Iteration of Rational Functions

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1   Overview

The iteration of rational maps on the Riemann sphere is one of the most attractive topics in the theory of dynamical systems. These maps are holomorphic which opens the door to study them with a rich set of tools from complex analysis, analysis and algebraic geometry. All rational maps of a fixed degree form a finite dimensional space, easily parametrized by the coefficients, which makes dealing with the totality of rational maps pleasantly approachable. Much is known about the iteration of rational maps and here we will describe the basic features of the dynamics. The remainder of this section is an informal overview of the topics covered. The reader can look ahead to find definitions of most unfamiliar terms.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere, or complex projective line, obtained by adding a single point at infinity to $\mathbb{C}$. From the point of view of the dynamics of f, the first distinction to make between points of the sphere is whether the point is "stable", i.e., nearby points behave similarly under iteration, or not, i.e., arbitrary close points can have drastically different fates. The set of stable points is called the *Fatou set*, $F(f)$, and the complement is the *Julia set*, $J(f)$. The Fatou set can be defined as the largest open set on which the collection of iterates of f is normal[1]. Both the Fatou and Julia sets are *fully invariant* meaning that they are closed under both forward and backward iteration; this in turn means that, at least set-theoretically, the dynamical system given by f on the sphere is the disjoint union of the dynamical systems given by f restricted to the Julia and Fatou sets.

The behavior of points in the Julia set is indeed "chaotic", for example, given any point z_0 in it, the set of its *iterated preimages* or *backward orbit*, $\{z \in \widehat{\mathbb{C}} : f^{\circ n}(z_0) = z \text{ for some } n\}$, is dense in $J(f)$. Something similar is true for forward orbits[2]: a generic point of the Julia set has a dense forward orbit[3], where "generic" means that the set of points having this property is an intersection of countably many dense open subsets of $J(f)$ —and by Baire's theorem, therefore dense.

The behavior on the open Fatou set is much more regular. Every connected component of the Fatou set, a *Fatou component*, is mapped *onto* another Fatou component, and one can study the dynamics induced by f on the collection of Fatou components. The remarkable *No Wandering Domains Theorem*, conjectured by Fatou and proved by Sullivan in (Sullivan

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[1] See the next section for a brief reminder about normal families and Montel's theorem.
[2] The forward orbit of z is $\{f^{\circ n}(z) : n \geq 0\}$.
[3] Dense in $J(f)$, that is.

1985) says that every Fatou component is *eventually periodic* meaning that some iterate of it *is* periodic. Moreover, it can be shown there are at most finitely many periodic Fatou components. In fact, Shishikura (Shishikura 1987) has shown there are at most $2d - 2$ such components for a rational map of degree d. There are usually infinitely many Fatou components[4], but all of them are preimages, under some iterate of f, of one of the at most $2d - 2$ periodic ones.

Given a Fatou component U of period p, f^p can be studied as a dynamical system on U in its own right and is always of one of the following types:

1. An *attractive basin*: f^p has a fixed point $z_0 \in U$, with $|(f^p)'(z_0)| < 1$, that attracts all points of U under iteration. If $|(f^p)'(z_0)| = 0$, U is called a *super-attractive basin*.

2. A *parabolic basin*: some point z_0 on the boundary of U attracts all points of U, in which case, necessarily $(f^p)'(z_0) = 1$.

3. A *Siegel disk*: the dynamical system (U, f^p) is *conformally isomorphic* to an irrational rotation on the unit disk.

4. A *Herman ring*: the dynamical system (U, f^p) is *conformally isomorphic* to an irrational rotation on some annulus. It is not trivial that Herman rings exist and, for example, no polynomial map can have one.

Iteration of rational maps is a large subject and we can only cover the basic features mentioned above. Good places to start reading further are (Blanchard 1984), (Milnor 2006), (McMullen 2011) and (Beardon 1991).

2 A few tools from complex analysis and hyperbolic geometry

Here we gather some important notions and results from complex analysis and hyperbolic geometry that we will need.

2.1 Normal families and Montel's theorem

Recall that a set of holomorphic functions with a common domain is called *normal* if it has compact closure in the topology of uniform convergence

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[4]A rational map can have 0, 1, 2 or infinitely many Fatou components, see Theorem 5.6.2 from (Beardon 1991).

on compact subsets, i.e., if every sequence of functions from the set has a subsequence that converges to some function uniformly on compact subsets of the domain. The limit function is necessarily holomorphic. We will use this definition for both holomorphic and meromorphic functions. In the meromorphic case, we regard the functions as taking values in the Riemann sphere and take uniform convergence with respect to the usual spherical metric on $\mathbb{C} = S^2$.

There are many tests for normality of a family of functions, but we won't bother with them and go straight for the big gun:

Theorem 1 (Montel's fundamental normality criterion)**.** *Any family of holomorphic functions from a domain $U \subseteq \mathbb{C}$ to $\mathbb{C} \setminus \{a, b\}$ (a and b two distinct points) is normal.*

2.2 Riemann surfaces and Uniformization

A Riemann surface is a connected one-dimensional complex manifold. Complex manifolds are defined in the same fashion as other sorts of manifolds with the transition functions between different charts required to be holomorphic. We won't really need much about Riemann surfaces that is not a straight forward extension of classical results from the theory of functions of one complex variable. In fact, for the most part we will just work with the Riemann sphere and its open subsets. Recall that the Riemann sphere, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, is topologically the one point compactification of the complex numbers (that is, homeomorphic to S^2), and is made into a one-dimensional complex analytic manifold by declaring that in a neighborhood of the point at infinity, the function $1/z$ is a holomorphic chart. Omitting any point from the Riemann sphere leaves you with a copy of $\mathbb{C}$, so many results extend quite simply to $\widehat{\mathbb{C}}$. Also, holomorphic functions from domains $U \subseteq \mathbb{C}$ to $\widehat{\mathbb{C}}$, apart from the constant function with value ∞, are the same thing as meromorphic functions on U (extended to take the value ∞ at the poles).

The one major result about Riemann surfaces we will use is the *Uniformization Theorem* that generalizes the Riemann mapping theorem to classify all simply connected Riemann surfaces.

Theorem 2 (Uniformization)**.** *Any simply connected Riemman surface is isomorphic to either $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

This result can be used to study an arbitrary Riemann surface through its universal cover. Start with any Riemann surface X and let $\pi : Y \to X$ be its universal cover. Initially Y is only a topological space, but one can use π to

give Y the structure of a complex manifold by locally copying the structure from X. Then by the uniformization theorem, Y must be $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$ and X is the quotient of Y by the action of the fundamental group of X as deck transformations (which are easily seen to be holomorphic automorphisms of Y).

- If $Y = \widehat{\mathbb{C}}$, then all the automorphisms of Y are given by Möbius transformations[5], all of which have fixed points and thus cannot be deck transformations of a cover. It follows that then $X = \widehat{\mathbb{C}}$ as well.

- If $Y = \mathbb{C}$, the automorphisms are the affine transformations $z \mapsto az+b$ ($a \neq 0$) and the ones with $a \neq 1$ have a fixed point. So, in this case $X = \mathbb{C}/\Gamma$ where Γ is a subgroup of $\mathbb{R}^2$. It must be discrete because deck transformation groups act properly discontinuously. So we get $X = \mathbb{C}$, $\mathbb{C}/\mathbb{Z}$ or a torus[6] depending on whether Γ has zero, one or two generators.

- If $Y = \mathbb{D}$, X is called *hyperbolic*. From the previous discussion we see almost all Riemann surfaces are hyperbolic. In particular, we'll use later on that any open $U \subset \widehat{\mathbb{C}}$ whose complement contains at least three points is hyperbolic. Indeed any map $\mathbb{C} \to U$ is constant by the Little Picard theorem, so the universal cover of U can be neither $\mathbb{C}$ nor S.

2.3   The hyperbolic metric

The *hyperbolic metric* on the unit disk $\mathbb{D}$ is given by the formula $|dz|/(1-|z|^2)$ and is characterized, up to multiplication by positive scalars, as the unique Riemannian metric on $\mathbb{D}$ invariant under all holomorphic automorphisms of $\mathbb{D}$.

This metric gives us a corresponding metric on any Riemann surface whose universal cover is $\mathbb{D}$: if $\pi : \mathbb{D} \to X$ is a covering map, we can "push forward" the hyperbolic metric on $\mathbb{D}$ to get a metric on X that is locally isometric to the hyperbolic metric. Normally one cannot push a metric forward since, given a point $x \in X$, there is an ambiguity in the choice of preimage of x to copy the metric from. In our case this is not a problem, because given any two preimages there is a deck transformation ϕ taking one to the other and the hyperbolic metric on $\mathbb{D}$ is invariant under ϕ. We call the

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[5]We will show this in the next section, on rational maps.

[6]All the tori $\mathbb{C}/\Gamma$ are homeomorphic to each other but they are not isomorphic as Riemann surfaces.

Riemannian metric constructed in this fashion the *hyperbolic metric on X*. Note that while all geodesics in $\mathbb{D}$ go off to infinity (i.e., $\partial\mathbb{D}$), a hyperbolic X can have closed geodesics: the projection to X of any geodesic of $\mathbb{D}$ passing through two points with the same image under π is a closed geodesic.

The main result about the hyperbolic metric we need is the following theorem, which is essentially a geometric manifestation of Schwarz's lemma:

Theorem 3 (Pick). *Let $f : X \to X'$ be a holomorphic map between two hyperbolic Riemann surfaces. Then either*

- *f is a covering map and a local isometry for the hyperbolic metric, or*

- *f strictly decreases all distances and on any compact subset of X, f is Lipschitz with Lipschitz constant less than one.*

See Theorem 2.11 in (Milnor 2006) for a proof.

Finally, we also need a result comparing the hyperbolic distance for a hyperbolic $U \subset \widehat{\mathbb{C}}$ with the spherical distance as we approach the boundary of U, which lies at infinite distance for the hyperbolic metric.

Proposition 1 (Hyperbolic vs spherical distance). *Let U be an open subset of $\widehat{\mathbb{C}}$ which is hyperbolic and let $z_n \in U$ be a sequence all of whose accumulation points lie on ∂U. Then for any $r > 0$, the diameter in the usual spherical distance on $\widehat{\mathbb{C}} = S^2$ of the hyperbolic ball $B(z_n, r)$ of radius r around z_n tends to 0.*

Proof. Let $p_n : \mathbb{D} \to U$ be a covering map such that $p_n(0) = z_n$. Since p_n is a local isometry, $p_n(B(0, r)) = B(z_n, r)$. Assume the spherical diameter of $B(z_n, r)$ does not tend to 0. Then, passing to a subsequence of z_n, there is some $\delta > 0$ such that $\mathrm{diam}_{\widehat{\mathbb{C}}}(B(z_n, r)) \geq \delta$. Since U is hyperbolic, it must be missing at least three points of $\widehat{\mathbb{C}}$ and therefore we can apply Montel's theorem to the family $\{p_n\}$. So after passing again to a subsequence, we can assume the p_n converge uniformly to some holomorphic function $f : \mathbb{D} \to \bar{U}$. If we showed that f is constant, we'd be done for then $B(z_n, r)$ for large n would lie in the spherical ball of radius $\delta/3$ around the value of f.

If f were not constant, $f(B(0, r))$ would be open and thus meet U. But if $f(w) \in U$ for some $w \in B(0, r) \subset \mathbb{D}$, we'd have that for large n, $d(z_n, f(w)) \leq d(z_n, p_n(w)) + d(p_n(w), f(w)) < r + 1$, where d is the hyperbolic distance on U. (We are using that d and the spherical distance $d_{\widehat{\mathbb{C}}}$ give the same topology on U, so that $d_{\widehat{\mathbb{C}}}(p_n(w), f(w)) \to 0$ implies $d(p_n(w), f(w)) \to 0$.) This contradicts that z_n goes off to the boundary of U. $\qquad\square$

3 Rational maps as holomorphic dynamical systems

We want to study rational functions with complex coefficients as dynamical systems, that is, as mappings from some space to itself. To handle poles gracefully we must regard rational functions as taking values not in the complex numbers, $\mathbb{C}$, but in the Riemann sphere. We can also include ∞ in the domain of a rational map in the usual way, to get a continuous map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, readily seen to be holomorphic.

Moreover, rational maps are *all* the holomorphic maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ other than the constant function with value ∞. Indeed, a non-constant holomorphic map can only take the value ∞ finitely many times, by the identity theorem and compactness of $\widehat{\mathbb{C}}$. The restriction of f to $\mathbb{C} \setminus f^{-1}(\infty)$ is holomorphic, and f only has poles at points of $f^{-1}(\infty)$, because $1/f$ is holomorphic on $\mathbb{C} \setminus f^{-1}(0)$. So, if f is not identically ∞, it is a meromorphic function on the plane with finitely many poles. Multiplying by a polynomial vanishing sufficiently at those poles we get an entire function, which must still have at worst a pole at ∞, forcing it to be a polynomial and thus forcing f to be rational.

Two basic things to know about a (smooth) dynamical system one is studying are the degree and the critical points. Let's say a few words about these for rational maps.

If $f(z) = p(z)/q(z)$ is a quotient of polynomial without any common roots, we say that f is a rational function of degree $d = \max(\deg p, \deg q)$. This agrees with the topological notion of degree, i.e., almost every point on $\mathbb{C}$ has d preimages, since the roots of the equation $f(z) = w$ are the roots of $p(z) - wq(z)$ which for almost all w is a polynomial of degree d without multiple roots. With slightly more care about the point at infinity one can show that every single value is taken d times counted with multiplicity.

It follows from this discussion about degree that the only automorphisms of the Riemann sphere are the rational maps of degree 1, the Möbius transformations. We will often slightly simplify a situation by conjugating f by an appropriate Möbius transformation, usually to place some important point, such as a fixed point, at infinity.

Any rational map of degree d has $2d - 2$ critical points counted with appropriate multiplicity. This is usually one of the first examples given of the Riemann-Hurwitz formula, but of course can easily be proved directly.

3.1 Iteration of Möbius transformations

The dynamics of a rational map of degree one is much, much simpler than that of rational maps of higher degree. Let's dispose of it first and consider maps of degree at least two in the sequel.

Looking at the equation $z = (az + b)/(cz + d)$ makes it clear that a Möbius transformation $f(z)$ has either one or two fixed points.

1. If f has only one fixed point z_0 conjugating, if necessary, by $1/(z - z_0)$, we can assume that the unique fixed point is ∞ and therefore f is a translation. The orbit of every point tends to ∞.

2. If f has two fixed points, conjugating by an appropriate Möbius transformation we can assume that the fixed points are 0 and ∞. Then f must be of the form $f(z) = az$ for some complex number a. If $|a| \neq 1$, all the orbits of points in $\mathbb{C} \setminus \{0, \infty\}$ tends to the same fixed point of f. If $|a| = 1$ there are two cases: either a is an n-th root of unity for some (minimal) n and every point that is not fixed has period n; or the orbit of any point $z_0 \neq 0, \infty$ is dense on the circle $|z| = |z_0|$ —in this case, f is called an *irrational rotation*.

4 The Julia and Fatou sets

Let's start by formally defining the Fatou and Julia sets of a map:

Definition 1. Let $f : S \to S$ be a non-constant holomorphic map from a compact Riemann surface S to itself. The *Fatou set* of f, denoted $F(f)$ is the domain of normality of $\{f^{on} : n \geq 0\}$, i.e., the largest open subset of S on which the family of iterates of f is normal. The Julia set $J(f)$ is defined simply to be the complement of $F(f)$.

Except for a few examples, we will only be interested in the case where f is a rational map of degree at least 2 on $S = \widehat{\mathbb{C}}$, the Riemann sphere.

Proposition 2. *Both the Fatou and Julia sets of f are* fully invariant *meaning that z belongs to one of them if and only if $f(z)$ belongs to it too.*

Proof. Since they are complementary, it is enough to show this for the Fatou set. A point $z \in F(f)$ if and only if there is a neighborhood U of z on which $\{f^{on}|_U : n \geq 1\}$ is a normal family. This happens if and only if $\{f^{on}|f(U) : n \geq 0\}$ is normal. $\qquad\square$

Proposition 3. *For a rational map of degree $d \geq 2$, the Julia set is never empty.*

Proof. If the iterates of f were normal on the whole Riemann sphere, we'd have a subsequence $f^{\circ n_j}$ converging uniformly on all of $\mathbb{C}$ to some holomorphic function g. Once j is large enough, $f^{\circ n_j}$ and g are homotopic (by going along the shortest arc of a great circle connecting their values), so $d^{n_j} = \deg(f^{\circ n_j}) = \deg g$, which is impossible if $d > 1$. □

We will call the connected components of $F(f)$, *Fatou components* of f.

Proposition 4. *For every Fatou component U of a rational map f, $f(U)$ is another Fatou component.*

Proof. Since f is open and the Fatou set is invariant, $f(U)$ is a connected open[7] domain of normality for the iterates of f and, as such, is contained in a single Fatou component V. Consider the closure $\bar{U}$ of U. It consists of U together with some points of the Julia set (because U is a component of the Fatou set), so $f(\bar{U})$ consists of $f(U)$ with some points of the Julia set. In particular, $f(U) = f(\bar{U}) \cap F(f)$. But $\bar{U}$ is compact, and so $f(\bar{U})$ is also compact and thus closed. So we have that $f(U)$ is closed in $F(f)$ showing it must be the whole component V. □

Remark 1. For holomorphic maps $\mathbb{C} \to \mathbb{C}$, the image of a Fatou component need not be a whole Fatou component. For example, for $z \mapsto \frac{1}{2}e^{z-1}$ one can check that 0 is in the Fatou set, but it's not in the image of the map at all. (The proof above is not applicable because $f(\bar{U})$ is not closed, a manifestation of the lack of compactness.) And it need not even be the case that, in the notation of the proof above, $V \setminus f(U)$ consists of values that f does not take. See Section 4.1 of (Bergweiler 1993) for more information.

4.1 Periodic points

Let's start our study of the Julia and Fatou sets of rational maps by figuring out in which of the two lie the fixed points and, more generally, the periodic points.

A *periodic point* of f is a point z such that for some $n > 0$, $f^{\circ n}(z) = z$. Of course, for $n = 1$ these are called *fixed points*. Many notions related to points of period n are simply the corresponding notions for fixed points of $f^{\circ n}$. For example, the *multiplier* of a point z of period n is $(f^{\circ n}(z))'$. If

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[7]Recall that holomorphic maps are open.

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the orbit of z is $z = z_1 \mapsto z_2 \mapsto \cdots z_{n+1} = z$, then the chain rule gives $(f^{\circ n}(z))' = f'(z_1)f'(z_2)\cdots f'(z_n)$, showing the multiplier of a periodic point depends only on its orbit. By definition a periodic orbit is *attracting, repelling* or *indifferent* according to whether the absolute value of its multiplier is less than 1, greater than 1 or equal to 1. If the multiplier is 0, the orbit is called *superattracting*.

The *basin of attraction* of a periodic orbit of period n is the set of all points z for which $\lim_{k \to \infty} f^{\circ nk}(z)$ is a point of the orbit.

Proposition 5. *The basin of attraction of any attracting periodic orbit is contained in the Fatou set, but all repelling periodic points lie in the Julia set.*

Proof. It is enough to prove the statements for fixed points, since it is easy to check that $J(f^{\circ n}) = J(f)$.[8] Let z_0 be a fixed point of f and let $\lambda = f'(z_0)$. If $|\lambda| > 1$, no sequence of iterates of f can converge uniformly on a neighborhood of z_0 as $|(f^{\circ n_j})'|(z_0) = \lambda^{n_j} \to \infty$.

Now assume $|\lambda| < c < 1$, so that $|f(z) - z_0| = |f(z) - f(z_0)| < c|z - z_0|$ on some neighborhood U of z_0. Then the iterates of f converge uniformly on U to the constant function with value z_0, so $U \subseteq F(f)$. Since $F(f)$ is fully invariant and some iterate of any point in the basin of attraction of z_0 will fall in U, we are done. $\square$

The case of indifferent periodic points is more complicated, and will not be dealt with here. We are also not including the local theory of fixed points which gives a complete description of a rational map in a neighborhood of the fixed point, up to the natural notion of isomorphism for dynamical systems: an isomorphism or *conjugacy* between two dynamical systems $f : X \to X$ and $g : Y \to Y$ is an invertible map $\phi : X \to Y$ such that $g = \phi \circ f \circ \phi^{-1}$. In our case f and g are holomorphic, so we ask ϕ to be holomorphic as well, and refer to it as a *conformal isomorphism* or *conformal conjugacy*. Some basic results are as follows, stated for simplicity assuming the fixed point is 0:

- Kœnigs' Linearization: if 0 is not superattracting or repelling, f is conjugate to multiplication by its derivative in a neighborhood of 0.

- Böttcher's Theorem: if 0 is superattracting but $f^{(n)}(0) \neq 0$, f is conjugate to z^n.

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[8]Basically, this is because if $\{f^{\circ nk} : k > 0\}$ has compact closure K, then every iterate of f belongs to the compact set of maps $\bigcup_{j=0}^{n-1}\{g \circ f^{\circ i} : g \in K\}$ —each term in the union is a continuous image of K, as composition from the right is clearly continuous in L^∞.

Again the case of indifferent fixed point is more complicated, and perhaps surprisingly number theoretic properties of the argument of the multiplier play a role. See chapter 11 of (Milnor 2006).

4.2 The Julia set is complicated

We will focus on the components of the Fatou set here, but here is a small taste of what happens on the Julia set.

Lemma 1. *Let f be a rational of degree at least two. Any finite set $S \subset \widehat{\mathbb{C}}$ that is closed under taking preimages under f is contained in the Fatou set.*

Proof. Let $z_0 \in S$. Any sequence $z_0, z_{-1}, z_{-2}, \ldots$ where each point is a preimage of the previous one is contained in S and thus must repeat a point. That repeated point is then periodic, and its orbit includes z_0, showing that S is also closed under forward iteration. Since $f : S \to S$ is surjective and S is finite, f must be a permutation of S. Now, counting multiplicities, every point has d preimages, so for $z_0 \in S$, the equation $f(z) = f(z_0)$ must have a root of multiplicity $d > 1$ at $z = z_0$, making z_0 a critical point of f and thus a superattracting periodic point —and we've seen all of those are in the Fatou set. $\qquad\square$

Corollary 1. *The Julia set of a rational map of degree at least two is infinite.*

Proposition 6. *For any point $z_0 \in J(f)$, the* backward orbit *of z_0, $O^-(z) := \{z : f^{\circ n}(z) = z_0 \text{ for some } n \geq 0\}$ is dense in $J(f)$.*

Proof. Let $z_1 \in J(f)$ be arbitrary. We must show that any open neighborhood U of z_1 contains points of $O^-(z)$. Let $V := \bigcup_{n \geq 0} f^{\circ n}(U)$, then we must show that $V \supset J(f)$. Notice that $f(V) \subseteq V$ by definition, so if $\mathbb{C} \setminus V$ contained three or more points, by Montel's theorem we would have that the iterates of f are normal on V and so $V \subseteq F(f)$, contradicting that $z_1 \in V$. So $\mathbb{C} \setminus V$ contains at most two points and we need only check that neither of them is in $J(f)$. Let $z \in \mathbb{C} \setminus V$, if there is such a point. No preimage w of z can lie in V, for then $z = f(w)$ would be in $f(V) \subseteq V$. By the lemma above $\mathbb{C} \setminus V \subset F(f)$. $\qquad\square$

Corollary 2. *The Julia set of a rational map of degree at least two has no isolated points.*

Proof. We've seen that $J(f)$ must be infinite. It therefore has at least one accumulation point z_0. Now $O^-(z_0)$ is a dense set of non-isolated points in $J(f)$. $\qquad\square$

Corollary 3. *The set of points of the Julia set whose forward orbit is dense (in $J(f)$) is a countable intersection of dense open subsets of $J(f)$, and therefore, by the Baire category theorem, dense in $J(f)$.*

Proof. Consider a countable basis for the topology of $\mathbb{C}$ and consider the collection $\mathcal{U}$ of those basic sets that meet $J(f)$. By the previous proposition, for every $U \in \mathcal{U}$ we have that $J(f) \cap \bigcup_{n \geq 0} f^{\circ - n}(U)$ is a dense open subset of the Julia set. If z is any point in the intersection of these countably many dense open subsets, then the forward orbit of z intersects every single $U \in \mathcal{U}$ and is therefore dense. $\qquad\square$

We will need the following result later on, in the proofs of the No Wandering Domains Theorem:

Theorem 4. *The Julia set is the closure of the set of repelling periodic points.*

For two interesting proofs, one due to Julia and one to Fatou, see chapter 14 of (Milnor 2006). There are many more basic facts about the Julia and Fatou sets that could be mentioned here. The reader is encouraged to look at the references.

5   No wandering domains

In this section we will prove the *No Wandering Domains* theorem, which says that every connected component of the Fatou set is eventually periodic. We'll start with a very rough sketch of the proof.

The idea is to attempt to modify the *conformal structure*[9] of the sphere on the Fatou component U so that the map f is still holomorphic in the new structure. Think of changing the conformal structure on U arbitrarily. To keep f holomorphic will require changing the conformal structure on $f(U)$ as well, and then on $f^{\circ 2}(U)$, and so on. If U is a periodic component, or even just an eventually periodic one, these changes will not be independent of one another and in fact there are strong conditions imposed on the allowable conformal strutures. But if U is a *wandering component*, that is one none of whose iterates is periodic, then the conformal structure on U can be chosen arbitrarily and then the conformal structures on the iterates of U can be repaired one at a time.

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[9]A *conformal structure* on a manifold is roughly a way to measure angles, we will discuss them later on.

But $\mathbb{C}$ really only has one conformal structure, that is, any two conformal structures for $\mathbb{C}$ are isomorphic. Let X be S^2 with some other conformal structure for which f is holomorphic, and let $\phi : X \to \widehat{\mathbb{C}}$ be an isomorphism from X to S^2 with the standard complex structure. Since $f : X \to X$ is holomorphic, we have that $\phi \circ f \circ \phi^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic too. Therefore $\phi \circ f \circ \phi^{-1}$ is some rational map on the Riemann sphere, and, moreover, it must be of the same degree as f since topological degree is clearly preserved by conjugation.

This means that we get a map from the space of conformal structures on U to the space of rational maps with the same degree as f. The space of complex structures on U is infinite dimensional while the space of rational maps of some fixed degree is clearly finite dimensional. So, if we can show that this map is injective, or at least injective on a subspace with high enough dimension, we are done.

Of course, we have glossed over many difficulties in this sketchiest of sketches:

- Conformal structures want to be pulled back, not pushed forward, so it's not that easy to start with one on U and then adjust it on $f(U)$, $f^{\circ 2}(U)$, etc. We will deal with this by finding a wandering component that maps isomorphically to its forward iterates. Related to this is the issue that to keep f holomorphic in the new conformal structure, we also need to modify it along iterated preimages of U. That is not a problem: we just pull back.

- A more serious issue is that at a limit point of $\bigcup_{n \in \mathbb{Z}} f^{\circ n}(U)$ it won't be possible to keep the modified conformal structure smooth or even continuous. To address this we will use a theory of measurable conformal structures (and just leave ours undefined at these limit points) developed by Ahlfors and Bers (Ahlfors and Bers 1960). Bers later noticed that part of their main result, a measurable version of the Riemann mapping theorem, was contained in previous work of Morrey (Morrey 1938). There is also an expository article on this topic (Zakeri and Zeinalian 1996).

- We haven't said anything about how to prove "sufficient injectivity" of the map from conformal structures on U to rational maps. In fact, we won't really prove the theorem using that map directly. Instead, following (McMullen 2011) we will look at the derivative of this map, whose target is then the space deformations of f. Sullivan's original proof goes along the lines suggested above.

Remark 2. Transcendental maps $\mathbb{C} \to \mathbb{C}$ *can* have wandering domains. See, for instance, section 4.5 of (Bergweiler 1993) where, among other things, Herman's examples from (Herman 1984), $z \mapsto z - 1 + e^{-z} + 2\pi i$ and $z \mapsto z + \lambda \sin(2\pi z) + 1$ (for appropriate λ), are described.

5.1 Reduction to simply connected wandering domains

The following observation of Baker's shows we need only worry about simply connected wandering domains.

Lemma 2 (Baker). *If U is a wandering domain for a rational map f, then $f^{\circ n}(U)$ is simply connected for all large n. Furthermore, each sufficiently high iterate of U is mapped homeomorphically onto the next.*

Our first step will be the following lemma:

Lemma 3. *If U is a wandering domain for f, and $K \subset U$ is compact, the diameter (in the spherical distance) of $f^{\circ n}(K)$ tends to zero.*

Proof. Suppose not. Then there is a positive ϵ and an infinite sequence of numbers (n_j) such that $\mathrm{diam}(f^{\circ n_j}(K)) \geq \epsilon$. Since the iterates of f are a normal family on U, passing to a subsequence of n_j we can assume that the $f^{\circ n_j}$ converges uniformly on K to some holomorphic function g.

This g cannot be a constant function, since if it had constant value w_0, for large enough j, $f^{\circ n_j}(K)$ would lie in the ball of radius $\epsilon/3$ around w_0 forcing its diameter to be smaller than ϵ.

Take any point $z_0 \in U$ and consider some small circle γ around z_0 whose interior is contained in U and on which g does not take the value $g(z_0)$ again. Then $|g(z) - g(z_0)|$ attains some positive minimum δ on γ and for large enough j we have $|f^{\circ n_j}(z) - g(z)| < \delta \leq |g(z) - g(z_0)|$ on γ. By Rouché's theorem, this implies all $f^{\circ n_j}$ for large j take the value $g(z_0)$ somewhere inside Γ, contracting the fact that the iterates of U are disjoint. $\square$

Now we can prove Baker's lemma:

Proof. Since f has only finitely many critical points, we can replace U by a high enough iterate to get that neither U nor any of its iterates contains a critical point. Then all the maps $f^{\circ n} : U \to f^{\circ n}(U)$ and $f : f^{\circ n}(U) \to f^{\circ (n+1)}(U)$ are covering maps.

Now take any simple closed curve $\gamma \subset U$ and consider its iterates $\gamma_n := f^{\circ n}(\gamma)$. By the above lemma, $\mathrm{diam}(\gamma_n) \to 0$, as does $\mathrm{diam}(A_n)$ where A_n is the union of the components of $\widehat{\mathbb{C}} \setminus \gamma_n$ that don't meet U, or, if we take

$\infty \in U$, then we mean the bounded components of $\widehat{\mathbb{C}} \setminus \gamma_n \subset \mathbb{C}$. Consider the open set $f(A_n)$. Its boundary is contained in $\gamma_{n+1} = f(\gamma_n)$ and thus $\mathrm{diam} f(A_n) \to 0$. This means that the components of $f(A_n)$ must be among the bounded components of $\mathbb{C} \setminus \gamma_{n+1}$, and so $f(A_n) \subseteq \overline{A_{n+1}}$, in particular, all iterates of $f(A_n)$ are contained in $\mathbb{C} \setminus U$. By Montel's theorem, this means the iterates of f are normal on A_n and so $A_n \subseteq F(f)$ and necessarily, $A_n \subseteq f^{\circ n}(U)$. But this says that γ_n is null-homotopic inside $f^{\circ n}(U)$, and since $f^{\circ n} : U \to U_n$ is a covering map, the null-homotopy can be lifted to U.

Thus, we have proved all high enough iterates of U are simply connected. And, since all of the maps $f : f^{\circ n}(U) \to f^{\circ(n+1)}(U)$, for large n, are now covering maps between simply connected domains, they are isomorphisms.
$\square$

5.2 Measurable Conformal Structures

In this section we describe measurable conformal structures, Beltrami differentials and the Measurable Riemann Mapping Theorem, but let's first discuss smooth conformal structures.

A *conformal structure* on a smooth manifold is a way of measuring angles without a specified companion way to measure distance. More precisely, it is a choice of *conformal class* of Riemannian metrics on the manifold, where two metrics g and h are conformally equivalent if $h = \lambda g$ for some positive smooth function λ. A map between two Riemannian manifolds $f : M \to N$ is *conformal* if the pull back of the metric on N is conformally equivalent to the metric on M. This means that f preserves oriented angles between tangent vectors.

For surfaces, a conformal structure is the same as giving a complex structure, because a function $U \subset \mathbb{C} \to \mathbb{C}$ is conformal if and only if it is holomorphic and has non-vanishing derivative on U. But the two notions differ in every other dimension and in particular, conformal structures exist on manifolds with odd real dimension.

How can we specify a conformal class of inner products in $\mathbb{R}^2$? Well, we need only specify the unit ball of the inner product, up to scaling, and this is some ellipse with center at the origin. To get a handy description of ellipses using complex coordinates, we can think of an ellipse as the inverse image of a circle under a linear transformation and notice that every linear function $\mathbb{R}^2 \to \mathbb{R}^2$ can be written in the form $z \mapsto az + b\bar{z}$ for some $a, b \in \mathbb{C}$. So any ellipse centered at the origin can be described, up to scaling, by an equation

of the form $|az + b\bar{z}| = 1$, or even just[10] $|z + \mu\bar{z}| = 1$ for some $\mu \in \mathbb{C}$. Let's compute the eccentricity of this ellipse. We want the maximum and minimum of $|z|$ when z satisfies $|z + \mu\bar{z}| = 1$. Well $|z| = |1 + \mu(\bar{z}/z)|^{-1}$ and $\bar{z}/z$ is an arbitrary complex number on the unit circle, so assuming $|\mu| < 1$, the maximum and minimum are $1/(1 - |\mu|)$ and $1/(1 + |\mu|)$, making the eccentricity $(1 + |\mu|)/(1 - |\mu|)$.

So to specify a conformal structure on an open domain $U \subset \mathbb{C}$, it is enough to give a complex number $\mu(z)$ for each $z \in U$. When is a smooth map $f : U \to \mathbb{C}$ conformal from the new conformal structure described by μ on U to the standard structure on $\mathbb{C}$? It's derivative must send the ellipses in the tangent space to U given by the choice of μ to circles in the tangent space to $\mathbb{C}$. In terms of the differential operators

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

we can write the linear operator Df (at some implied point z) as

$$Df(w) = w\frac{\partial f}{\partial z} + \bar{w}\frac{\partial f}{\partial \bar{z}}.$$

Comparing to the description above of the ellipses, we see that f will be conformal from the μ-structure on U to the standard structure on $\mathbb{C}$ if the *Beltrami equation* is satisfied:

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}.$$

Note that $\mu \equiv 0$ corresponds to the usual structure on U, and that then the Beltrami equation, reasonably enough, reduces to the Cauchy-Riemann equation. Solutions to the Cauchy-Riemann equation are conformal[11], so we will call a solution to the Beltrami equation for some μ satisfying $\mu(z) < c < 1$ (some constant c), a *quasiconformal* map $U \to \mathbb{C}$ with multiplier μ.

We can also figure out when a map $f : U \to V$ is conformal for conformal structure given by $\mu : U \to \mathbb{C}$ and $\nu : V \to \mathbb{C}$. We need $Df|_z$ to take ellipses $|w + \mu(z)\bar{w}| = \text{const}$ to ellipses $|w + \nu(f(z))\bar{w}| = \text{const}$. Plugging the formula for Df into the equation of the target ellipse and comparing with the equation for the domain ellipse, we get that f is μ-ν-conformal if

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[10]The case $a = 0$ gives a circle, same as $\mu = 0$, so we don't need it.

[11]Well, when they are locally injective at least.

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and only if[12]

$$\frac{\partial f}{\partial \bar{z}} + \nu(f(z))\frac{\partial \bar{f}}{\partial \bar{z}} = \mu(z)\left(\frac{\partial f}{\partial z} + \nu(f(z))\frac{\partial \bar{f}}{\partial z}\right).$$

Notice that when $\nu \equiv 0$ this reduces, as it should, to the Beltrami equation.

Now we need to generalize in two directions: we need to *globalize*, that is describe conformal structures on Riemann surfaces, not just domains in $\mathbb{C}$; and we need to *weaken the smoothness assumption*, which we will do by considering distributional derivatives. Let's handle the second direction first:

Definition 2. A continuous function $f : U \to \mathbb{C}$ has *distributional derivatives* in L^1 if there are functions f_z and $f_{\bar{z}}$ in $L^1(U)$ so that for any smooth function $g : U \to \mathbb{C}$ with compact support we have

$$\iint_U \left(f_z(z)g(z) + h(z)\frac{\partial g}{\partial z}\right) dx\, dy = 0,$$

and the analogous equation for $\bar{z}$

Of course f_z is meant to stand for $\frac{\partial f}{\partial z}$, and indeed, if f is C^1, we can take it to be so: the integrand is just the partial derivative $\frac{\partial}{\partial z}(fg)$, the integral vanishes because fg has compact support.

For any bounded measurable function μ, we can pose the Beltrami equation for this more general kind of derivative: $f_{\bar{z}}(z) = \mu(z)f_z$. (The product of a bounded measurable fucntion and an L^1 function is L^1, so this is a reasonable equation.) The fundamental result about solutions to this equations is the following:

Theorem 5 (Measurable Riemann mapping theorem). *Let $U \subset C$ be open, and let $\mu : U \to \mathbb{C}$ be a measurable function satisfying $|\mu(z)| < c < 1$ for some constant c almost everywhere. Then there is a solution $f : U \to \mathbb{C}$ to the Beltrami equation $f_{\bar{z}} = \mu f_z$. Any solution is a homeomorphism onto its image, and for any two solutions f_1 and f_2, $f_2 \circ f_1^{-1}$ is holomorphic.*

Notice that the condition $|\mu(z)| < c < 1$, says that the eccentricities of the ellipses giving the conformal structure are bounded.

Now, to describe a conformal structure on a Riemann surface X, we can give bounded measurable functions on patches of X, but they need to satisfy some compatibility on overlapping patches which we will see shortly.

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[12]The reader checking the computation should remember the identities $\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$ and $\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}$.

We want to build a new Riemann surface X_μ reflecting the new conformal structures on the patches of X, so we take X_μ to be the same topological space as X but use the solutions to the Beltrami equations on individual patches as charts. Let z and z' two holomorphic coordinates on overlapping patches where we have given the conformal structure by functions μ and μ' respectively. From our equation above for μ-μ'-conformal maps we see that for the change of coordinate map $(f = z' \circ z^{-1})$ to be holomorphic we must have

$$\mu'(z') = \mu(z)\frac{\partial z'}{\partial z}\frac{\partial \bar{z}}{\partial \bar{z'}}.$$

Such a collection of measurable functions will be called a *Beltrami differential* (it transforms as a $(-1, 1)$-form would, in some sense). Notice that just to define a Beltrami differential on X we don't really need to worry about this compatibility condition if we define it on a single coordinate patch that covers all of X except for a set of measure 0.

So given a Beltrami differential μ on X we obtain a new Riemannian surface homeomorphic to X but in general, conformally different. However, in the special case of $X = \widehat{\mathbb{C}}$, it follows from the Uniformization theorem that $\widehat{\mathbb{C}}_\mu$ actually does have to be conformally isomorphic to $\widehat{\mathbb{C}}$, and in particular, there is a unique conformal isomorphism $f : \widehat{\mathbb{C}}_\mu \to \widehat{\mathbb{C}}$ fixing 0, 1 and ∞. As above we will also say that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiconformal, without the μ on the first $\widehat{\mathbb{C}}$. We will need one further result: the theorem of Ahlfors and Bers that this isomorphism varies holomorphically in an appropriate sense:

Theorem 6 (Ahlfors-Bers). *For any Beltrami differential μ on $\widehat{\mathbb{C}}$ with $|\mu|_\infty < 1$, there is a unique quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with $f_{\bar{z}} = \mu f_z$ and such that f fixes 0, 1 and ∞. Moreover, the mapping $f_t(z)$ obtained for the Beltrami differential $t\mu$ (for small enough complex t), depends holomorphically on t for fixed z.*

5.3 The space of rational maps of a fixed degree

A rational map $p(z)/q(z)$ of degree d has $2d + 2$ coefficients, but these only matter up to rescaling, so we can regard rational maps as points in projective space $\mathbb{CP}^{2d+1}$. Not all points there correspond to a rational map of degree d: at least one of p and q must have non-zero leading coefficient, and they must be relatively prime. Both of these are open conditions[13] (and even Zariski open), so that Rat_d is an open subset of $\mathbb{CP}^{2d+1}$.

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[13]Being relative prime means having disjoint sets of roots, which is clearly preserved under slight perturbations.

Now let's figure out how to represent tangent vectors to Rat_d. A holomorphic curve in Rat_d is a family of rational maps $g_t(z)$ that depend holomorphically on t. The derivative $w(z) = \frac{\partial}{\partial t}|_{t=0} f_t(z)$ is a tangent vector to $\widehat{\mathbb{C}}$ at the point $f(z)$, so w is a global holomorphic section of the pull back bundle $f^*(T\widehat{\mathbb{C}})$. Thus $T_f\mathrm{Rat}_d = \Gamma(\widehat{\mathbb{C}}, f^*(T\widehat{\mathbb{C}}))$.

5.4 The proof

Now we can prove the No Wandering Domains theorem!

The setup. Assume f is a rational map of degree at least two with a wandering domain U. We've already seen that we can assume that U is simply-connected and that f maps each iterate $f^{\circ n}(U)$ isomorphically onto the next iterate. Let V be the set of all points such that some iterate of them lies in some iterate of U, this is the set along which we wil varying the conformal structure of $\widehat{\mathbb{C}}$. By our assumption on U, V is the disjoint union $\cup_{n\in\mathbb{Z}} f^{\circ n}(U)$. We will assume $\infty \in J(f)$ and freely regard $V \subset \mathbb{C}$.

Spreading Beltrami forms from U to all of $\widehat{\mathbb{C}}$. Given any measurable $\mu : U \to \mathbb{C}$ with $\|\mu\|_\infty < \infty$, we can define a Beltrami differential $\hat{\mu}$ on all of $\widehat{\mathbb{C}}$ that is f-*invariant*, that is, for which $f : \widehat{\mathbb{C}}_{\hat{\mu}} \to \widehat{\mathbb{C}}_{\hat{\mu}}$ is conformal. Using the equation from the previous section that tells when maps are conformal for structures given by Beltrami differentials, we see that for a holomorphic map f, we just need to ensure $\hat{\mu}(f(z))\overline{f'(z)} = \hat{\mu}(z)f'(z)$. To accomplish this we set

- $\hat{\mu} = 0$ outside V

- $\hat{\mu} = \mu$ on U,

- define $\hat{\mu}$ on the forward iterates of U recursively: $\hat{\mu}(z) := \mu(w)f'(w)/\overline{f'(w)}$ for $z \in f^{\circ(n+1)}(U)$ where $w = f^{-1}(z) \in f^{\circ n}(U)$,

- define $\hat{\mu}$ on (most of the points in) the iterated preimages of U again recursively: $\hat{\mu}(z) := \mu(f(z))\overline{f'(z)}/f'(z)$ for $z \in f^{\circ -n}(U)$.

Note that this defines $\hat{\mu}$ at all points except those that are *precritical*, i.e., some of iterate of which is a critical point of f. This doesn't matter, we can just leave $\hat{\mu}$ undefined at those countably many points since Beltrami differentials are only defined up to sets of measure zero.

The formulas above make it clear that $\|\hat{\mu}\|_\infty = \|\mu\|_\infty < \infty$. So we have defined an injective linear map $M(U) \hookrightarrow M(\widehat{\mathbb{C}})^f$ where $M(X)$ is the space of essentially bounded Beltrami differentials on X and $M(\mathbb{C})^f$ is the subspace of f-invariant Beltrami differentials.

Constructing deformations of f. Now we define a map $M(\widehat{\mathbb{C}})^f \to T_f\mathrm{Rat}_d$. Let $\mu \in M(\widehat{\mathbb{C}})^f$. Then for small enough t, $\|t\mu\|_\infty < 1$ and by the Measureable Riemann Mapping Theorem, we get a familiy of quasi-conformal maps $\phi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ solving the Beltrami equation for $t\mu$ and depending holomorphically on t. Since the equation for f-invariance is linear in μ, $t\mu$ will also be f-invariant, so $f : \widehat{\mathbb{C}}_{t\mu} \to \widehat{\mathbb{C}}_{t\mu}$ is conformal. Conjugating by the conformal isomorphism $\phi_t : \widehat{\mathbb{C}}_{t\mu} \to \widehat{\mathbb{C}}$, gives the rational map $f_t := \phi_t \circ f \circ \phi_t^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, depending holomorphically on t. Then $w_\mu(z) := \frac{\partial}{\partial t}|_{t=0} f_t(z)$ can be identified with a tangent vector in $T_f\mathrm{Rat}_d$.

It is not immediately clear from this description that this map $M(\widehat{\mathbb{C}})^f \to T_f\mathrm{Rat}_d$ is in fact linear. Let $v_\mu(z) := \frac{\partial}{\partial t}|_{t=0}\phi_t(z)$, which is a (merely) continuous vector field on $\widehat{\mathbb{C}}$. Then

- w_μ depends linearly on v_μ, because $w_\mu(z) = v_\mu(f(z)) - f'(z)v_\mu(z)$, and,

- v_μ depends linearly on μ, because it is the unique solution to the equation $\bar{\partial}v = \mu$ that vanishes at 0, 1 and ∞.

The proof of both of these equations is a simple calculation involving the chain rule. The uniqueness in the second statement is an infinitesimal version of the Measurable Riemann Mapping Theorem (see theorem 5.28 in (McMullen 2011)).

The "sufficiently injective" bit. Now let's find this large dimensional subspace on which our map will be injective.

Lemma 4. *There is an infinite dimensional subspace V of $M(U)$ of compactly supported Beltrami differentials with the following property: if $\mu \in V$ satisfies $\bar{\partial}v = \mu$ for some continuous vector field with $v|_{\partial U} = 0$, then $\mu = 0$.*

Proof. Let's consider an analogous problem for $M(\mathbb{D})$ first. Consider the subspace $V' \subset M(\mathbb{D})$ spanned by the Beltrami differentials

$$\mu_k(z) = \begin{cases} \bar{z}^k & \text{if } |z| \leq 1/2 \\ 0 & \text{if } |z| > 1/2. \end{cases}$$

for $k \geq 0$. These have the following particular solutions for the $\bar{\partial}$ equation:

$$v_k(z) = \begin{cases} \frac{1}{k+1}\bar{z}^{k+1}\frac{\partial}{\partial z} & \text{if } |z| \leq 1/2 \\ \frac{1}{k+1}(4z)^{-(k+1)}\frac{\partial}{\partial z} & \text{if } |z| > 1/2 \end{cases}.$$

Suppose $\mu = \bar{\partial}v \in V'$ with $v|_{\partial\mathbb{D}} = 0$. Then v is holomorphic on $1/2 < |z| < 1$, so being zero on $|z| = 1$, it is also zero on $|z| = 1/2$. Let $w = \sum \lambda_k v_k$ where the λ_k are the coefficients in $\mu = \sum \lambda_k \mu_k$. Since $\bar{\partial}(w - v) = 0$, $w - v$ is holomorphic throughout $\mathbb{D}$, but it also agrees with w on $|z| = 1/2$, which is a contradiction, since there w is a polynomial in z^{-1}.

Now let's deal with U. Take a conformal isomorphism $\pi : \mathbb{D} \to U$. Let V the subspace of $M(U)$ corresponding to the above subspace V' of $M(\mathbb{D})$ under the induced isomorphism $M(\mathbb{D}) \simeq M(U)$. Suppose $\mu = \bar{\partial}v \in V \subset M(U)$ for some $v = v(z)\frac{\partial}{\partial z}$ with $v|_{\partial U} = 0$. Then $(\pi^{-1})_*(v) = \frac{v(\pi(z))}{\pi'(z)}\frac{\partial}{\partial z}$ and the numerator, $v(\pi(z))$ is holomorphic outside a compact subset of $\mathbb{D}$ and tends to zero as $z \to \partial\mathbb{D}$. By the Schwarz reflection principle, $v(\pi(z))$ is identically zero outside that compact subset of $\mathbb{D}$, and therefore $(\pi^{-1})_*v$ is a compactly supported vector field on $\mathbb{D}$, which by consruction satisfies $\bar{\partial}(\pi^{-1})_*v = \pi^*\mu$. Thus $\pi^*\mu = 0$, which forces $\mu = 0$. $\qquad\square$

Concluding. We can now finish the proof. Consider the composition $V \hookrightarrow M(U) \hookrightarrow M(\widehat{\mathbb{C}})^f \to T_f\mathrm{Rat}_d$. If some $\mu \in V$ maps to 0 under the composition, i.e., if $w_\mu = 0$, we'd have $v_\mu(f(z)) = f'(z)v_\mu(z)$ for all z. Let z be a periodic point of period n with multiplier λ, and multiply[14] the n equations $v_\mu(f^{\circ(j+1)}(z)) = f'(f^{\circ j}(z))v_\mu(f^{\circ j}(z))$ for $j = 0, \ldots, n-1$. We get

$$(\lambda - 1)\prod_{j=0}^{n-1} v(f^{\circ j}(z)) = 0.$$

So, if we had chosen z to be a repelling periodic point, we'd get that one, and therefore all, of the $v(f^{\circ j}(z))$ ($j = 0, 1, \ldots, n-1$) are 0. As the Julia set is the closure of the repelling periodic points and v is continuous, we see that v vanishes on $J(f)$ and, in particular, on ∂U. By the lemma, $\mu = 0$. Thus the composite map $V \hookrightarrow T_f\mathrm{Rat}_d$ is injective and this is the desired contradiction: V is infinite-dimensional and $T_f\mathrm{Rat}_d$ has dimension $2d + 1$.

5.5 Sullivan's original proof

The original proof in (Sullivan 1985) was a little more complicated for two reasons. First, it does not use Baker's observation; instead, Sullivan splits the proof into cases. If the wandering domain doesn't have the property that for all large n the restriction $f : f^{\circ n}(U) \to f^{\circ(n+1)}(U)$ is an isomorphism, he takes the direct limit of the system $U \to f(U) \to f^{\circ 2}(U) \to \cdots$, shows

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[14] Here we take $v(z)$ to be, not quite a tangent vector, but the coefficient of $\frac{\partial}{\partial z}$ in such a vector.

it is a Riemann surface of infinite type and that then it has an infinite dimensional family of (conformal equivalence classes of) conformal structures quasi-conformally equivalent to each other. Even if f is eventually an isomorphism on iterates of U, U still might not be simply connected and small modifications of the argument are required.

Second, the original proof does not use McMullen's linearized argument with deformations. Instead it shows, as we mentioned in the sketch, "sufficient injectivity" of the map from conformal structures to rational maps of degree d. This can be done relatively easily if ∂U is a Jordan curve, but in general deals with the boundary of U using Caratheodory's theory of prime ends. See either (Sullivan 1985) directly or appendix F of (Milnor 2006).

6 Classification of periodic domains

Here we prove the classification of periodic Fatou components stated in the overview. Recall the different types:

Definition 3. A periodic Fatou component U of f with period p is called

- an *attractive basin* if f^p has a fixed point $z_0 \in U$, with $|(f^p)'(z_0)| < 1$, that attracts all points of U under iteration,

- a *parabolic basin* if some point z_0 on the boundary of U attracts all points of U, in which case, necessarily $(f^p)'(z_0) = 1$,

- *Siegel disk* if the dynamical system (U, f^p) is conformally isomorphic to an irrational rotation on the unit disk, and

- A *Herman ring* if the dynamical system (U, f^p) is conformally isomorphic to an irrational rotation on some annulus

Theorem 7. *Every periodic Fatou component of a rational map of degree at least two is either an attractive basin, a parabolic basin, a Siegel disk or a Herman ring.*

Proof. Replacing f by f^p, where p is the period of the Fatou component U, we can assume that $f(U) = U$. Considered as a Riemann surface in its own right, U is hyperbolic (because the Julia set is infinite), so by Pick's theorem, f does not increase the hyperbolic distance on U, which we will denote d. We will also use the usual spherical distance on $\mathbb{C}$ and denote it $d_{\widehat{\mathbb{C}}}$.

For any point $z \in U$, let $A(z) \subseteq \bar{U}$ be the set of accumulation points of the orbit of z. We say that the orbit of z *tends to infinity* if $A(z) \subseteq \partial U$.

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Proposition HvsS implies that if one orbit tends to infinity, then all do, because $d(f^{\circ n}(z), f^{\circ n}(w))$ is bounded by $d(z, w)$ which does not depend on n. So we have two cases:

Case 1: All orbits tend to infinity. Let $z_0 \in U$ and $z_n = f^{\circ n}(z_0)$. Then we must have $d_{\widehat{\mathbb{C}}}(z_n, z_{n+1}) \to 0$ by Proposition HvsS, which is applicable since $d(z_n, z_{n+1})$ is bounded by $d(z_0, z_1)$. This tells us something about $A(z_0)$. First, it consists of fixed points of f lying on ∂U. Second, it must be connected, for if it had at least two components A_1 and A_2, these would be two disjoint compact subsets of $\widehat{\mathbb{C}}$ and therefore some (spherical) distance $\delta > 0$ apart; for large n, $d_{\widehat{\mathbb{C}}}(z_n, z_{n+1}) < \delta$, contradicting that the orbit accumulates on both components. Since f is not the identity the only connected sets of fixed points it can have are singletons, $A = \{p\}$. This fixed point p attracts every point in $w \in U$, because $d(f^{\circ n}(w), z_n)$ is bounded so, again by Proposition HvsS, $d_{\widehat{\mathbb{C}}}(f^{\circ n}(w), z_n) \to 0$. Since p attracts U but is in the Julia set it must be an indifferent fixed point. It still remains to show that $\lambda := f'(p) = 1$. This should be geometrically plausible by the following argument:

Choose a path connecting z_0 and z_1. Its image under f connects z_1 and z_2, and its image under $f^{\circ 2}$ connects z_2 and z_3, etc. Glueing all of these pieces together, taking a unit of time for each piece, we get a path $\alpha : [0, \infty) \to U \setminus \{p\}$ such that $\alpha(t+1) = f(\alpha(t))$ and $\alpha(t) \to p$. The closer we get to p the closer f is to multiplication by λ, so, if $\lambda \neq 1$, this path looks more and more like a simple spiral: $\alpha(t+1) \sim \lambda\alpha(t)$. Assume this picture is accurate: that α really does like a spiral around p, and that it has no self-intersections. Define a region W by making a "cross-cut": start at a point $\alpha(t_0)$ near p, walk radially away from p until you meet α again, and then follow α back to where you started. This region W gets mapped by f onto the analogous region starting at $\alpha(t_0 + 1)$ which is properly contained in W. By the Schwarz Lemma, the fixed point must be attracting, which is a contradiction.

This is just a sketch of the proof of a result called the *Snail Lemma*, which says that given a fixed point p with multiplier λ and a path α satisfying i) and ii), then either $|\lambda| < 1$ or $\lambda = 1$. See Lemma 16.2 in (Milnor 2006).

Case 2: Every orbit has an accumulation point in U. By Pick's theorem there are just these two subcases:

Case 2a: f decreases hyperbolic distance. As before, let z_n be an orbit in U. There is some compact $K \subset U$ such that $z_n \in K$ for infinitely many n. This is called *recurrence*. Recall that in this subcase, f is Lipschitz on the compact set $K \cup f(K)$ with Lipschitz constant $c < 1$. We claim that $d(z_n, z_{n+1}) \to 0$. Indeed, it a decreasing sequence of numbers and addition-

ally, whenever $z_n \in K$, we have $d(z_{n+1}, z_{n+2}) \leq c \, d(z_n, z_{n+1})$. Therefore any point $p \in A(z_0) \cap K$ must be a fixed point and, because f decreases the hyperbolic distance, it must be the only one in U. It attracts every point because all hyperbolic balls $B(p, r)$ are invariant, and in fact, f is Lipschitz on such a ball with some constant less than 1, so that p attracts the whole ball uniformly.

Finally we prove that $|f'(p)| < 1$. Consider a small *spherical* disk D around p: it is contained in some hyperbolic ball $B(p, R)$ and contains some hyperbolic ball $B(p, r)$. Say f is Lipschitz on $B(p, R)$ with constant $k < 1$. Then, as soon as $k^n < r/R$, $f^{\circ n}(D) \subset B(p, r) \subset D$. By Schwarz's Lemma, $|(f^{\circ n})'(p)| < 1$, so $|f'(p)| < 1$ as well.

Case 2b: f is a covering map and a local isometry. If $\pi_1(U)$ is abelian, then U is a disk, a punctured disk or an annulus. Since the Julia set has no isolated points, the case of a punctured disk cannot occur. By explicitly looking at the self-coverings of disks and annuli it is easy to check that the only recurrent ones are rotations. Since the iterate of f have larger and larger degree, f cannot be of finite order, and thus the rotation must be irrational.

If $\pi_1(U)$ is nonabelian, then $U = \mathbb{D}/\Gamma$ where $\pi_1(U) \cong \Gamma \subset \mathrm{Aut}(\mathbb{D})$ is the group of deck transformations of the universal cover $\pi : \mathbb{D} \to U$. Let $g_n : \mathbb{D} \to \mathbb{D}$ be a lift of $f^{\circ n}$. There is freedom in choosing the g_n: namely we can pick $g_n(0) \in \pi^{-1}(f^{\circ n}(\pi(0)))$ arbitrarily; let's pick it to lie as close to 0 in the hyperbolic metric as possible. This ensures that the g_n lie in a compact subset of $\mathrm{Aut}(\mathbb{D})$, since whenever the n-th iterate of $\pi(0)$ under f returns to some compact set, so does $g_n(0)$.

Now notice that for every $\gamma \in \Gamma$ and every $n \geq 0$, $g_n \circ \gamma \circ g_n^{-1}$ covers the identity on U, and thus is a deck transformation and belongs to Γ. But it turns out the set $E := \{h \in \mathrm{Aut}(\mathbb{D}) : h\Gamma h^{-1} \subset \Gamma\}$ is discrete by a very clever argument from (McMullen and Sullivan 1998). Indeed, if $h_n \to h$ is a convergent sequence of elements in that set, look at the sequence $k_n := h^{-1} \circ h_n$: it converges to the identity and has the property that $k_n \Gamma k_n^{-1} \subset h^{-1}\Gamma h =: \Gamma'$. Since Γ is discrete, so is its conjugate Γ'. Now, for any $\gamma \in \Gamma$, the sequence $k_n \circ \gamma \circ k_n^{-1} \in \Gamma'$ converges to γ (because $k_n \to \mathrm{id}$) and thus γ commutes with k_n for large enough n. Finally given *two elements* $\gamma_1, \gamma_2 \in \Gamma$, they both commute with the same k_n (for any n large enough) which implies they commute with each other by the following classical result: /two elements of $\mathrm{Aut}(\mathbb{D})$ commute with each other if and only if their extensions to $\overline{\mathbb{D}}$ have the same fixed point set/. But since $\Gamma \cong \pi_1(U)$ is nonabelian, this is a contradiction and we conclude E is discrete.

Now we know that the g_n lie in a compact subset of $\mathrm{Aut}(\mathbb{D})$ and also in

E, a discrete subset. It follows there are only finitely many distinct g_n, which means there are only finitely many distinct iterates of f, i.e., f is of finite order. This contradiction shows that nonabelian $\pi_1(U)$ cannot occur. $\qquad\square$

6.1 There are finitely many periodic domains

As mentioned in the overview, Shishikura has proved a sharp bound of $2d-2$ for the number of periodic Fatou components. It is easier to show just finiteness and this is done, for example, in (McMullen and Sullivan 1998). We won't show even this here, but we will very briefly describe their proof. There are classical results saying that each attracting, superattracting and parabolic cycle attracts a critical point, so there are at most $2d-2$ of those. Also, Fatou showed the number of Siegel disks is bounded by $4d-4$ (the proof is roughly that a suitable perturbation of f makes at least half of the indifferent cycles attracting).

So it remains to bound the Herman rings. In their paper, they study the Teichmüller space $\mathrm{Teich}(\widehat{\mathbb{C}}, f)$ of a rational map f, which is the quotient of the space $M_1(\widehat{\mathbb{C}})^f := \{\mu \in M(\widehat{\mathbb{C}})^f : \|\mu\|_\infty < 1\}$ of conformal structures on $\widehat{\mathbb{C}}$ for which f is holomorphic, modulo the group $QC_0(\widehat{\mathbb{C}}, f)$, which very roughly consists of quasiconformal conjugacies isotopic to the identity[15]. They prove that $\mathrm{Teich}(\mathbb{C}, f)$ is a connected complex manifold of dimension at most $2d-2$ (where $d = \deg f$), and that each Herman ring contributes a one dimensional factor to $\mathrm{Teich}(\widehat{\mathbb{C}}, f)$, so there are at most $2d-2$ Herman rings.

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[15]To make this correct, one needs to add that the isotopies are relative to the *ideal boundary* of X, and they should be through quasiconformal maps of uniformly bounded dilatation.

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