Proceedings of the Edinburgh Mathematical Society

Submitted Paper
Paper 30 July 2020

NILPOTENT N-TUPLES IN $SU(2)$

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(Received 8 May 2019)

Abstract We describe the connected components of the space $\mathrm{Hom}(\Gamma, SU(2))$ of homomorphisms for a discrete nilpotent group Γ. The connected components arising from homomorphisms with non-abelian image turn out to be homeomorphic to $\mathbb{RP}^3$. We give explicit calculations when Γ is a finitely generated *free nilpotent* group. In the second part of the paper we study the filtration $B_{\mathrm{com}}SU(2) = B(2, SU(2)) \subset \cdots \subset B(q, SU(2)) \subset \cdots$ of the classifying space $BSU(2)$ (introduced by Adem, Cohen and Torres-Giese), showing that for every $q \geq 2$, the inclusions induce a homology isomorphism with coefficients over a ring in which 2 is invertible. Most of the computations are done for $SO(3)$ and $U(2)$ as well.

Keywords: spaces of representations; nilpotent groups; classifying spaces

AMS 2010 *Mathematics subject classification:* Primary 22E99
Secondary 55R35

1. Introduction

Given a discrete group Γ and a topological group G, we topologize the set of homomorphisms $\mathrm{Hom}(\Gamma, G)$ as a subspace of G^Γ, where the latter space is given the product topology. In the most commonly studied case, Γ is finitely generated. If Γ has k generators, $\mathrm{Hom}(\Gamma, G)$ is in one to one correspondence with the subset of the product G^k consisting of k-tuples satisfying the relations that hold among the generators, and the topology we gave $\mathrm{Hom}(\Gamma, G)$ agrees with its topology as a subspace of G^k. In this paper we give a full description of the spaces of homomorphisms of a discrete nilpotent group Γ, to the Lie group $SU(2)$. We do this by showing that all non-abelian nilpotent subgroups of $SU(2)$ are conjugate to one of the generalized quaternions Q_{2^q} of order 2^q.

Theorem 3.1. Let Γ be a discrete nilpotent group, and for every $q \geq 3$ let $\mathrm{Epi}(\Gamma, Q_{2^q})$ denote the set of surjective homomorphisms $\Gamma \to Q_{2^q}$. Then there is a homeomorphism

$$\mathrm{Hom}(\Gamma, SU(2)) \cong \mathrm{Hom}(\Gamma^{\mathrm{ab}}, SU(2)) \sqcup \bigsqcup_{q\geq 3} \bigsqcup_{\mathcal{O}} PU(2)$$

where $\mathcal{O}$ runs through the orbits $\mathrm{Epi}(\Gamma, Q_{2^q})/N_{SU(2)}(Q_{2^q})$ induced by the action of conjugation of the normalizer in $SU(2)$.

To have a full description of $\mathrm{Hom}(\Gamma, SU(2))$ one also needs a description of the space $\mathrm{Hom}(\Gamma^{\mathrm{ab}}, SU(2))$, which is afforded by the following proposition for a general discrete abelian group.

Proposition 4.1. Let A be a discrete abelian group. There is a pushout square which is also a homotopy pushout square,

$$\begin{tikzcd}
SU(2)/N(T) \times \mathrm{Hom}(A, \{\pm I\}) \arrow[r] \arrow[d] & SU(2)/T \times_{\mathbb{Z}/2} \mathrm{Hom}(A, T) \arrow[d] \\
\mathrm{Hom}(A, \{\pm I\}) \arrow[r] & \mathrm{Hom}(A, SU(2))
\end{tikzcd}$$

where T is a maximal torus of $SU(2)$, $N(T)$ its normalizer; the horizontal arrows are induced by the inclusions $\{\pm I\} \hookrightarrow T$, $\{\pm I\} \hookrightarrow SU(2)$; the vertical arrows are induced by conjugation, and the action of $\mathbb{Z}/2$ on $SU(2)/T \times \mathrm{Hom}(A, T)$ is diagonal, given by the antipodal action on $SU(2)/T \cong S^2$ and by complex conjugation on $\mathrm{Hom}(A, T)$.

As an application of Theorem 3.1 we compute the number of connected components of the space $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, G)$ where F_n/Γ_n^{q+1} is the free nilpotent group on n generators of nilpotency class q, and $G = SU(2), SO(3)$ or $U(2)$. For example:

Corollary 3.2. Let $q \geq 2$ and $n \geq 1$. Then

$$\mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \cong \mathrm{Hom}(\mathbb{Z}^n, SU(2)) \sqcup \bigsqcup_{C(n,q+1)} PU(2)$$

where

$$C(n, q+1) = \frac{2^{n-2}(2^n - 1)(2^{n-1} - 1)}{3} + (2^n - 1)(2^{(q-2)(n-1)} - 1)2^{2n-3}.$$

In section 5, we cover the cases $G = SO(3)$ and $U(2)$.

Section 6 contains a description of $\mathrm{Rep}(F_n/\Gamma_n^3, U(m))$, the orbit space associated to the conjugation action of $U(m)$ over $\mathrm{Hom}(F_n/\Gamma_n^3, U(m))$ —sometimes called the the representation space. We express it as a union of almost commuting tuples $B_n(U(\mathbf{a}), Z_{\mathbf{a}})$ (i.e. n-tuples whose commutators are contained in the center $Z(U(\mathbf{a})) =: Z_{\mathbf{a}}$) of block matrix subgroups $U(\mathbf{a}) \subset U(m)$. For a partition $\mathbf{a}$ of m this subgroup $U(\mathbf{a})$ is isomorphic to the product $U(m_1) \times \cdots \times U(m_k)$, where $m_1 + \cdots + m_k = m$ and the m_i's are the sizes of the parts of $\mathbf{a}$.

As any almost commuting tuple generates a subgroup of nilpotency class at most 2,

$$\bigcup_{\mathbf{a} \vdash m} B_n(U(\mathbf{a}), Z_{\mathbf{a}}) \subset \mathrm{Hom}(F_n/\Gamma_n^3, U(m)).$$

Proposition 6.1. The above inclusion is surjective upon passing to orbits, that is, if

$$\pi : \mathrm{Hom}(F_n/\Gamma_n^3, U(m)) \to \mathrm{Rep}(F_n/\Gamma_n^3, U(m))$$

denotes the quotient map, then:

$$\pi\left(\bigcup_{\mathbf{a}\vdash m} B_n(U(\mathbf{a}), Z_{\mathbf{a}})\right) = \mathrm{Rep}(F_n/\Gamma_n^3, U(m)).$$

The spaces $\mathrm{Hom}(F_n/\Gamma_n^q, G)$ assemble into a simplicial space whose geometric realization is denoted by $B(q,G)$. These spaces yield a filtration

$$B(2,G) \subseteq B(3,G) \subseteq \cdots \subseteq BG$$

originally introduced by Adem, Cohen and Torres-Giese [2]. Using our work in section 1, and [2, Theorem 4.6] applied to Q_{2^n} we see that

$$B(r, Q_{2^n}) \simeq B\left(*_{\mathbb{Z}/2^{r-1}}^{2^{n-r}} Q_{2^r} *_{\mathbb{Z}/2^{r-1}} \mathbb{Z}/2^{n-1}\right)$$

where $*_A$ denotes the amalgamated product along A. The above formula allowed us to do some computations in cohomology for $r=2$. In this case, the space $B(2,G)$ is denoted by $B_{\mathrm{com}}G$.

Proposition 8.2.

1. There is an isomorphism of graded rings $H^*(B_{\mathrm{com}}Q_8;\mathbb{F}_2) \cong \mathbb{F}_2[y_1,y_2,y_3,z]/(y_iy_j, y_1^2+y_2^2+y_3^2, i\neq j)$ where y_i has degree 1 and z degree 2.

2. Let $n\geq 4$. Then the $\mathbb{F}_2$-cohomology ring of $B_{\mathrm{com}}Q_{2^n}$ has a presentation given by

$$H^*(B_{\mathrm{com}}Q_{2^n};\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2,y_1,\ldots,y_{2^{n-2}},z]/(x_1^2, x_ky_i, y_iy_j, i\neq j, x_2+\sum_{i=1}^{2^{n-2}} y_i^2),$$

   where x_1, y_j have degree 1 and x_2, z have degree 2.

Increasing the nilpotency class makes cohomology calculations considerably harder as we illustrate for the case of $B(3,Q_{16})$ (part of our calculation uses Singular [12]).

In the last section, we describe the homotopy type of the spaces $B(q,SU(2))$ —which classify principal $SU(2)$-bundles with transitional nilpotency class less than q, as defined by Adem, Gómez, Lind and Tillman [6]. We show there is a homotopy pushout square

$$\begin{tikzcd}
PU(2)\times_{N_{q+1}} B(q,Q_{2^{q+1}}) \arrow[r] \arrow[d] & B(q,SU(2)) \arrow[d] \\
PU(2)\times_{N_{q+1}} BQ_{2^{q+1}} \arrow[r] & B(q+1,SU(2)),
\end{tikzcd}$$

where N_{q+1} is the normalizer of the dihedral group D_{2^q} in $PU(2)$.

Theorem 10.1. The inclusions in the filtration

$$B_{\mathrm{com}}SU(2) \subset B(3, SU(2)) \subset \cdots \subset B(q, SU(2)) \subset \cdots$$

are homology isomorphisms with coefficients over a ring R where 2 is invertible. Moreover, the cohomology ring of each term of the filtration is given by $R[c_2, y_1]/(y_1^2)$, where $\deg(c_2) = \deg(y_1) = 4$, and c_2 is the pullback of the second Chern class $c_2 \in H^4(BSU(2); R)$ under the inclusion $B(q, SU(2)) \to BSU(2)$.

For $q \geq 2$, the 2-fold covering map $SU(2) \to SO(3)$ induces $B\mathbb{Z}/2$-principal bundles $B(q+1, SU(2)) \to B(q, SO(3))$, and we conclude:

Corollary 10.4. Let $q \geq 2$. Then the maps $B(q, SU(2)) \to B(q, SO(3))$ induced by the double cover $SU(2) \to SO(3)$ are homology isomorphisms with coefficients over a ring R in which 2 is invertible.

Remark 1.1. It is shown in [9, Theorem 1] that given a finitely generated nilpotent group Γ, a complex reductive linear group G and $K \subset G$ a maximal compact subgroup, the inclusion induces a strong deformation retract $\mathrm{Hom}(\Gamma, G) \simeq \mathrm{Hom}(\Gamma, K)$. Applying this to $G = SL(2, \mathbb{C}), SL(3, \mathbb{R})$ or $GL(2, \mathbb{C})$ and $K = SU(2), SO(3)$ or $U(2)$ respectively, we get descriptions of $\mathrm{Hom}(\Gamma, G)$, and $B(q, G)$ for all $q \geq 2$, as well.

2. Nilpotent subgroups of $SU(2)$

In this section we record some purely algebraic facts about $SU(2)$, chief among them a description of all of its non-abelian nilpotent subgroups. Let $T \subset SU(2)$ denote the maximal torus

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} : \lambda \in \mathbb{C} \text{ and } |\lambda| = 1 \right\}$$

and let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the commutator $[x, y] = xyx^{-1}y^{-1}$. A straightforward calculation shows:

Lemma 2.1. *Let $x, y \in T$. Then $wx = \overline{x}w$ and*

1. $[x, y] = 1$;
2. $[x, wy] = x^2$;
3. $[wx, y] = \overline{y}^2$;
4. $[wx, wy] = \overline{x}^2 y^2$.

Definition 2.2. Let Q be a group, define inductively $\Gamma^1(Q) = Q$; $\Gamma^{q+1}(Q) = [\Gamma^q(Q), Q]$. This is called the descending central series of Q

$$\cdots \subset \Gamma^q(Q) \subset \cdots \subset \Gamma^2(Q) \subset \Gamma^1(Q) = Q.$$

A group Q is *nilpotent* if $\Gamma^{q+1}(Q) = 1$ for some q. The least such integer q is called the *nilpotency class* of Q.

Let $\xi_n = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix} \in T$ be an n-th root of unity and let μ_n stand for the subgroup generated by ξ_n. The general quaternions

$$Q_{2^{q+1}} := \mu_{2^q} \cup w\mu_{2^q}$$

have nilpotency class q for any $q \geq 1$. Indeed, for every $n \geq 1$ the commutators $[\xi_{2^n}, w] = \xi_{2^n}^2 = \xi_{2^{n-1}}$, then one can show inductively that for $r \geq 1$ the $(r+1)^{\mathrm{st}}$ stage in the descending central series is $\Gamma^{r+1}(Q_{2^{q+1}}) = \mu_{2^{q-r}}$, and hence the descending central series of $Q_{2^{q+1}}$

$$1 \subset \mu_2 \subset \cdots \subset \mu_{2^{q-2}} \subset \mu_{2^{q-1}} \subset Q_{2^{q+1}}$$

has q non-trivial terms.

Lemma 2.3. *Let $H \subset T \cup wT$ be a subgroup. Suppose $\Gamma^r(H) = \mu_{2^n}$ for some $n > 0$. Then if $r > 2$, $\Gamma^{r-1}(H) = \mu_{2^{n+1}}$, and for $r = 2$ there exists $t \in T$ such that $tHt^{-1} = Q_{2^{n+2}}$.*

Proof. We prove the case $r = 2$. Suppose $\Gamma^2(H) = [H, H] = \mu_{2^n}$. Since the commutator generates μ_{2^n}, there must exist $z_0, wx_0 \in H$, with $x_0 \in T$, such that $[z_0, wx_0] = \xi_{2^n}^k$ for some odd k (otherwise we would have $[H, H] \subseteq \mu_{2^{n-1}}$). If $z \in H \cap T$, then $[z, wx_0] = z^2$ is a power of ξ_{2^n}, that is, z is a power of $\pm\xi_{2^{n+1}}$ and $z \in \mu_{2^{n+1}}$. And for $wy \in H \cap wT$, where $y \in T$, the commutator $[wy, wx_0] = \overline{y}^2 x_0{}^2 = \xi_{2^n}^p$ for some $p > 0$, by Lemma 2.1, hence $y = \pm\xi_{2^{n+1}}^p x_0$. Therefore $H \subseteq \mu_{2^{n+1}} \cup w\mu_{2^{n+1}} x_0$.

To get the equality, we turn back to the element $z_0 \in H$ chosen above. There are two possibilities:

- If $z_0 \in T$, then $z_0^2 = [z_0, wx_0] = \xi_{2^n}^k$, so $z_0 = \pm\xi_{2^{n+1}}^k$.
- If $z_0 \in wT$, say $z_0 = wz_0'$, then $\overline{z_0'}^2 x_0^2 = [wz_0', wx_0] = \xi_{2^n}^k$ so that $z_0' = \pm\xi_{2^{n+1}}^k x_0$.

In either case z_0 and $wx_0 z_0$ generate $\mu_{2^{n+1}} \cup w\mu_{2^{n+1}} x_0$. Conjugating any element $w\xi_{2^{n+1}}^p x_0$ by $t = \sqrt{x_0} \in T$ we obtain

$$\sqrt{x_0} w \xi_{2^{n+1}}^p x_0 \overline{\sqrt{x_0}} = w\overline{\sqrt{x_0}} \xi_{2^{n+1}}^p x_0 \overline{\sqrt{x_0}} = w\xi_{2^{n+1}}^p.$$

This is independent of the choice of the branch cut for $\sqrt{x_0}$. Hence $\Gamma^1(tHt^{-1}) = tHt^{-1} \subseteq \mu_{2^{n+1}} \cup w\mu_{2^{n+1}}$.

For $r > 2$ it's easier: the same arguments without the anti-diagonal matrix cases prove the result, since $\Gamma^{r-1}(H) \subset T$. $\square$

Lemma 2.4. *Let $X, Y \in SU(2)$. Then*

1. *Suppose $X \neq \pm I$. If $[X, Y] = I$ and $g \in SU(2)$ diagonalizes X, then it also diagonalizes Y.*
2. *If $[X, Y] = -I$ and $g \in SU(2)$ diagonalizes X, then $gXg^{-1} = \pm\xi_4$ and $gYg^{-1} \in wT$.*

Proof.

1. When $X \neq \pm I$ its eigenvalues are distinct conjugate numbers. So gYg^{-1} commutes with a non-scalar diagonal matrix and must also be diagonal.

2. Let g be a matrix that conjugates X to a diagonal matrix with eigenvalues $\lambda, \overline{\lambda}$. Let $X' = gXg^{-1}$ and $Y' = gYg^{-1}$. Choose a non-zero vector $v \in E_\lambda$, the eigenspace of X associated to λ. Since $[X', Y'] = -I$ we get

$$X'Y'v = -Y'X'v = -\lambda Y'v.$$

   Hence $-\lambda$ is an eigenvalue of X so that $-\lambda = \overline{\lambda}$, which implies $\lambda = \pm i \in \mathbb{C}$. Therefore $X' = \pm\xi_4$. Now, $v \in E_i$ and $Y'v \in E_{-i}$ tells us that Y is conjugated by g into an anti-diagonal matrix.

□

Proposition 2.5. *Let $H \subset SU(2)$ be a nilpotent subgroup. Then either H is abelian or there exists an $r \geq 2$ and an element $g \in SU(2)$ such that*

$$gHg^{-1} = Q_{2^{r+1}}.$$

In particular, any non-abelian nilpotent subgroup of $SU(2)$ is automatically finite.

Proof. Suppose H is non-abelian. Then, there exists a unique $r > 1$ such that $\Gamma^{r+1}(H) = I$ and $\Gamma^r(H) \neq I$. This implies that $\Gamma^r(H)$ sits inside the center of H. Non-abelian subgroups of $SU(2)$ have center contained in $\{\pm I\}$ (for a proof of this statement see [19, Example 2.22, p. 16]) and therefore $\Gamma^r(H) = [H, \Gamma^{r-1}(H)] = \mu_2$. By hypothesis, there exists a non-central element X in $\Gamma^{r-1}(H)$. Then for every $h \in H$, the commutator $[X, h]$ is inside $\{\pm I\}$. Let $g \in SU(2)$ be an element that diagonalizes X. By Lemma 2.4, for every $h \in H$, $ghg^{-1} \in T \cup wT$ and thus H can be conjugated to a subgroup of $T \cup wT$. Now, $\Gamma^r(gHg^{-1}) = \mu_2$ and applying Lemma 2.3 inductively we get that there exists $t \in T$, such that $tgH(tg)^{-1} = Q_{2^{r+1}}$. □

In the next section we will need to know a little bit more about the generalized quaternion groups: their normalizers in $SU(2)$ and a list of their subgroups.

Lemma 2.6. *The normalizer of Q_{2^n} in $SU(2)$ is:*

1. *The Binary Octahedral group which is generated by* $\left\{\xi_8, w, \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}\right\}$ *for $n = 3$;*

2. *$Q_{2^{n+1}}$ for $n > 3$.*

Proof. For this it is more convenient to think of $SU(2)$ as the group of unit quaternions, which enables one to think geometrically in terms of rotations in the space of purely imaginary quaternions, a space we shall identify with $\mathbb{R}^3$. If $q = \cos(\theta) + \sin(\theta)\hat{q}$ where $\hat{q}$

is a unit length imaginary quaternion, then $v \mapsto qvq^{-1}$ maps imaginary quaternions to imaginary quaternions and is a rotation around $\hat{q}$ of angle 2θ. An arbitrary quaternion can be written as $a+v$ where a is the real part and v is purely imaginary. We have $q(a+v)q^{-1} = a + qvq^{-1}$, so that conjugation by q preserves the real part and rotates the imaginary part. This rule that associates a rotation of $\mathbb{R}^3$ to each unit quaternion is the double cover homomorphism $SU(2) \to SO(3)$.

The normalizer of Q_{2^n} consists of those unit quaternions whose corresponding rotations are the orientation preserving symmetries of the set of imaginary parts of the elements of Q_{2^n}. As quaternions, the elements of Q_{2^n} are $\cos(2\pi m/2^{n-1}) + i\sin(2\pi m/2^{n-1})$ and $j\cos(2\pi m/2^{n-1}) + k\sin(2\pi m/2^{n-1})$ for $m = 0, 1, \ldots, 2^{n-1}-1$. The imaginary parts are then 2^{n-2} points on the i-axis together with the vertices of a regular 2^{n-1}-gon in the jk-plane. For $n > 3$, any orientation preserving isometry of $\mathbb{R}^3$ preserving this set of points must separately preserve the 2^{n-1}-gon and the points on the i-axis, and thus must be a rotation of $\mathbb{R}^3$ whose restriction to the plane of the 2^{n-1}-gon is a symmetry of the 2^{n-1}-gon. Each of the 2^n elements of the dihedral group of symmetries of the 2^{n-1}-gon extends to a unique rotation of $\mathbb{R}^3$ (the rotations in the dihedral group extend to rotations around the i-axis, while the reflections extend to rotations of angle π around the axis of the reflection), and each such rotation of $\mathbb{R}^3$ has two quaternions inducing it; these 2^{n+1} quaternions are easily seen to be the elements of $Q_{2^{n+1}}$.

Now, if $n = 3$, there is extra symmetry because the 2^{n-1}-gon in the jk-plane is just a square, and the $2^{n-2} = 2$ points on the i-axis together with that square form a regular octahedron. Therefore the normalizer of Q_{2^3} is the preimage in $SU(2)$ of the group of orientation preserving symmetries of the octahedron in $SO(3)$; this preimage is known as the Binary Octahedral group. The generator $\frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$ listed in the statement of the proposition (which is the only generator not in Q_{2^4}), corresponds to a rotation of $2\pi/3$ around a line connecting the centers of two opposite faces of the octahedron. This extra symmetry does *not* separately preserve the i-axis and the square $\{\pm j, \pm k\}$. $\square$

Lemma 2.7. *Let $n \geq 2$. Then*

1. *The abelian subgroups of $Q_{2^{n+1}}$ are all subgroups of μ_{2^n} or $\{\pm I, \pm wx\}$ where x is an element of μ_{2^n}.*

2. *The non-abelian subgroups of $Q_{2^{n+1}}$ are of the form $\mu_{2^r} \cup wx\mu_{2^r}$ where $x = (\xi_{2^n})^p$, for some $2 \leq r \leq n$ and some $0 \leq p < 2^{n-r}$.*

Proof.

1. By Lemma 2.1, any subgroup containing an element of $\mu_{2^n} - \{\pm I\}$ and one of $w\mu_{2^n}$ has non-trivial commutator.

2. By the proof of Lemma 2.3 any non-abelian nilpotent subgroup of $T \cup wT$ has the form $\mu_{2^r} \cup wx\mu_{2^r}$ for some $x \in T$ and $r \geq 2$. Therefore the non-abelian subgroups of Q_{2^n} are $\mu_{2^r} \cup wx\mu_{2^r}$ where x ranges over representatives for the cosets of μ_{2^r} in μ_{2^n}.

□

3. Spaces of homomorphisms from nilpotent groups to $SU(2)$

Now that we know what all the non-abelian nilpotent subgroups of $SU(2)$ are, we can describe spaces of homomorphisms from discrete nilpotent groups into $SU(2)$.

Theorem 3.1. *Let Γ be a discrete nilpotent group, and for every $q \geq 3$ let $\mathrm{Epi}(\Gamma, Q_{2^q})$ denote the set of surjective homomorphisms $\Gamma \to Q_{2^q}$. Then there is a homeomorphism*

$$\mathrm{Hom}(\Gamma, SU(2)) \cong \mathrm{Hom}(\Gamma^{\mathrm{ab}}, SU(2)) \sqcup \bigsqcup_{q\geq 3} \bigsqcup_{\mathcal{O}} PU(2)$$

where $\mathcal{O}$ runs through the orbits $\mathrm{Epi}(\Gamma, Q_{2^q})/N_{SU(2)}(Q_{2^q})$ induced by the action of conjugation of the normalizer in $SU(2)$.

Proof. Let $\rho \in \mathrm{Hom}(\Gamma, SU(2)) - \mathrm{Hom}(\Gamma^{ab}, SU(2))$. By Proposition 2.5, $\rho(\Gamma)$ is conjugate to Q_{2^q} for some $q \geq 3$. Then, ρ factors through $\Gamma \xrightarrow{\varphi} Q_{2^q} \hookrightarrow SU(2) \xrightarrow{c_g} SU(2)$, where φ is a surjective homomorphism and c_g is conjugation by some $g \in SU(2)$. Thus, we have a surjective map

$$SU(2) \times \bigsqcup_{q\geq 3} \mathrm{Epi}(\Gamma, Q_{2^q}) \to \mathrm{Hom}(\Gamma, SU(2)) - \mathrm{Hom}(\Gamma^{ab}, SU(2))$$

given by $(g, \varphi) \mapsto g\varphi g^{-1}$. Now two different elements (g, φ) and (h, ψ) have the same image if and only if $\varphi, \psi \in \mathrm{Epi}(\Gamma, Q_{2^q})$ for the same $q \geq 3$, and $g^{-1}h\psi h^{-1}g = \varphi$; in this case, conjugation by $g^{-1}h$ takes elements in $\operatorname{im}\psi = Q_{2^q}$ to elements of $\operatorname{im}\varphi = Q_{2^q}$ and therefore $g^{-1}h \in N_{SU(2)}(Q_{2^q})$. Thus, after modding out by the action of $N_{SU(2)}(Q_{2^q})$ by conjugation over $\mathrm{Epi}(\Gamma, Q_{2^q})$, we can assume that $\varphi = \psi$, which implies $g^{-1}h \in Z(\langle Q_{2^q} \cup \{g^{-1}h\}\rangle) = \{\pm I\}$ (see the proof of Lemma 2.4). It follows that $h = -g$. Modding out by the left translation action of $\{\pm I\}$ on $SU(2)$ gives the desired homeomorphism. □

Our main family of examples will be the finitely generated free nilpotent groups. By definition, these are the groups F_n/Γ_n^k, where $\Gamma_n^k := \Gamma^k(F_n)$.

Corollary 3.2. *Let $q \geq 2$ and $n \geq 1$. Then*

$$\mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \cong \mathrm{Hom}(\mathbb{Z}^n, SU(2)) \sqcup \bigsqcup_{C(n,q+1)} PU(2)$$

where

$$C(n, q+1) = \frac{2^{n-2}(2^n - 1)(2^{n-1} - 1)}{3} + (2^n - 1)(2^{(q-2)(n-1)} - 1)2^{2n-3}.$$

Proof.

In order to have an epimorphism $F_n/\Gamma_n^{q+1} \to Q_{2^r}$ we must have $r \leq q$. And when $r \leq q$, we need only choose n-tuples in Q_{2^r} that generate the whole group. Let us identify $\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^r})$ with the subset of $(Q_{2^r})^n$ consisting of n-tuples whose entries generate

Q_{2^r}. The action of $N_{SU(2)}(Q_{2^r})$ on $\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^r})$ is free once we take quotient by the center of the group, which acts trivially. By Theorem 3.1, the number of connected components homeomorphic to $PU(2)$ (which arise from non-abelian representations) is

$$\sum_{r=3}^{q+1} \frac{|\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^r})|}{|N_{SU(2)}(Q_{2^r})|/2}.$$

By Lemma 2.6 we know the order of $N_{SU(2)}(Q_{2^r})$. It remains to count the number of elements in $\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^r})$, for which we use an inclusion–exclusion argument: a tuple generates Q_{2^r} if and only if its elements don't all come from a single maximal proper subgroup of Q_{2^r}. So if $M_1, M_2, \ldots, M_m$ are the maximal subgroups of Q_{2^r}, we have

$$|\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^r})| = |Q_{2^r}|^n - \sum_i |M_i|^n + \sum_{i,j} |M_i \cap M_j|^n - \sum_{i,j,k} |M_i \cap M_j \cap M_k|^n + \cdots$$

For Q_8, there are three maximal subgroups, all isomorphic to $\mathbb{Z}/4$. The intersection of any two of them is the center of Q_8, $\pm I$. So we get

$$|\mathrm{Epi}(F_n/\Gamma_n^q, Q_8)| = 8^n - 3 \cdot 4^n + 3 \cdot 2^n - 2^n = 2^{n+1}(2^n - 1)(2^{n-1} - 1).$$

For $r \geq 4$, it follows from Lemma 2.7 that things are pretty much the same: there are just three maximal subgroups of Q_{2^r}, namely, $\mu_{2^{r-1}}$, $\mu_{2^{r-2}} \cup w\mu_{2^{r-2}}$ and $\mu_{2^{r-2}} \cup w\xi_{2^{r-1}}\mu_{2^{r-2}}$. And the intersection of any pair of them is $\mu_{2^{r-2}}$, so we get

$$|\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^r})| = (2^r)^n - 3\cdot(2^{r-1})^n + 3\cdot(2^{r-2})^n - (2^{r-2})^n = 2^{(r-2)n+1}(2^n - 1)(2^{n-1} - 1),$$

and a quick calculation verifies the formula claimed. □

Remark 3.3. As noted in [19], the space $\mathrm{Hom}(F_n/\Gamma_n^3, SU(2))$ is the same as the space of almost commuting tuples $B_n(SU(2), \{\pm I\})$. In [4] they describe this space, and it agrees with our computation for $q = 2$.

We finish this section with a geometric application of Theorem 3.1.

Corollary 3.4. *For any $q \geq 2$, consider the 3-manifolds $M_q = SU(2)/Q_{2^q}$. Then there are $3 + 2^{q-2}$ isomorphism classes of flat principal $SL(2, \mathbb{C})$-bundles over M_q.*

Proof. For any Lie group G, the number of isomorphism classes of flat G-bundles is equal to the number of points in the space $\mathrm{Rep}(\pi_1(M_q), G) = \mathrm{Hom}(\pi_1(M_q), G)/G$ (where the orbits are taken modulo the conjugation action of G). Our results will let us easily count this number for $G = SU(2)$, and then we shall argue that the answer is the same for $G = SL(2, \mathbb{C})$.

According to Theorem 3.1, there are two types of components in $\mathrm{Hom}(\pi_1(M_q), SU(2))$: the components of $\mathrm{Hom}(\pi_1(M_q)^{\mathrm{ab}}, SU(2))$, and several components homeomorphic to $PU(2)$; each of the latter is a conjugacy class of some epimorphism $\pi_1(M_q) \to Q_{2^r}$. Since $\pi_1(M_q)^{\mathrm{ab}} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and the only element of order 2 in $SU(2)$ is $-I$ (and the same is true in $SL(2, \mathbb{C})$), $\mathrm{Hom}(\pi_1(M_q)^{\mathrm{ab}}, SU(2))$ consists of 4 points. The other components contribute a point each to $\mathrm{Rep}(\pi_1(M_q), SU(2))$, so it is a finite discrete space of cardinality given by:

$$|\mathrm{Hom}(Q_{2^q}^{\mathrm{ab}}, SU(2))| + \sum_{r=3}^{q} |\mathrm{Epi}(Q_{2^q}, Q_{2^r})/N_{SU(2)}(Q_{2^r})|.$$

Let's show that for $r \geq 4$ we have $|\mathrm{Epi}(Q_{2^q}, Q_{2^r})/N_{SU(2)}(Q_{2^r})| = 2^{2r-3}/2^r = 2^{r-3}$, and that for $r = 3$, $|\mathrm{Epi}(Q_{2^q}, Q_8)/N_{SU(2)}(Q_8)| = 1$ —which gives the stated count, but for $G = SU(2)$.

Let $r \geq 4$. A homomorphism $Q_{2^q} \to Q_{2^r}$ is determined by where it sends the generators $w \mapsto x$ and $\xi_{2^{q-1}} \mapsto y$, and we must count the number of choices for which the homomorphism is well-defined and surjective. First of all, since $x^4 = I$ we cannot have $x \in \mu_{2^{r-1}}$. Indeed, in that case we would need to have $y \in w\mu_{2^{r-1}}$ (to avoid having $\langle x, y\rangle \subseteq \mu_{2^{r-1}}$), but then $\langle x, y\rangle \cap \mu_{2^{r-1}} = \langle x, y^2\rangle \subseteq \mu_4 \subsetneq \mu_{2^{r-1}}$. So $x = wx'$ for $x' \in \mu_{2^{r-1}}$ and there are 2^{r-1} choices for x'.

We can either have $y \in \mu_{2^{r-1}}$ or $y \in w\mu_{2^{r-1}}$. In the first case, y alone is responsible for generating $\mu_{2^{r-1}}$, which leaves 2^{r-2} possibilities for y. In the second case, say $y = wy'$ with $y' \in \mu_{2^{r-1}}$, we have $\langle x, y\rangle \cap \mu_{2^{r-1}} = \langle xy\rangle = \langle -\overline{x'}y'\rangle$, which again gives 2^{r-2} possibilities for y'. (It remains to check all these possibilities are realized which is straightforward using the presentation $Q_{2^r} = \langle \xi_{2^{r-1}}, w \mid w^4 = 1, \xi_{2^{r-1}}^{2^{r-1}} = 1, w\xi_{2^{r-1}} = \xi_{2^{r-1}}^{-1}w\rangle$.) All together we have 2^{2r-3} epimorphisms. Again the normalizer acts freely after modding out the center which acts trivially, so $|\mathrm{Epi}(Q_{2^q}, Q_{2^r})/N_{SU(2)}(Q_{2^r})| = 2^{2r-3}/2^r = 2^{r-3}$ as claimed.

The case $r = 3$ is straightforward. One can verify that there are 24 pairs (x, y) that generate Q_8, all of these pairs define homomorphisms from Q_{2^q}, and $N_{SU(2)}(Q_8)/\{\pm I\}$ acts simply transitively on the 24 pairs.

Now it only remains to show that the count is the same for $G = SL(2, \mathbb{C})$. By [9, Theorem 1], the inclusion $SU(2) \to SL(2, \mathbb{C})$ induces a homotopy equivalence between $\mathrm{Hom}(\pi_1(M_q), SU(2))$ and $\mathrm{Hom}(\pi_1(M_q), SL(2, \mathbb{C}))$, so in particular these spaces have the same number of connected components. Since both $SU(2)$ and $SL(2, \mathbb{C})$ are connected, this implies that the corresponding Rep spaces also have the same number of connected components, so we need only show that for $G = SL(2, \mathbb{C})$ each component of $\mathrm{Rep}(\pi_1(M_q), G)$ is a single point, as was the case for $G = SU(2)$. Consider the map $\mathrm{Rep}(\pi_1(M_q), SU(2)) \to \mathrm{Rep}(\pi_1(M_q), SL(2, \mathbb{C}))$ induced by the inclusion. Since any homomorphism from the finite group $\pi_1(M_q)$ to $SL(2, \mathbb{C})$ can be conjugated into $SU(2)$ (by the classical averaging argument producing a $\pi_1(M_q)$-invariant Hermitian metric), this map is surjective. Thus the components of $\mathrm{Rep}(\pi_1(M_q), SL(2, \mathbb{C}))$ must also be points, as required. $\square$

4. Spaces of homomorphisms from abelian groups to $SU(2)$

For a complete description of spaces of homomorphisms from nilpotent groups to $SU(2)$, it only remains to describe the space $\mathrm{Hom}(A, SU(2))$ for a discrete abelian group A.

Proposition 4.1. *Let A be a discrete abelian group. There is a pushout square which is also a homotopy pushout square,*

$$\begin{tikzcd} SU(2)/N(T) \times \mathrm{Hom}(A, \{\pm I\}) \arrow[r] \arrow[d] & SU(2)/T \times_{\mathbb{Z}/2} \mathrm{Hom}(A, T) \arrow[d] \\ \mathrm{Hom}(A, \{\pm I\}) \arrow[r] & \mathrm{Hom}(A, SU(2)) \end{tikzcd}$$

where T is a maximal torus of $SU(2)$, $N(T)$ its normalizer; the horizontal arrows are induced by the inclusions $\{\pm I\} \hookrightarrow T$, $\{\pm I\} \hookrightarrow SU(2)$; the vertical arrows are induced by conjugation, and the action of $\mathbb{Z}/2$ on $SU(2)/T \times \mathrm{Hom}(A, T)$ is diagonal, given by the antipodal action on $SU(2)/T \cong S^2$ and by complex conjugation on $\mathrm{Hom}(A, T)$.

Proof. It is enough to show the square is a pushout since $SU(2)/T \times \mathrm{Hom}(A, T)$ has a $\mathbb{Z}/2$-CW-complex structure for which $SU(2)/T \times \mathrm{Hom}(A, \{\pm I\})$ is the sub-complex of fixed points, so the top map is a cofibration.

The right vertical map is induced by the map $G/T \times \mathrm{Hom}(A, T) \to \mathrm{Hom}(A, SU(2))$ given by $(gT, \phi) \mapsto (a \mapsto g\phi(a)g^{-1})$. This descends to the quotient by the described $\mathbb{Z}/2$-action because that action has a more conceptual description, as can easily be checked: $\mathbb{Z}/2$ arises here as the Weyl group, $N(T)/T$; and it acts on $\mathrm{Hom}(A, T)$ by conjugation on the codomain of the homomorphisms, namely $nT \cdot \phi = a \mapsto n\phi(a)n^{-1}$ for $nT \in N(T)/T$, $\phi \in \mathrm{Hom}(A, T)$, and it acts on G/T by $nT \cdot gT = gn^{-1}T$ for $nT \in N(T)/T$, $gT \in G/T$.

In $SU(2)$ every abelian subgroup is conjugate to a subgroup of T, and therefore the right vertical map is surjective. Let's figure out how non-injective it is. If (gT, ϕ) and $(g'T, \phi')$ have the same image in $\mathrm{Hom}(A, SU(2))$, then $h\phi'(a)h^{-1} = \phi(a)$ for all $a \in A$ where $h = g^{-1}g'$. This implies that $\phi(A) \subseteq T \cap hTh^{-1}$. There are two cases:

- If $T = hTh^{-1}$, then $h \in N(T)$ and $hT \cdot (g'T, \phi') = (gT, \phi)$, so the two pairs have the same image in the quotient by the Weyl group and we don't really have an instance of non-injectivity.
- If $T \neq hTh^{-1}$, then $T \cap hTh^{-1} = \{\pm I\}$, since geometrically T and hTh^{-1} are two great circles on $SU(2) \cong S^3$. So, in this case, $\phi\colon A \to \{\pm I\}$ which implies in turn that $\phi' = \phi$, but leaves g and g' unconstrained.

In summary, all the fibres of the right vertical map are singletons except for the fibres above points $\phi \in \mathrm{Hom}(A, \{\pm I\})$, which are of the form $SU(2)/N(T) \times \{\phi\}$. The pushout of the top and left maps in the square precisely collapses each of those fibres down to a single point, so the pushout is the bottom right corner as desired. $\square$

Remark 4.2. The (homotopy) pushout square for $A = \mathbb{Z}^n$ first appeared in [7] and indeed, the proof above is a straightforward adaptation of our proof there. The homotopy type of $\mathrm{Hom}(\mathbb{Z}^n, SU(2))$ after one suspension has been described independently in [4, 8] and [11] (in fact, the title of this paper was motivated by the one in [8]). To be precise, for $n \geq 2$

$$\Sigma \mathrm{Hom}(\mathbb{Z}^n, SU(2)) \simeq \Sigma \left((S^3)^{\vee n} \vee \bigvee_{k=2}^{n} \left((\mathbb{RP}^2)^{k\lambda_2}/s_k(\mathbb{RP}^2)\right)^{\vee \binom{n}{k}} \right),$$

where λ_2 is the canonical line bundle over $\mathbb{RP}^2$, X^λ is the Thom space of a vector bundle λ over X, and s_k is the zero section of $k\lambda_2$. For an arbitrary finitely generated abelian group A, the homotopy type of $\mathrm{Hom}(A, SU(2))$ after one suspension is given in more detail in [5, Example 4.3].

Example 4.3. As an example of a non-finitely generated group, take $A = \mathbb{Q}$ — with the discrete topology. Since every element of $\mathbb{Q}$ is divisible by 2, there are no non-trivial homomorphisms $\mathbb{Q} \to \{\pm I\}$, which simplifies the pushout somewhat. The space $\mathrm{Hom}(\mathbb{Q}, T)$ is the Pontryagin dual of $\mathbb{Q}$ as a discrete group, which is well-known to be $\mathbb{A}/\mathbb{Q}$. Here $\mathbb{A}$ is the group of adeles, built out of the p-adic numbers $\mathbb{Q}_p$ and p-adic integers $\mathbb{Z}_p$, as the subgroup of $\mathbb{R} \times \prod_p \mathbb{Q}_p$ consisting of sequences such that for almost all primes p, the p-th coordinate lies in $\mathbb{Z}_p$. The homomorphism $\psi_a \colon \mathbb{Q} \to T$ corresponding to $a = (a_\infty, (a_p)_p) \in \mathbb{R} \times \prod_p \mathbb{Q}_p$ is $\psi_a(r) = e^{-2\pi i a_\infty r} \prod_p e^{-2\pi i \{a_p r\}_p}$, where $\{x\}_p$ denotes the p-adic fractional part of $x \in \mathbb{Q}_p$, i.e., the rational number with p-th power denominator that one obtains as the part of the p-adic expansion of x coming after the decimal point. From that description of ψ_a, one sees that the complex conjugation action on $\mathrm{Hom}(\mathbb{Q}, T)$ corresponds to sending a to $-a$ in $\mathbb{A}/\mathbb{Q}$. Putting this all together, we get that the space $\mathrm{Hom}(\mathbb{Q}, SU(2))$ is homeomorphic to $\left(S^2 \times_{\mathbb{Z}/2} \mathbb{A}/\mathbb{Q}\right)/\mathbb{RP}^2$, where the copy of $\mathbb{RP}^2$ inside $S^2 \times_{\mathbb{Z}/2} \mathbb{A}/\mathbb{Q}$ that gets collapsed to a point is $S^2 \times_{\mathbb{Z}/2} \{0\}$.

By the same token, for $A = \mathbb{Q}^n$ we get $\mathrm{Hom}(\mathbb{Q}^n, SU(2)) \cong \left(S^2 \times_{\mathbb{Z}/2} (\mathbb{A}/\mathbb{Q})^n\right)/\mathbb{RP}^2$, where the copy of $\mathbb{RP}^2$ collapsed to a point is $S^2 \times_{\mathbb{Z}/2} \{(0, \dots, 0)\}$.

Example 4.4. As a more exotic example of Theorem 3.1, consider the Heisenberg group over the rationals:

$$H_\mathbb{Q} := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Q} \right\}.$$

There are no epimorphisms $H_\mathbb{Q} \to Q_{2^q}$ for any $q \geq 3$ at all. Indeed, any element in $H_\mathbb{Q}$ has a square root, and the same is not true of some elements in Q_{2^q}. Therefore Theorem 3.1 implies that $\mathrm{Hom}(H_\mathbb{Q}, SU(2)) \cong \mathrm{Hom}(H_\mathbb{Q}^{\mathrm{ab}}, SU(2))$. We have $H_\mathbb{Q}^{\mathrm{ab}} \cong \mathbb{Q} \oplus \mathbb{Q}$, so that $\mathrm{Hom}(H_\mathbb{Q}^{\mathrm{ab}}, SU(2)) \cong \mathrm{Hom}(\mathbb{Q} \times \mathbb{Q}, SU(2))$. From Example 4.3, we get $\mathrm{Hom}(H_\mathbb{Q}, SU(2)) \cong \left(S^2 \times_{\mathbb{Z}/2} (\mathbb{A}/\mathbb{Q})^2\right)/\mathbb{RP}^2$.

5. Spaces of homomorphisms from F_n/Γ_n^q to $SO(3)$ and $U(2)$

Corollary 3.2 can be used to describe the spaces of nilpotent n-tuples in $SO(3)$ and $U(2)$. We study first $SO(3)$, which we will identify with $SU(2)/\{\pm I\} = PU(2)$.

Let $\pi \colon SU(2) \to PU(2)$ be the quotient homomorphism, and for the induced map write $\pi_* \colon \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \to \mathrm{Hom}(F_n/\Gamma_n^{q+1}, PU(2))$. We claim that for any $q \geq 2$, the image of π_* is precisely $\mathrm{Hom}(F_n/\Gamma_n^q, PU(2))$. Clearly any commuting tuple in $SU(2)$ is mapped to a commuting tuple in $PU(2)$. Now, let H be a non-abelian subgroup of $SU(2)$ of nilpotency class q. As showed before this implies that $\Gamma^q(H) = \{\pm I\}$ and hence $\Gamma^q(\pi(H)) = I$. Therefore $\pi(H)$ has nilpotency class $q-1$. It remains to show that any nilpotent subgroup in $PU(2)$ of nilpotency class $(q-1)$ has a lift to a nilpotent subgroup of $SU(2)$ of nilpotency class q. This follows from the fact that π is an epimorphism and the preimage of any trivial commutator is $\{\pm I\}$, and thus central in $SU(2)$. Therefore, for any $q \geq 2$

$$\pi_*\colon \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \to \mathrm{Hom}(F_n/\Gamma_n^q, PU(2))$$

is surjective as claimed.

A result of W. M. Goldman [13, Lemma 2.2] implies that the restriction $\pi_*^{-1}(C) \to C$ to any connected component C of $\mathrm{Hom}(F_n/\Gamma_n^q, PU(2))$, is a 2^n-fold covering map (the fiber is in bijection with $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, \ker\pi) = \{\pm I\}^n$). From Theorem 3.2 we know the connected components of $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2))$: one is $\mathrm{Hom}(\mathbb{Z}^n, SU(2))$ and the others, which we call K_i, are all homeomorphic to $PU(2)$. We will describe the correspondence between components K_i and the connected components $\pi_*(K_i)$ in $\mathrm{Hom}(F_n/\Gamma_n^q, PU(2))$. Each of the components K_i consists of representations (by elements of $PU(2)$) conjugate to a fixed surjective homomorphism $\rho\colon F_n/\Gamma_n^{q+1} \to Q_{2^{r+1}}$ with $2 \leq r \leq q$. The image under π_* of these components are the conjugated homomorphisms (also by elements of $PU(2)$) of the surjective homomorphism $\pi_*(\rho)\colon F_n/\Gamma_n^q \to D_{2^r}$. Since π_* is open, $\pi_*|\colon K_i \to \pi_*(K_i)$ is a covering map. Fix a homomorphism $\sigma\colon F_n/\Gamma_n^q \to D_{2^r}$ in $\pi_*(K_i)$. Then the lifts of σ in K_i are among the homomorphisms $g\rho g^{-1}\colon F_n/\Gamma_n^{q+1} \to gQ_{2^{r+1}}g^{-1}$ for $g \in PU(2)$. To have $\pi \circ g\rho g^{-1} = \sigma$, g must be in $Z(D_{2^r})$. We have two different cases. First, if $r \geq 3$, then $Z(D_{2^r}) = \langle \pi(\xi_4)\rangle$ and $\pi_*(K_i) \cong PU(2)/\pi(\mu_4)$, where μ_4 is the cyclic group generated by ξ_4 as before. Then the restriction $\pi_*|_{K_i}\colon K_i \to \pi_*(K_i)$ is a 2-fold covering map. When $r = 2$, $Z(D_4) = D_4$, and $\pi|_{K_i}\colon K_i \to \pi_*(K_i) \cong PU(2)/\pi(Q_8)$ is a 4-fold covering map.

With these covering spaces, we can easily count the connected components of the space $\mathrm{Hom}(F_n/\Gamma_n^q, SO(3))$. Indeed, let $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}}$ denote the connected component containing $(I, ..., I)$. The connected component corresponding to $\mathrm{Hom}(\mathbb{Z}^n, SU(2))$ is mapped under π_* to $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}}$. For components of commuting tuples, other than the component $\mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}}$, the correspondence is $\frac{2^n}{4}$ to 1, and for components consisting of n-tuples that generate a subgroup of nilpotency class at most r, with $r \geq 2$, is $\frac{2^n}{2}$ to 1. Thus, we only need to divide the numbers $C(n,2)$ and $C(n,r) - C(n,2)$ of Corollary 3.2 by 2^{n-2} and 2^{n-1} respectively.

Corollary 5.1. *Let $q \geq 2$, $n \geq 2$. Then*

$$\mathrm{Hom}(F_n/\Gamma_n^q, SO(3)) \cong \mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}} \sqcup \bigsqcup_{M(n)} SU(2)/Q_8 \sqcup \bigsqcup_{M(n,q)} SU(2)/\mu_4,$$

where $M(n) = \frac{(2^n-1)(2^{n-1}-1)}{3}$ and $M(n,q) = (2^n-1)(2^{(q-2)(n-1)}-1)2^{n-2}$.

Remark 5.2. The case $q = 2$ which corresponds to commuting elements was originally computed in [18], and the number $M(n)$ agrees with their calculation.

Now we discuss the situation for $U(2)$. Any matrix in $X \in U(2)$ can be written as $\sqrt{\det(X)}X'$, where $X' \in SU(2)$. In this decomposition, $[X,Y] = [X',Y']$ for any X,Y in $U(2)$. Consider the map

$$(S^1)^n \times \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \to \mathrm{Hom}(F_n/\Gamma_n^{q+1}, U(2))$$

given by $(\lambda_1,\ldots,\lambda_n,x_1,\ldots,x_n) \mapsto (\lambda_1 x_1,\ldots,\lambda_n x_n)$. By the previous observation this a surjective map. This representation for an n-tuple is uniquely determined up to signs, that is, we have a homeomorphism

$$(S^1)^n \times_{(\mathbb{Z}/2)^n} \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \cong \mathrm{Hom}(F_n/\Gamma_n^{q+1}, U(2)).$$

Using Theorem 3.2 we see that the connected components of this space are homeomorphic to the space $(S^1)^n \times_{(\mathbb{Z}/2)^n} \mathrm{Hom}(\mathbb{Z}^n, SU(2)) \cong \mathrm{Hom}(\mathbb{Z}^n, U(2))$ or $(S^1)^n \times_H PU(2)$ where H is a subgroup of $(\mathbb{Z}/2)^n$. The first component corresponds to the commuting n-tuples in $U(2)$ and the latter to the n-tuples that generate a non-abelian subgroup of nilpotency class at most q. We can see the $(\mathbb{Z}/2)^n$ action as an action on the indexing set of the connected components, which can be represented by elements of $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, Q_{2^{q+1}})$. To count the number of connected components, let $\vec{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_n)$ with each $\varepsilon_i = \pm 1$ be an arbitrary element in $(\mathbb{Z}/2)^n$ and $\vec{x} = (x_1,\ldots,x_n)$ a non-commutative n-tuple in $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, Q_{2^{q+1}})$. Then the stabilizer of this element consists of

$$\mathrm{Stab}(\vec{x}) = \{\vec{\varepsilon} \mid (\varepsilon_1 x_1,\ldots,\varepsilon_n x_n) = (gx_1g^{-1},\ldots,gx_ng^{-1}) \text{ for some } g \in SU(2)\}.$$

That is, such g either commutes or anti-commutes with all x_i. We have several cases. Let $\vec{\varepsilon} \in \mathrm{Stab}(\vec{x})$ be a non trivial element.

Case 1: Suppose some x_i lies in $T - \{\pm I\}$.

- If $\varepsilon_i = 1$, $x_i = gx_ig^{-1}$ and thus g must also lie in T. To generate a non-abelian nilpotent subgroup of $Q_{2^{q+1}}$ with x_i, we need at least one more element of the form $\xi_{2^q}^k w$ for some k. This element does not commute with any element of $T - \{\pm I\}$ and only anti-commutes with $\pm\xi_4$. Thus, the only choices for g are $\pm I$ and $\pm\xi_4$.
- If $\varepsilon_i = -1$, $-x_i = gx_ig^{-1}$ and by Lemma 2.4, $x_i = \pm\xi_4$ and $g \in Tw$. Again, in the n-tuple there must be an element of the form $x_j = \xi_{2^q}^k w$ for some k. The only elements in Tw that commute or anti-commute with x_j are $g = \pm\xi_{2^q}^k w$ or $g = \pm\xi_4\xi_{2^q}^k w$. Note that in this case, the remaining elements in the n-tuple can only be of the form $\pm I, \pm\xi_4, \pm\xi_{2^q}^k w$ or $\pm\xi_4\xi_{2^q}^k w$, which generates a copy of Q_8.

Case 2: Suppose all x_i lie in $Tw \cup \{\pm I\}$.

Not all x_i can be $\pm I$, since $\vec{x}$ is a non-commutative tuple, so some x_i is of the form $x_i = \xi_{2^q}^k w$.

- Suppose $\vec{x}$ generates a nilpotent group of nilpotency class $r \geq 3$. Then there is at least one $x_j = \xi_{2^q}^l w$ that is different from $\pm x_i$ or $\pm\xi_4 x_i$. Thus the only choices for g are $\pm I$ and $\pm\xi_4$.
- Suppose $\vec{x}$ generates a nilpotent group of nilpotency class 2. Then the only other choices for the remaining x_j's are $\pm\xi_{2^q}^k w$ or $\pm\xi_4\xi_{2^q}^k w$. Hence g can only be $\pm I, \pm\xi_4, \pm\xi_{2^q}^k w$

or $\pm\xi_4\xi_{2^q}^k w$.

We can conclude that if $\vec{x}$ generates a copy of Q_8, then $|\mathrm{Stab}(\vec{x})| = 4$ and $|\mathrm{Stab}(\vec{x})| = 2$ in any other case.

Corollary 5.3. *Let $q \geq 2$ and $n \geq 1$, then*

$$\mathrm{Hom}(F_n/\Gamma_n^{q+1}, U(2)) \cong \mathrm{Hom}(\mathbb{Z}^n, U(2)) \sqcup \bigsqcup_{M(n)} (S^1)^n \times_{(\mathbb{Z}/2)^2} PU(2) \sqcup \bigsqcup_{M(n,q)} (S^1)^n \times_{(\mathbb{Z}/2)} PU(2),$$

where $M(n)$ and $M(n,q)$ are as in Corollary 5.1.

6. Nil-2 tuples in $U(m)$

What about $m > 2$, what are the connected components of $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, U(m))$ then? We can not give an answer to this in its full generality, but we can at least say something about the case $q = 2$.

Our description of $\mathrm{Hom}(F_n/\Gamma_n^q, U(2))$ relies on the fact that we can classify non-abelian nilpotent subgroups in $SU(2)$ and then lift them to $U(2)$. For $m > 2$, such a classification for $SU(m)$ seems hard to obtain, even in the simplest case for subgroups of nilpotency class at most 2. For example, the first step in the classification for $SU(2)$ uses the fact that the center of any non-abelian nilpotent subgroup is actually contained in the center $\{\pm I\}$ of $SU(2)$; one can then describe those pairs whose commutator is $-I$, and this can be used to show we can conjugate any tuple of the subgroup to $N(T)$. The fact about the center of nilpotent subgroups already fails for $SU(m)$.

In the $q = 2$ case, although we have no control over tuples $(x_1, ..., x_n)$ in $U(m)$ whose commutators are central in $\langle x_1, ..., x_n\rangle$, we can still simultaneously diagonalize the commutators. Then our strategy will be to place the x_i's in smaller block matrix subgroups of $U(m)$ up to conjugation.

Let $\mathbf{a}$ be a partition of $\{1, 2, \ldots, m\}$ into disjoint non-empty subsets. Define $U(\mathbf{a})$ as the subgroup of $U(m)$ consisting of $m \times m$ "block diagonal matrices with blocks indexed by $\mathbf{a}$", by which we mean matrices $A \in U(m)$ whose (i, j)-th entry is 0 whenever i and j are in different parts of the partition $\mathbf{a}$. To explain our terminology, notice that when each part of $\mathbf{a}$ consists of consecutive numbers, say, if the parts are $\{1, \ldots, m_1\}$, $\{m_1+1, \ldots, m_1+m_2\}$, $\{m_1+m_2+1, \ldots, m_1+m_2+m_3\}, \ldots$, then A is what is traditionally called a block diagonal matrix:

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix}; \qquad A_i \in U(m_i)$$

The conjugacy class of the subgroup $U(\mathbf{a})$ depends only on the sizes of the parts of $\mathbf{a}$. To be specific, if π is any permutation of $\{1, \ldots, m\}$ such that the image of each part of $\mathbf{a}$ consists of consecutive numbers, then $U(\mathbf{a})$ is conjugate, via the permutation matrix associated to π, to the subgroup of traditional block diagonal matrices as above (where the

m_i are the sizes of the parts of $\mathbf{a}$). In particular, the subgroup $U(\mathbf{a})$ is always isomorphic to $\prod_{i=1}^k U(m_i)$ where the m_i are the sizes of the parts of $\mathbf{a}$ but the isomorphism is far from unique.

Let $Z_{\mathbf{a}}$ denote the center of $U(\mathbf{a})$ which consists of "block scalar matrices": diagonal matrices $\operatorname{diag}(\lambda_1, \dots, \lambda_m) \in U(m)$ such that $\lambda_i = \lambda_j$ whenever i and j are in the same part of $\mathbf{a}$. For example, if the parts consist of consecutive numbers, the elements of $Z_{\mathbf{a}}$ are of the form:

$$\begin{pmatrix} \lambda_1 I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_{m_k} \end{pmatrix}.$$

Given any diagonal matrix $D = \operatorname{diag}(\lambda_1, \dots, \lambda_m) \in U(m)$ there is a coarsest partition $\mathbf{a}(D)$ such that $D \in Z_{\mathbf{a}(D)}$, namely, the partition where i and j are in the same part *if and only if* $\lambda_i = \lambda_j$. One can easily check that the centralizer of D is precisely $U(\mathbf{a}(D))$.

Our goal is to interpret 2-nilpotent tuples of $U(m)$ as *almost commuting* elements of the subgroups $U(\mathbf{a})$. What we mean by "almost commuting" is as follows: for any topological group G and any closed subgroup $K \subset G$ contained in the center of G, the space of K-almost commuting n-tuples is defined to be

$$B_n(G, K) := \{(x_1, \dots, x_n) \in G^n \mid [x_i, x_j] \in K \quad (1 \le i, j \le n)\}.$$

Thus, we have an inclusion $B_n(G, Z(G)) \subset \operatorname{Hom}(F_n/\Gamma_n^3, G)$.

In particular, for $U(m)$,

$$\bigcup_{\mathbf{a} \vdash m} B_n(U(\mathbf{a}), Z_{\mathbf{a}}) \subset \operatorname{Hom}(F_n/\Gamma_n^3, U(m)),$$

where we've borrowed the notation $\mathbf{a} \vdash m$ typically used for partitions of the *number* m to indicate that the union is over all partitions of the *set* $\{1, \dots, m\}$ where each part consists of consecutive numbers and the parts are ordered by size.

Let $\operatorname{Rep}(\Gamma, G)$ denote the orbit space of the action of G on $\operatorname{Hom}(\Gamma, G)$ by conjugation.

Proposition 6.1. *The above inclusion is surjective upon passing to orbits, that is, if*

$$\pi : \operatorname{Hom}(F_n/\Gamma_n^3, U(m)) \to \operatorname{Rep}(F_n/\Gamma_n^3, U(m))$$

denotes the quotient map, then:

$$\pi\left(\bigcup_{\mathbf{a} \vdash m} B_n(U(\mathbf{a}), Z_{\mathbf{a}})\right) = \operatorname{Rep}(F_n/\Gamma_n^3, U(m)).$$

Proof. Let $(x_1, \dots, x_n)$ be an element of $\operatorname{Hom}(F_n/\Gamma_n^3, U(m))$. Then every commutator $[x_i, x_j]$ is central in the group generated by $\{x_1, \dots, x_n\}$. In particular, all commutators commute with each other. Since each x_i is in $U(m)$ and hence diagonalizable, we can simultaneously diagonalize all commutators by an element $g \in U(m)$. Let $y_i := g x_i g^{-1}$ and $y_{ij} := g[x_i, x_j]g^{-1}$. Now, each y_{ij} lies in the center $Z_{\mathbf{a}_{ij}}$ for some coarsest partition

$\mathbf{a}_{ij}$. Choose $\mathbf{a}$ as the infimum of all the $\mathbf{a}_{ij}$, that is, as the coarsest partition refining all $\mathbf{a}_{ij}$. We have by construction $y_{ij} \in Z_{\mathbf{a}_{ij}} \subseteq Z_{\mathbf{a}}$; and for each k, we have that y_k is in the centralizer of each y_{ij}, so $y_k \in \bigcap_{ij} U(\mathbf{a}_{ij}) = U(\mathbf{a})$.

The last remaining detail is that this partition $\mathbf{a}$ may not have parts that consist of consecutive numbers, or those parts may not be ordered by size, but, as explained above, a further conjugation fixes that. □

Remark 6.2. The same argument works for $SU(m)$ and its subgroups $SU(\mathbf{a})$. There are only two minor differences: the first is that $SU(\mathbf{a})$ is identified with the subgroup of matrices in $\prod_{i=1}^{k} U(m_i)$ of determinant 1; the second is that for the last bit of the proof, the "consecutivization", one needs to observe that for every partition one can always find an *even* permutation such that the image of each part consists of consecutive numbers and permutation matrices for even permutations lie in $SU(m)$.

Note that we already know the smaller pieces in Proposition 6.1, as $B_n(U(1), 1) = U(1)^n$ and $B_n(U(2), S^1) = \mathrm{Hom}(F_n/\Gamma_n^3, U(2))$ was described in Section 5. We do an analysis for 2-step nilpotent tuples in $U(3)$ in the following example.

Example 6.3. As an application of Proposition 6.1, we calculate the number of connected components of $\mathrm{Rep}(F_n/\Gamma_n^3, U(3))$. The union is indexed by the partitions 3, $1+2$ and $1+1+1$, where we are using partitions of the number 3 as shorthand for partitions of the set $\{1, 2, 3\}$ whose parts are of the given sizes and consist of consecutive numbers. So $\mathrm{Rep}(F_n/\Gamma_n^3, U(3))$ is the union of the images under π of $B_n(U(3), S^1)$, $B_n(U(1) \times U(2), \{1\} \times S^1) \cong U(1)^n \times B_n(U(2), S^1)$, and $(U(1)^3)^n$. The intersection of those first two B_n's is clearly the last one, but not only that: we claim the same is true for their images under π. This is clear once we characterize the n-tuples $x = (x_1, \ldots, x_n) \in \mathrm{Hom}(F_n/\Gamma_n^3, U(3))$ such that $\pi(x) \in \pi(B_n(U(\mathbf{a}), Z_{\mathbf{a}}))$ by means of a conjugation-invariant property shared by the commutators $[x_i, x_j]$ $(1 \leq i, j \leq n)$, in each of those three cases:

- For $\mathbf{a} = 3$, all commutators are scalar matrices.
- For $\mathbf{a} = 1 + 2$, all commutators share an eigenvector with eigenvalue 1.
- For $\mathbf{a} = 1 + 1 + 1$, all commutators are the identity.

We conclude that:

$$\mathrm{Rep}(F_n/\Gamma_n^3, U(3)) = B_n(U(3), S^1)/U(3) \cup_{(U(1)^3)^n/U(3)} (U(1)^n \times B_n(U(2), S^1))/U(3).$$

With this one can count the number of connected components. First, recall that the quotient map $U(3) \to U(3)/S^1 = PU(3)$ induces a principal bundle,

$$(S^1)^n \to B_n(U(3), S^1) \to \mathrm{Hom}(\mathbb{Z}^n, PU(3)),$$

so $B_n(U(3), S^1)$ has the same number of connected components as $\mathrm{Hom}(\mathbb{Z}^n, PU(3))$, which was calculated in [3, Theorem 1]. Passing to orbits induced by conjugation does not change the number of components, because $U(3)$ is path connected; so, because $B_n(U(2), S^1) = \mathrm{Hom}(F_n/\Gamma_n^3, U(2))$, the total number of connected components is

$$\frac{(3^n-1)(3^{n-1}-1)}{8} + \frac{(2^n-1)(2^{n-1}-1)}{3} + 1.$$

Remark 6.4. The spaces $\mathrm{Hom}(\Gamma, U(m))$ where Γ is a central extension of the form $1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^n \to 1$ were studied in [1]. The case where $r = 1$, has a very detailed description, and they give a formula for the number of connected components for the example case $\mathrm{Hom}(H_{\mathbb{Z}}, U(m))$, where $H_{\mathbb{Z}}$ is the Heisenberg group (over the integers). Taking $n = 2$ in our previous example, says that the space of homomorphisms from the Heisenberg group $H_{\mathbb{Z}} \cong F_2/\Gamma_2^3$ to $U(3)$ has 4 connected components, which agrees with their formula when $m = 3$.

7. Homotopy type of $B(r, Q_{2^n})$

Now we turn our attention to the classifying spaces of principal G-bundles of transitional nilpotency class less than q, for a topological group G. As described in [2], the spaces $\mathrm{Hom}(F_n/\Gamma_n^q, G) \subset G^n$ give rise to a simplicial subspace of the nerve of G. The geometric realizations $B(q, G) := |\mathrm{Hom}(F_*/\Gamma_*^q, G)|$ fit into a natural filtration of BG

$$B_{\mathrm{com}}G := B(2, G) \subset B(3, G) \subset \cdots \subset B(q, G) \subset \cdots \subset BG.$$

So far we have completely described the subgroups of $Q_{2^{q+1}}$. This will allow us to compute the homotopy type of $B(r, Q_{2^{q+1}})$ as follows. Let G be a finite group. Consider the poset $\mathcal{N}_r(G)$ of all subgroups of G of nilpotency class less than r, ordered by inclusion, and the subposet $\mathcal{P}_r(G) = \{M_\alpha\} \cup \{M_\alpha \cap M_\beta\}$ where the M_α are the maximal subgroups in $\mathcal{N}_r(G)$. It was proved in [2, Theorem 4.6] that when $\mathcal{P}_r(G)$ is a tree, there is a homotopy equivalence

$$B(r, G) \simeq B\left(\operatorname*{colim}_{A \in \mathcal{N}_r(G)} A\right).$$

Let $q \geq 2$. By Lemma 2.7 we conclude that for $2 \leq r \leq q$ the maximal subgroups of $Q_{2^{q+1}}$ of nilpotency class less than r are μ_{2^q} and the 2^{q+1-r} subgroups isomorphic to Q_{2^r}; any two of them have the same intersection, namely, $\mu_{2^{r-1}}$. Therefore we have:

$$B(r, Q_{2^{q+1}}) \simeq B\left(\ast_{\mu_{2^{r-1}}}^{2^{q+1-r}} Q_{2^r} \ast_{\mu_{2^{r-1}}} \mu_{2^q}\right)$$

where $\ast$ denotes the amalgamated product of groups.

8. Cohomology of $B_{\mathrm{com}}Q_{2^n}$.

Taking $r = 2$ we see that $B_{\mathrm{com}}Q_{2^n} \simeq B(\mathbb{Z}/2^{n-1} \ast_{\mathbb{Z}/2} (\ast_{\mathbb{Z}/2}^{2^{n-2}} \mathbb{Z}/4))$ for any $n \geq 3$. Applying the associated Mayer-Vietoris sequence inductively we obtain

$$H^i(B_{\mathrm{com}}Q_{2^n}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2^{n-1} \oplus (\mathbb{Z}/2)^{2^{n-2}} & i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 8.1. The integral homology groups $H_i(B_{\mathrm{com}}Q_8;\mathbb{Z})$ were originally computed in [2, Example 8.12], where the authors were studying the homology of the spaces $B_{\mathrm{com}}A$, where A is a *transitively commutative group* (that is, a group where all centralizers of non-identity elements are abelian; also know as a CA-group).

Proposition 8.2.

1. *There is an isomorphism of graded rings $H^*(B_{\mathrm{com}}Q_8;\mathbb{F}_2) \cong \mathbb{F}_2[y_1,y_2,y_3,z]/(y_iy_j, y_1^2+y_2^2+y_3^2, i\neq j)$ where y_i has degree 1 and z degree 2.*

2. *Let $n \geq 4$. Then the $\mathbb{F}_2$-cohomology ring of $B_{\mathrm{com}}Q_{2^n}$ has a presentation given by*

$$H^*(B_{\mathrm{com}}Q_{2^n};\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2,y_1,\dots,y_{2^{n-2}},z]/(x_1^2, x_ky_i, y_iy_j, i\neq j, x_2+\sum_{i=1}^{2^{n-2}} y_i^2),$$

where x_1, y_j have degree 1 and x_2, z have degree 2.

Proof. We work out the case $n \geq 4$. Let $\Gamma = \mathbb{Z}/2^{n-1} *_{\mathbb{Z}/2} (*_{\mathbb{Z}/2}^{2^{n-2}}\mathbb{Z}/4)$. We use the central extension

$$\mathbb{Z}/2 \lhd \Gamma \to \mathbb{Z}/2^{n-2} * *^{2^{n-2}}\mathbb{Z}/2.$$

Recall that for $j>1$, $H^*(B\mathbb{Z}/2^j;\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2]/x_1^2$ where $\deg(x_1)=1$ and $\deg(x_2)=2$. Thus, $H^*(B(\mathbb{Z}/2^{n-2} * *^{2^{n-2}}\mathbb{Z}/2);\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2,y_1,\dots,y_{2^{n-2}}]/(x_1^2, x_ky_i, y_iy_j, i\neq j)$. The k-invariant of the associated Serre spectral sequence of this extension is $x_2 + \sum_{i=1}^{2^{n-2}} y_i^2$. Therefore the $E_3^{*,*}$ page is

$$\mathbb{F}_2[z]\otimes\mathbb{F}_2[x_1,x_2,y_1,\dots,y_{2^{n-2}}]/(x_1^2, x_ky_i, y_iy_j, i\neq j, x_2+\sum_{i=1}^{2^{n-2}} y_i^2).$$

The Steenrod square $\mathrm{Sq}^1(x_2+\sum_{i=1}^{2^{n-2}} y_i^2) = \mathrm{Sq}^1(x_2)$ is 0 since it can be expressed only in terms of x_2. That is, $d_3=0$ and thus the spectral sequence abuts to $E_3^{*,*}$.

So we have found the E_∞-page along with its ring structure. Since all the relations involve only generators from the base of the fibration, these relations hold in the cohomology ring as well. □

Remark 8.3. Recall that $H^*(BQ_8;\mathbb{F}_2) = \mathbb{F}_2[x,y,t]/(x^2+xy+y^2, x^2y+xy^2)$ where x, y have degree 1 and t has degree 4. Consider the inclusion $\iota\colon B_{\mathrm{com}}Q_8 \to BQ_8$ and $\iota^*\colon H^*(BQ_8;\mathbb{F}_2) \to H^*(B_{\mathrm{com}}Q_8;\mathbb{F}_2)$. By Proposition 8.2, $\iota^*(x^2y)=0$ since $\iota^*(x)$ and $\iota^*(y)$ are linear combinations of the y_j and all monomials of degree 3 in the y_j are 0. Thus ι^* is not injective (this was also pointed out in [14, Proposition 6.8]). It is unclear for which groups G and with which coefficients $\iota^* : H^*(B_{\mathrm{com}}G) \to H^*(BG)$ is injective. In [7] it is shown that ι^* is injective both with $\mathbb{F}_2$ and with integral coefficients for the groups $SU(2), U(2), O(2), SO(3)$.

9. Cohomology of $B(3, Q_{16})$.

By the above discussion, $B(3, Q_{16})$ is the classifying space of the group $\Gamma := Q_8 *_{\mu_4} Q_8 *_{\mu_4} \mu_8$. To compute it's $\mathbb{F}_2$ cohomology, we'll use the Lyndon-Hochschild-Serre spectral sequence for the central extension $1 \to \mu_2 \to \Gamma \to \Gamma/\mu_2 \to 1$, so we first need the cohomology of the group Γ/μ_2. We have:

$$\Gamma/\mu_2 = (Q_8/\mu_2) *_{\mu_4/\mu_2} (Q_8/\mu_2) *_{\mu_4/\mu_2} (\mu_8/\mu_2) \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) *_{\mathbb{Z}/2} (\mathbb{Z}/2 \times \mathbb{Z}/2) *_{\mathbb{Z}/2} \mathbb{Z}/4.$$

We are lucky that the first amalgamated product simplifies:

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) *_{\mathbb{Z}/2} (\mathbb{Z}/2 \times \mathbb{Z}/2) \cong (\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2.$$

This simplification is easiest to see if we use the shear automorphism of the $\mathbb{Z}/2 \times \mathbb{Z}/2$'s to replace the diagonal embeddings of $\mathbb{Z}/2$ with the inclusions into, say, the first factor. Under this isomorphism, the inclusion $\mathbb{Z}/2 \to (\mathbb{Z}/2 \times \mathbb{Z}/2) *_{\mathbb{Z}/2} (\mathbb{Z}/2 \times \mathbb{Z}/2)$ corresponds to the inclusion $\mathbb{Z}/2 \to (\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2$ into the second factor.

To compute the cohomology of Γ/μ_2, we can now use the Mayer-Vietoris sequence for the homotopy pushout square

$$\begin{tikzcd}
B\mathbb{Z}/2 \arrow[r] \arrow[d] & B\left((\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2\right) \arrow[d] \\
B\mathbb{Z}/4 \arrow[r] & B(\Gamma/\mu_2).
\end{tikzcd}$$

The top map is surjective on cohomology which implies all the connecting homomorphisms are zero; this in turn means that $H^*(B(\Gamma/\mu_2)) \to H^*(B\left((\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2\right) \vee B\mathbb{Z}/4)$ is an injective ring homomorphism. The groups in the pushout (and thus their wedge) have the following cohomology rings, where t_2 has degree 2 and all other variables have degree 1:

$$\begin{aligned}
H^*(B\left((\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2\right)) &= \mathbb{F}_2[y_1, y_2, t_1]/(y_1 y_2) \\
H^*(B\mathbb{Z}/4) &= \mathbb{F}_2[y_3, t_2]/(y_3^2) \\
H^*(B\left((\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2\right) \vee B\mathbb{Z}/4) &= \mathbb{F}_2[y_1, y_2, t_1, y_3, t_2]/(y_1 y_2, y_3^2, uv),
\end{aligned}$$

where u ranges over $\{y_1, y_2, t_1\}$, and v over $\{y_3, t_2\}$.

If t denotes the cohomology generator of the $B\mathbb{Z}/2$ in the top left corner, then the top horizontal map on cohomology is given by $y_1, y_2 \mapsto 0$, $t_1 \mapsto t$, and the left vertical map by $y_3 \mapsto 0$, $t_2 \to t^2$. Therefore, in the Mayer-Vietoris sequence, the kernel of the homomorphism

$$H^*(B\left((\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2\right)) \oplus H^*(B\mathbb{Z}/4) \to H^*(B\mathbb{Z}/2)$$

is generated by the linear subspace spanned by $\{(t_1^{2k}, t_2^k) : k \geq 0\}$ together with $K_1 \times K_2$, where $K_1 = (y_1, y_2)$ is the kernel of the ring homomorphism $H^*(B\left((\mathbb{Z}/2 * \mathbb{Z}/2) \times \mathbb{Z}/2\right)) \to H^*(B\mathbb{Z}/2)$ and $K_2 = (y_3)$ is the kernel of the ring homomorphism $H^*(B\mathbb{Z}/4) \to H^*(B\mathbb{Z}/2)$.

From this one can verify that $H^*(B\Gamma/\mu_2)$ is the subring of $\mathbb{F}_2[y_1, y_2, t_1, y_3, t_2]/(y_1y_2, y_3^2, uv)$ generated by $y_1, y_2, y_3, \beta_1 := y_1t_1, \beta_2 := y_2t_1, z := t_1^2 + t_2$. This subring has the following presentation:

$$H^*(B(\Gamma/\mu_2)) = \mathbb{F}_2[y_1, y_2, y_3, \beta_1, \beta_2, z]/(y_iy_j, y_3^2, y_i\beta_j, \beta_1\beta_2, \beta_j^2 - y_j^2 z),$$

where i and j range over all indices such that $i \in \{1, 2, 3\}$, $j \in \{1, 2\}$ and $i \neq j$. Here y_i has degree 1, and z and the β_j have degree 2.

Now we can go back to the group Γ. The E_2-page of the Lyndon-Hochschild-Serre spectral sequence for the central extension $1 \to \mu_2 \to \Gamma \to \Gamma/\mu_2 \to 1$ is $\mathbb{F}_2[u] \otimes_{\mathbb{F}_2} H^*(B\Gamma/\mu_2)$ and $d_2(u)$ is given by the k-invariant of the extension, which we must figure out now. This k-invariant must restrict to the correct k-invariant for the Q_8 summands and for the μ_8 summand. The k-invariant for the extension $1 \to \mu_2 \to Q_8 \to \mathbb{Z}/2\times\mathbb{Z}/2 \to 1$ is given by $x^2 + xy + y^2$ where x and y are the generators of the cohomology of $\mathbb{Z}/2 \times \mathbb{Z}/2$. And the k-invariant for the extension $1 \to \mu_8 \to \mathbb{Z}/4$ is t_2, again using the presentation $H^*(B\mathbb{Z}/4) = \mathbb{F}_2[y_3, t_2]/(y_3^2)$. To have those restrictions to the extension for the 3 summands, we must have $d_2(u) = z + \beta_1 + y_1^2 + \beta_2 + y_2^2$.

We are in the situation where a page of the spectral sequence is polynomials in one variable, namely u, over the base ring A (here, $A := H^*(B\Gamma/\mu_2)$), in this situation the entire differential is determined by $d_2(u)$ and the next page is easily written in terms of the ideal generated by $d_2(u)$ and its annihilator. Indeed, given an element $au^q \in E_2^{p,q}$, we have $d_2(au^q) = qad_2(u)u^{q-1}$, from which we see that (1) the element is in the kernel of d_2 if either q is even or q is odd and $ad_2(u) = 0$, and (2) the image of d_2 is spanned by monomials bu^j where b is a multiple of $d_2(u)$ and j is even. Therefore, $E_3^{*,*} = A/(d_2(u))[u^2] \oplus u\mathrm{ann}_A(d_2(u))[u^2]$, where the multiplication is given by the canonical $A/(d_2(u))$-module structure on $\mathrm{ann}_A(d_2(u))$.

Now we claim that $d_2(u)$ is not a zero divisor in the ring A, so that $\mathrm{ann}_A(d_2(u)) = 0$. This should be plausible because of the "leading" term z, and we verified this claim using Singular in the appendix. That tells us that the E_3-page is also polynomials in u^2 over the new base, $A/(d_2(u))$ and to continue we need to compute $d_3(u^2)$. By the Kudo transgression theorem, $d_3(u^2)$ is the image of $\mathrm{Sq}^1(d_2(u))$ from the E_2 page. We can compute this Steenrod square by restricting to the Q_8 and μ_8 summands. For the Q_8's, using the notation above, $\mathrm{Sq}^1(x^2 + xy + y^2) = x^2y + xy^2$ which is the image of $y_jz + y_j\beta_j$. For the μ_8 we have $\mathrm{Sq}^1(t_2) = 0$. Together these facts tells us $\mathrm{Sq}^1(d_2(u))$ must be $y_1z + y_1\beta_1 + y_2z + y_2\beta_2$. This *is* a zero divisor, even on the E_2-page, since it is annihilated by y_3. In the appendix we use Singular to verify that y_3 generates the entire annihilator in the E_3-page (this is even true in the E_2-page). Thus, we get that the E_4-page is:

$$E_4^{*,*} = A/(d_2(u), \mathrm{Sq}^1(d_2(u)))[u^4] \oplus u^2(y_3)[u^4],$$

where the ideal (y_3) is taken in the ring $A/(d_2(u))$.

The E_5-page is the same as the E_4-page because d_4 vanishes for degree reasons on ring generators (u^4, u^2y_3) of the above ring. Again using the Kudo transgression theorem, the next possible differential is $d_5(u^4) = \mathrm{Sq}^2\mathrm{Sq}^1 d_2(u)$. By a similar computation as for

Sq^1, we get $d_5(u^4) = y_1z^2 + y_1^3\beta_1 + y_2z^2 + y_2^3\beta_2$. This, fortunately, is 0 on the E_3-page as we verify in the appendix. Now the spectral sequence collapses: for degree reasons all remaining differentials are zero on the ring generators for the E_4-page and thus on their entire pages of definition. As a result, the $E_4^{*,*}$ obtained above gives $H^*(B(3,Q_{16});\mathbb{F}_2)$ as an $\mathbb{F}_2$-vector space. We are not claiming that the ring structure we found for the E_4-page is the correct cohomology ring for $B(3,Q_{16})$ as we were unable to decide the multiplicative extension problems.

10. Homology of $B(q, SU(2))$

We finish with a recursive description of the homotopy type of $B(q,SU(2))$ which yields homology isomorphisms between the $B(q,SU(2))$, when homology is taken with coefficients for which 2 is invertible. For $q \geq 2$ consider the map $PU(2) \times (Q_{2^{q+1}})^n \to \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2))$ where $(g,x_1,\dots,x_n) \mapsto (gx_1g^{-1},\dots,gx_ng^{-1})$. It is well defined in the quotient

$$PU(2) \times_{N_{q+1}} (Q_{2^{q+1}})^n \to \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2))$$

where the normalizer $N_{q+1} := N_{PU(2)}(D_{2^q})$ acts by translation on $PU(2)$ and by conjugation on $(Q_{2^{q+1}})^n$. Consider the subsets $\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}}), \mathrm{Hom}(F_n/\Gamma_n^q, Q_{2^{q+1}}) \subset (Q_{2^{q+1}})^n$ and the restrictions of the above map to the respective subspaces

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & PU(2) \times_{N_{q+1}} \mathcal{H}_{q+1} \arrow[r] \arrow[d, hook] & \mathrm{Hom}(F_n/\Gamma_n^q, SU(2)) \arrow[d, hook] \\
PU(2) \times_{N_{q+1}} \mathcal{E}_{q+1} \arrow[r, hook] & PU(2) \times_{N_{q+1}} (Q_{2^{q+1}})^n \arrow[r] & \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)),
\end{tikzcd}$$

where we have set $\mathcal{E}_{q+1} = \mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}})$ and $\mathcal{H}_{q+1} = \mathrm{Hom}(F_n/\Gamma_n^q, Q_{2^{q+1}})$, solely to make the diagram fit the page.

We claim that all the above squares are pushouts. From the proof of Theorem 3.1 one can deduce the homeomorphism

$$PU(2) \times \mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}})/N_{q+1} \cong \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) - \mathrm{Hom}(F_n/\Gamma_n^q, SU(2)).$$

Since $\mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}})$ is discrete; N_{q+1} is finite and acts freely, we have that

$$PU(2) \times \mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}})/N_{q+1} \cong PU(2) \times_{N_{q+1}} \mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}})$$

which proves the outside square to be a pushout in sets. But, $\mathrm{Hom}(F_n/\Gamma_n^q, SU(2))$ is closed and $PU(2) \times_{N_{q+1}} \mathrm{Epi}(F_n/\Gamma_n^q, Q_{2^{q+1}})$ is the image of a compact space, so that $\mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2))$ has the disjoint union topology. This proves our claim for the outside square, and a similar argument can be used for the square on the left side. Now we look at the right square. The outside and left squares being pushouts imply the right square is also a pushout. The middle arrow is a closed cofibration, and thus the right square is also a homotopy pushout. Moreover, since the maps are either inclusions or given by conjugation, all arrows are simplicial maps. Geometric realization commutes with colimits and homotopy colimits, and hence

$$\begin{tikzcd} PU(2) \times_{N_{q+1}} B(q, Q_{2^{q+1}}) \arrow[r] \arrow[d] & B(q, SU(2)) \arrow[d] \\ PU(2) \times_{N_{q+1}} BQ_{2^{q+1}} \arrow[r] & B(q+1, SU(2)) \end{tikzcd}$$

is a pushout of topological spaces and a homotopy pushout.

Theorem 10.1. *The inclusions in the filtration*

$$B_{\mathrm{com}}SU(2) \subset B(3, SU(2)) \subset \cdots \subset B(q, SU(2)) \subset \cdots$$

are homology isomorphisms with coefficients over a ring R where 2 is invertible. Moreover, the cohomology ring of each term of the filtration is given by $R[c_2, y_1]/(y_1^2)$, where $\deg(c_2) = \deg(y_1) = 4$, *and c_2 is the pullback of the second Chern class $c_2 \in H^4(BSU(2); R)$ under the inclusion $B(q, SU(2)) \to BSU(2)$.*

Proof. Since we have shown that $B(q, Q_{2^{q+1}})$ is the classifying space of an amalgamated product of finite 2-groups, its homology with coefficients in any ring in which 2 is invertible, vanishes, as does that of $BQ_{2^{q+1}}$. This implies that over such a ring the left vertical arrow is a homology isomorphism and thus so is the right vertical arrow. The second part of the statement follows from the integral cohomology ring computations in [7, Theorem 4.2] $\square$

Remark 10.2. The filtration $B(2, SU(2)) \subseteq B(3, SU(2)) \subseteq \cdots \subseteq BSU(2)$ is not exhaustive. Indeed, by the previous proposition, the union $\bigcup_{q\geq 2} B(q, SU(2))$ has the rational homology of any of the terms in the union, which differs from the rational homology of $BSU(2)$. This is to be expected, since $SU(2)$ has finitely generated subgroups (even finite subgroups) which are not nilpotent.

Remark 10.3. Recently D. Ramras and M. Stafa in [16, Corollary 11.5] gave conditions on m for rational homological stability for $m \mapsto B(q, SU(m))_{\mathbb{1}}$ with $q \geq 2$. This sounds similar to our Theorem 10.1, but beyond the fact that we only deal with $m = 2$, there are two further differences in the setup: we vary q, not m, and we deal with the full $B(q, SU(m))$, not $B(q, SU(m))_{\mathbb{1}}$; both of these differences are significant and it seems there is not much relation between our result and their work.

There is no easy relationship between $B(q, SU(m))_{\mathbb{1}}$ and $B(q, SU(m))$, and it is an interesting question if the same stability found in [16] holds for $m \mapsto B(q, SU(m))$. Also, for a fixed $m > 2$ one could ask for stability conditions on $q \mapsto B(q, SU(m))$. For the reduced version, $B(q, SU(m))_{\mathbb{1}} \hookrightarrow B(q+1, SU(m))_{\mathbb{1}}$ is a homeomorphism for every $q \geq 2$ by a result of M. Bergeron and L. Silberman [10]. But this is no longer the case for the full non-$(-)_{\mathbb{1}}$ version. Indeed, for $SU(2)$, one can deduce from the $B\mathbb{Z}/2$-bundle $B(3, SU(2)) \to B_{\mathrm{com}}SO(3)$ shown bellow and [17, Proposition 4.4] that $\pi_2(B(3, SU(2))) = \mathbb{Z}/2$, but $\pi_2(B_{\mathrm{com}}SU(2)) = 0$.

Finally, here are some relations with the classifying spaces $B(q, SO(3))$ and $B(q, U(2))$. Let $q \geq 2$ and consider the map $\pi_n := \pi_* \colon \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \to \mathrm{Hom}(F_n/\Gamma_n^q, SO(3))$. In section 3 we said that π_n is a 2^n-fold covering map for every $n \geq 0$. Moreover, we have the pullback diagrams

$$\begin{tikzcd} \mathrm{Hom}(F_n/\Gamma_n^{q+1}, SU(2)) \arrow[r, hook] \arrow[d, "\pi_n"] & SU(2)^n \arrow[d, "(\pi)^n"] \\ \mathrm{Hom}(F_n/\Gamma_n^q, SO(3)) \arrow[r, hook] & SO(3)^n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Hom}(\mathbb{Z}^n, SU(2)) \arrow[r, hook] \arrow[d, "\pi_n"] & SU(2)^n \arrow[d, "(\pi)^n"] \\ \mathrm{Hom}(\mathbb{Z}^n, SO(3))_{\mathbb{1}} \arrow[r, hook] & SO(3)^n. \end{tikzcd}$$

Again, all maps in the squares are simplicial. Since geometric realization of simplicial spaces commutes with taking pullbacks, we obtain pullback squares of topological spaces:

$$\begin{tikzcd} B(q+1, SU(2)) \arrow[r] \arrow[d] & BSU(2) \arrow[d] \\ B(q, SO(3)) \arrow[r] & BSO(3) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_{\mathrm{com}}SU(2) \arrow[r] \arrow[d] & BSU(2) \arrow[d] \\ B_{\mathrm{com}}SO(3)_{\mathbb{1}} \arrow[r] & BSO(3). \end{tikzcd}$$

The right hand side arrow is a $B\mathbb{Z}/2$-bundle, hence the two arrows on the left are also principal $B\mathbb{Z}/2$-bundles.

Corollary 10.4. *Let $q \geq 2$. Then the maps induced by the double cover $SU(2) \to SO(3)$*

1. *$B(q, SU(2)) \to B(q, SO(3))$, and*
2. *$B_{\mathrm{com}}SU(2) \to B_{\mathrm{com}}SO(3)_{\mathbb{1}}$*

are homology isomorphisms with coefficients over a ring R in which 2 is invertible.

Proof. Consider the composition $B(q, SU(2)) \to B(q+1, SU(2)) \to B(q, SO(3))$. By Theorem 10.1, the first arrow is an $H_*(-; R)$ isomorphism, and the above argument shows that $B(q+1, SU(2)) \to B(q, SO(3))$ is an $H_*(-; R)$ isomorphism as well, since $H_*(B\mathbb{Z}/2; R) = H_*(\mathrm{pt}; R)$. □

Remark 10.5. The case $q = 2$ of Corollary 10.4 was recently proved in [15, Theorem 4.7]

Now, let $q \geq 2$ and consider the group homomorphism $\psi \colon S^1 \times SU(2) \to U(2)$ given by $(\lambda, x) \mapsto \lambda x$, where S^1 represents the scalar matrices in $U(2)$. The kernel of ψ is the group of order 2 generated by $(-1, -I)$. For each n we have the pullback square

$$\begin{tikzcd} (S^1)^n \times \mathrm{Hom}(F_n/\Gamma_n^q, SU(2)) \arrow[r, hook] \arrow[d, "\pi_n"] & (S^1 \times SU(2))^n \arrow[d, "(\psi)^n"] \\ \mathrm{Hom}(F_n/\Gamma_n^q, U(2)) \arrow[r, hook] & U(2)^n \end{tikzcd}$$

which for the same reasons as before, after taking geometric realizations gives a $B\mathbb{Z}/2$-bundle

$$B\mathbb{Z}/2 \to BS^1 \times B(q, SU(2)) \to B(q, U(2)).$$

Under the same hypotheses as in Corollary 10.4, we have that $BS^1 \times B(q, SU(2)) \to B(q, U(2))$ is an $H_*(-; R)$ isomorphism.

Appendix A. Computing annihilators in Singular

We made two claims about annihilators in the calculation of $H^*(B(3, Q_{16}))$, namely, that $\mathrm{ann}_A(k) = 0$ and $\mathrm{ann}_{A/(k)}(y_1z + y_1\beta_1 + y_2z + y_2\beta_2) = (y_3)$ where $A = H^*(B\Gamma/\mu_2)$ and $k = z+\beta_1+y_1^2+\beta_2+y_1^2$ is the k-invariant of the central extension $1 \to \mu_2 \to \Gamma \to \Gamma/\mu_2 \to 1$. We verified these claims with help from the Singular computer algebra system [12]. We include the code used for the benefit of potential readers facing similar computations but unfamiliar with what computer algebra systems can offer and how easy to use they can be.

```
ring P = 2, (y1,y2,y3,b1,b2,z), dp;
ideal I = y1*y2, y1*y3, y2*y3, y3^2, y2*b1, y3*b1, y1*b2, y3*b2,
          b1*b2, b1^2-y1^2*z, b2^2-y2^2*z;
qring A = std(I);
poly k = z + b1 + y1^2 + b2 + y2^2;
quotient(0, k);
```

This defines P as the polynomial ring $\mathbb{F}_2[y_1, y_2, y_3, \beta_1, \beta_2, z]$; the 2 specifies the coefficients are from $\mathbb{F}_2$. Then it defines A as the quotient P/I (when A is defined, P is the "current ring" and thus the quotient is implicitly a quotient of A), and asks for generators of the ideal $(0 : (k))$. Singular responds `_[1]=0`, indicating that k is not a zero divisor. To compute the annihilator of $y_1z + y_1\beta_1 + y_2z + y_2\beta_2$ in $A/(k)$, we can do:

```
qring A_mod_k = std(k);
quotient(0, y1*z + y1*b1 + y2*z + y2*b2);
```

Singular responds `_[1]=y_3` indicating the annihilator is generated by y_3.

We also needed to show that $y_1z^2 + y_1^3b_1 + y_2z^2 + y_2^3b_2$ is zero in $A/(k)$. We can check this by asking for `reduce(y1*z^2 + y1^3*b1 + y2*z^2 + y2^3*b2, std(z^3));` when `A_mod_k` is the current ring.

Acknowledgements. We would like to thank José Manuel Gómez for helpful suggestions about $U(2)$ and for motivating us to look into $SU(m)$ and $U(m)$ for $m > 2$. In an earlier version of this paper, during his PhD, the second author was supported by a CONACYT (Mexico) scholarship in "Programa de Apoyos para el Fomento, la Formación, el Desarrollo y la Consolidación de Científicos y Tecnólogos y de Recursos Humanos de Alto Nivel." In the latest version of this manuscript, the second author was supported by CONACYT **265667**.

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