The following examples have been sometimes given in lectures and so the fractions are rather unpleasant for testing purposes. Note that each question is imagined to be independent; the changes are not cumulative.

We wish to consider a desk manufacturer who can choose to produce three types of desks from 3 raw materials

	Desk 1	Desk 2	Desk 3	availability
carpentry	4	6	8	600 hours
finishing	1	3.5	2	300 hours
space	2	4	3	550 sq. m.
net profit	12	20	18	

Now setting x_i = number of desks of type i to be produced we have the LP:

$$\begin{array}{lllll} \max & 12x_1 & +20x_2 & +18x_3 & & \\ & 4x_1 & +6x_2 & +8x_3 & \leq & 600 \\ & x_1 & +3.5x_2 & +2x_3 & \leq & 300 \\ & 2x_1 & +4x_2 & +3x_3 & \leq & 550 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

We get the final dictionary:

$$\begin{array}{lllllll} x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\ x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\ z & = & 1950 & -6x_3 & -\frac{11}{4}x_4 & -x_5 \end{array}$$

a) Give A_B^{-1}, appropriately labelled:

$$A_B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array}$$

b) Give the marginal values associated with carpentry,finishing and space:

carpentry: $\frac{11}{4}$, finishing: 1, space: 0

i.e. extra hours carpentry worth $\frac{11}{4}$, extra hours finishing worth 1, extra space not helpful.

c) Give a range on b_3 (space) so $\{x_1, x_2, x_6\}$ still yields an optimal basis:

In this case $c_N^T - c_B^T A_B^{-1} A_N \leq \mathbf{0}$ as before so we need $A_B^{-1}\mathbf{b} \geq \mathbf{0}$.

$$A_B^{-1}\begin{pmatrix} 600 \\ 300 \\ b_3 \end{pmatrix} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} \frac{7}{16} & -\frac{3}{4} & 0 \\ -\frac{1}{8} & \frac{1}{2} & 0 \\ -\frac{3}{8} & -\frac{1}{2} & 1 \end{pmatrix} \end{array} \begin{pmatrix} 600 \\ 300 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{75}{2} \\ 75 \\ b_3 - 375 \end{pmatrix} \geq \mathbf{0}$$

Thus for $b_3 \geq 375$, we still have the same optimal basis.

d) Predict value of the optimal solution when $\mathbf{b} = (610, 310, 500)^T$:

Thus $\Delta b_1 = 10, \Delta b_2 = 10, \Delta b_3 = -50$ and so the new value of z is the old value of z plus $10 \times \frac{11}{4} + 10 \times 1 - 50 \times 0$ which is $1950 + \frac{75}{2} = 1987.5$. We check that

$$A_B^{-1}\begin{pmatrix}610\\310\\500\end{pmatrix} = \begin{pmatrix}\frac{75}{2}\\75\\175\end{pmatrix} + \begin{matrix} \\ x_1\\x_2\\x_6\end{matrix}\overset{\begin{matrix}x_4 & x_5 & x_6\end{matrix}}{\begin{pmatrix}\frac{7}{16} & -\frac{3}{4} & 0\\ -\frac{1}{8} & \frac{1}{2} & 0\\ -\frac{3}{8} & -\frac{1}{2} & 1\end{pmatrix}}\begin{pmatrix}10\\10\\-50\end{pmatrix} = \begin{pmatrix}\frac{550}{16}\\ \frac{315}{4}\\ \frac{930}{8}\end{pmatrix} \geq \mathbf{0}$$

e) Determine the range for c_3 so that the basis $\{x_1, x_2, x_6\}$ remains optimal:

$$c_N^T - c_B^T A_B^{-1} A_N = \overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}c_3 & 0 & 0\end{pmatrix}} - \overset{\begin{matrix}x_1 & x_2 & x_6\end{matrix}}{\begin{pmatrix}12 & 20 & 0\end{pmatrix}} \begin{matrix}x_1\\x_2\\x_6\end{matrix}\overset{\begin{matrix}x_4 & x_5 & x_6\end{matrix}}{\begin{pmatrix}\frac{7}{16} & -\frac{3}{4} & 0\\ -\frac{1}{8} & \frac{1}{2} & 0\\ -\frac{3}{8} & -\frac{1}{2} & 1\end{pmatrix}} \begin{matrix}x_4\\x_5\\x_6\end{matrix}\overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}8 & 1 & 0\\ 2 & 0 & 1\\ 3 & 0 & 0\end{pmatrix}}$$

$$= \overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}c_3 - 24 & -\frac{11}{4} & -1\end{pmatrix}}$$

We are optimal for $c_3 \leq 24$.

A much quicker and more reasonable approach is to note the -6 as the coefficient of x_3 in the z row and so deduce that the current c_3 can rise by as much as 6, i.e. $c_3 \leq 18 + 6 = 24$. Note how our sensitivity output from LINDO gives this as a reduced cost.

We can check our bound by noting that $18 \leq 24$. Note also that for $c_3 > 24$, we know that x_3 will be in the basis since apart from c_3 the problem is unchanged so if x_3 is not in the basis then we just have the original solution.

f) Determine the range for c_1 so that the basis $\{x_1, x_2, x_6\}$ remains optimal:

$$c_N^T - c_B^T A_B^{-1} A_N = \overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}18 & 0 & 0\end{pmatrix}} - \overset{\begin{matrix}x_1 & x_2 & x_6\end{matrix}}{\begin{pmatrix}c_1 & 20 & 0\end{pmatrix}} \begin{matrix}x_1\\x_2\\x_6\end{matrix}\overset{\begin{matrix}x_4 & x_5 & x_6\end{matrix}}{\begin{pmatrix}\frac{7}{16} & -\frac{3}{4} & 0\\ -\frac{1}{8} & \frac{1}{2} & 0\\ -\frac{3}{8} & -\frac{1}{2} & 1\end{pmatrix}} \begin{matrix}x_4\\x_5\\x_6\end{matrix}\overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}8 & 1 & 0\\ 2 & 0 & 1\\ 3 & 0 & 0\end{pmatrix}}$$

$$= \overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}18 - 2c_1 & \frac{5}{2} - \frac{7}{16}c_1 & \frac{3}{4}c_1 - 10\end{pmatrix}}$$

Thus we are optimal for $c_1 \geq 9, c_1 \geq \frac{40}{7}, c_1 \leq \frac{40}{3}$, i.e. $9 \leq c_1 \leq \frac{40}{3}$. Note $12 \in [9, \frac{40}{3}]$ which is a good check on our work.

g) Determine an optimal solution if $c_1 = 8$:

$$c_N^T - c_B^T A_B^{-1} A_N = \overset{\begin{matrix}x_3 & x_4 & x_5\end{matrix}}{\begin{pmatrix}2 & -1 & -4\end{pmatrix}}$$

Thus x_3 becomes an entering variable. New dictionary for basis $\{x_1, x_2, x_6\}$ is

$$\begin{array}{rclllll} x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\ x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\ z & = & * & 2x_3 & -1x_4 & -4x_5 \end{array}$$

x_3 enters and x_1 leaves.

$$\begin{array}{rcl} x_3 &=& \frac{75}{4} \\ x_2 &=& 75 & & & * \\ x_6 &=& 175+\frac{75}{4} \\ z &=& * & -x_1 & -\frac{23}{16}x_4 & -\frac{13}{4}x_5 \end{array}$$

$$\text{optimal solution: } x_3 = \frac{75}{4}, x_2 = 75, x_6 = 193.25$$

h) What is the optimal solution if $b_3 = 365$ (outside of the range given in c)). We compute

$$A_B^{-1}\begin{pmatrix}600\\300\\365\end{pmatrix} = \begin{pmatrix}\frac{75}{2}\\75\\-10\end{pmatrix}$$

The final dictionary becomes:

$$\begin{array}{rcl} x_1 &=& 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\ x_2 &=& 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 &=& -10 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\ z &=& * & -6x_3 & -\frac{11}{4}x_4 & -x_5 \end{array}$$

We do a dual simplex pivot. We have x_6 leave. The largest t such that $(-6 \quad -\frac{11}{4} \quad -1) + (1 \quad \frac{3}{8} \quad \frac{1}{2})t \leq \mathbf{0}$ is $t = 2$ and x_5 enters:

$$\begin{array}{rcl} x_1 &=& \frac{105}{2} \\ x_2 &=& 65 & & * \\ x_5 &=& 20 \\ z &=& * & -4x_3 & -2x_4 & -2x_6 \end{array}$$

$$\text{optimal solution: } x_1 = \frac{105}{2}, x_2 = 65, x_5 = 20$$

The new marginal values are carpentry: 2, finishing 0, space 2.

i) Consider a new desk with requirements of 8 hours of carpentry, 2 hours finishing, and 6 sq. m of space with a net profit of \$26 per desk. Is it profitable to produce this desk?

Let x_7 denote the number of desks produced of this new type. We compute

$$c_7 - c_B^T A_B^{-1} A_7 = 26 - (\tfrac{11}{4} \quad 1 \quad 0)\begin{pmatrix}8\\2\\6\end{pmatrix} = 2 > 0$$

Thus we will produce the new desk at optimality. Here is the final dictionary with variable x_7 added (we needed to compute $A_B^{-1}A_7$).

$$\begin{array}{rcl} x_1 &=& 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 & -2x_7 \\ x_2 &=& 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\ x_6 &=& 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 & -2x_7 \\ z &=& 1950 & -6x_3 & -\frac{11}{4}x_4 & -x_5 & +2x_7 \end{array}$$

We have x_7 enter and x_1 leave:

$$\begin{array}{rcl} x_7 &=& \frac{75}{4} \\ x_2 &=& 75 & & & * \\ x_6 &=& 137\frac{1}{2} \\ z &=& 1987\frac{1}{2} & -7x_3 & -\frac{51}{16}x_4 & -\frac{1}{4}x_5 & -\frac{1}{2}x_1 \end{array}$$

$$\text{optimal solution: } x_7 = \frac{75}{4}, x_2 = 75, x_6 = 137.5$$

j) What is the optimal solution if we add the constraint $x_1 + x_2 + x_3 \le 100$. Think of this as a constraint on market size. Obviously the current solution is no longer feasible. We add a slack variable x_7 to get $x_7 = 100 - x_1 - x_2 - x_3$ and then reexpress in terms of non basic variables to get $x_7 = -\frac{25}{2} + x_3 + \frac{5}{16}x_4 - \frac{1}{4}x_5$ and get the final dictionary:

$$\begin{array}{rclllll}
x_1 & = & 37.5 & -2x_3 & -\frac{7}{16}x_4 & +\frac{3}{4}x_5 \\
x_2 & = & 75 & & +\frac{1}{8}x_4 & -\frac{1}{2}x_5 \\
x_6 & = & 175 & +x_3 & +\frac{3}{8}x_4 & +\frac{1}{2}x_5 \\
x_7 & = & -\frac{25}{2} & +x_3 & +\frac{5}{16}x_4 & -\frac{1}{4}x_5 \\
z & = & 1950 & -6x_3 & -\frac{11}{4}x_4 & -x_5
\end{array}$$

We do a dual simplex pivot. We have x_7 leave. The largest t such that $(\,-6 \quad -\frac{11}{4} \quad -1\,) + (\,1 \quad \frac{5}{16} \quad -\frac{1}{4}\,)\,t \le \mathbf{0}$ is $t = 6$ and x_3 enters:

$$\begin{array}{rclllll}
x_1 & = & \frac{25}{2} & & & \\
x_2 & = & 75 & & * & \\
x_6 & = & 187\frac{1}{2} & & & \\
x_3 & = & \frac{25}{2} & 77 & & \\
z & = & * & -6x_7 & -\frac{7}{8}x_4 & -\frac{5}{2}x_5
\end{array}$$

$$\text{optimal solution: } x_1 = \frac{25}{2}, x_2 = 75, x_3 = \frac{25}{2}, x_6 = 162\frac{1}{2}$$

The new marginal values are carpentry $\frac{7}{8}$, finishing $\frac{5}{2}$, space 0, market 6.

Many other questions can be asked such as changing an entry in A, the matrix of the constraints. For a nonbasic variable this is reasonable (try it!). In a test environment, only one pivot suffices to get you to optimality but this is unrealistic and for some changes it may be advisable to start from scratch.