13$ and then using $x_3^* \geq 1$ we see that the inequality must be a strict inequality (i.e. the third primal slack is at least .13 and hence > 0). By Complementary Slackness, $y_3^* = 0$. We try again in the fourth inequality of the primal and deduce the fourth slack is $3 - x_1^* + 2x_2^* + 2x_3^*$. Substituting $x_1^* - x_2^* = 2$, $x_2 \geq 0$ and $x_3^* \geq 1$ we get that the fourth slack is at least $3 + x_2^* > 0$ and so by Complementary Slackness we find that $y_4^* = 0$. Using the equation $-y_2^* - 2y_3^* - 2y_4^* = -2$ we deduce that $y_2^* = 2$. Now we have a complete optimal dual solution $y_1^* = 1, y_2^* = 3, y_3^* = 0, y_4^* = 0$ of value $2 \cdot 1 - 3 \cdot 2 + 2.13 \cdot 0 + 3 \cdot 0 = -4$ and so by Strong Duality, the maximum value of z in the primal is -4.

A reasonable question is to determine if there are any values for c_1, c_2 such that the answer above is true. One possibility is to take $c_1 = -1, c_2 = 1$ and then take $x_1^* = 2, x_2^* = 0, x_3^* = 1$.